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Comments

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Comment on "Cosmological solution of Einstein's equations with uniform density and nonuniform pressure"

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The exact solution presented by Wesson and Ponce de Leon [Phys. Rev. D **39**, 420 (1989)] is a specific spherically symmetric particular case of the general, nonstatic, conformally flat perfect-fluid metric. The global and causal structure of this specific solution is discussed.

The spherically symmetric, perfect-fluid, exact solution recently presented by Wesson and Ponce de Leon[1] is described by the metric [their Eqs. (3a)–(3c)]

$$ds^2=\frac{c\,dt^2}{(1-\alpha\beta^2r^2t^{2/3})^2}-\frac{\beta^2c^2t^{4/3}}{(1-\alpha\beta^2r^2t^{2/3})^2}(dr^2+r^2d\Omega^2)\ , \qquad (1)$$

where $d\Omega^2\equiv d\theta^2+\sin^2\theta\,d\phi^2$, and α and β are arbitrary constants. This solution is a particular case of the well-known, spherically symmetric, nonstatic subclass of conformally flat, perfect-fluid solutions,[2,3] whose metric has the general form[4]

$$ds^2=\left[\frac{1-\frac{r^2}{4}\Gamma}{1+\frac{r^2}{4}LH_0^2}\right]^2dt^2-\frac{H_0^2}{\left[1+\frac{r^2}{4}LH_0^2\right]^2}(dr^2+r^2d\Omega^2)\ , \qquad (2)$$

where $L=L(t)$ and $H_0=H_0(t)$ are arbitrary functions, and

$$\Gamma(t)\equiv LH_0^2\left[1+\frac{(d/dt)\ln L}{(d/dt)\ln H_0}\right]\ .$$

The metric (1) is the particular case of (2) obtained from setting $c=1$, $\beta=\frac{1}{2}$, $H_0=t^{2/3}$, and $L=-\alpha t^{-2/3}$. Wesson and Ponce de Leon mention (their Ref. 5) a solution similar to theirs found by Henriksen, Emslie, and Wesson.[5] The metric found by the latter authors, which follows from their Eqs. (6), (8), (9), and (60), is also a particular case of (2). This is easily verified by rescaling the time and radial coordinate of (2) as $t=(1-C)T$ and $r=2X^C$, where C is an arbitrary constant. Hence, by setting $H_0=T^{1-C}$ and $L=T^{-2}$, one arrives at the metric of Henriksen, Emslie, and Wesson (with T and X denoting their time and radial coordinates).

The conformally flat class of solutions described by the metric (2) were discovered originally by Kustaanheimo and Qvist,[3,6,7] and have been used mostly as models of collapsing stars[8–13] and (less so) as cosmological spacetimes.[4,14,15] For a survey of their global properties, singularities, and causal structure, see Sec. IX of Ref. 4.

In their Sec. I Wesson and Ponce de Leon suggest a cosmological interest in the metric (1) [with $\alpha<0$ (since Wesson and Ponce de Leon are concerned mostly with the case $\alpha<0$ in (1); the case $\alpha>0$ will not be considered)] because in the coordinates they use $\rho=\rho(t)$ and also because it "departs modestly enough from the standard ones." Presumably, they mean a modest departure from an Einstein–de Sitter spacetime (hence their choice $H_0=t^{2/3}$) to which their solution is manifestly conformal. However, even if the solution (1) somehow generalizes the Einstein–de Sitter spacetime [the particular case of (1) with $\alpha=0$], and its study merits the work invested in it, its global and local properties are radically different from an Einstein–de Sitter model. Wesson and Ponce de Leon hinted this fact, by remarking that [from their Eq. (3e)] "pressure increases away from the origin" and, hence, recommended truncating the solution in order to use it as a model of a "bubble" matched to a suitable cosmological background. However, these authors did not elaborate any further on the asymptotical properties, nor mention the existence of a scalar curvature singularity of the "finite density" type as $r\to\infty$. The latter singularities, characterized by a finite density with a diverging pressure, appear frequently in many spherically symmetric shear-free solutions.[16]

The global view of the solution (1) can be better appreciated by using other coordinates. Since the simple time rescaling $t'=3c\beta t^{1/3}$ transforms (1) into a manifestly conformally flat metric, the conformal compactification technique presented in Sec. 5.1 of Hawking and Ellis[17] can be applied to it in order to visualize its singularities and its causal and global structure. Taking $c=\beta=1$, and following Hawking and Ellis, the coordinate transformation

$$6t^{1/3}=\tan[\tfrac{1}{2}(\eta+\chi)]+\tan[\tfrac{1}{2}(\eta-\chi)]\ ,$$
$$2r=\tan[\tfrac{1}{2}(\eta+\chi)]-\tan[\tfrac{1}{2}(\eta-\chi)]$$

brings the metric (1) into the form

$$ds^2=\Phi^2(\eta,\chi)(d\eta^2-d\chi^2-\sin^2\chi\, d\Omega^2) \tag{3}$$

with

$$\Phi(\eta,\chi)=\frac{\{\tan[\frac{1}{2}(\eta+\chi)]+\tan[\frac{1}{2}(\eta-\chi)]\}^2}{18\cos[\frac{1}{2}(\eta+\chi)]\cos[\frac{1}{2}(\eta+\chi)]\left[1+\frac{|\alpha|}{36}\{\tan^2[\frac{1}{2}(\eta+\chi)]-\tan^2[\frac{1}{2}(\eta-\chi)]\}^2\right]}\ ,$$

which is conformal to the section bounded by $\chi=0$, $\eta=0$, and $\eta+\chi=\pi$ of the Einstein static Universe. The (η,χ) coordinate representation of this section (displayed in Fig. 1) is the "Penrose diagram" of the manifold.

Apparently, the Penrose diagram of Fig. 1 coincides with that of Fig. 21 (iii) of Hawking and Ellis.[17] A "big-bang" singularity, characterized by $p\to\infty$, $\rho\to\infty$, and $R=(-g_{\theta\theta})^{1/2}=0$, and marked by $t=0$ in the coordinates of (1), also appears as the spacelike singularity labeled by $\eta=0$, $0<\chi<\pi$. The locus $\chi=0$, $0<\eta<\pi$ also marks the only regular symmetry center ($r=0$). The world lines of observers comoving with the fluid ($r=$const) and the three-surfaces of constant t are marked by the same vertical and horizontal dotted curves as in Fig. 21 (iii) of Hawking and Ellis. The former world lines are seen to "converge" towards the locus $i_+=\{\eta=\pi,\chi=0\}$ ($t\to\infty, r$ finite), at which the proper time diverges [i.e., $\tau=\int(g_{tt})^{1/2}\to\infty$]. Hence, as in the Einstein–de Sitter case, this locus can be identified with a regular ($\rho\to 0$, $p\to|\alpha|r$) "future timelike infinity."

However, there are significant differences in comparison with the Einstein–de Sitter case. (1) At i_+, one has $R\to 1/|\alpha|r$ as $t\to\infty$ for $r>0$, which is a different behavior from that of an Einstein–de Sitter spacetime. (2) The integral $\int(-g_{rr})^{1/2}dr$ (t fixed) converges as $r\to\infty$ indicating that infinite values of r (unlike the Einstein–de Sitter case) correspond to finite proper distances along the three-surfaces of constant t (orthogonal to the four-velocity). (3) At the locus $\eta+\chi=\pi$ ($t\to\infty$, $r\to\infty$), one has $R\to 0$ and $p\to\infty$, instead of finite p and $R\to\infty$ as would be the case in an Einstein–de Sitter spacetime. Hence, a null singularity of the "finite density" type seems to replace the regular "spacelike infinity" and "future null infinity" depicted in Fig. 21 (iii) of Ref. 17.

The currently accepted criterion defining scalar curvature singularities requires (besides diverging curvature scalars) that the affine parameter of well-defined causal curves be finite as the singularity is approached along these curves. A suitable parameter for this purpose is the affine parameter ϑ along null geodesics of Fig. 1 whose tangent vectors are given by $d/dv=\partial/\partial\eta+\partial/\partial\chi$ ("outgoing") and $d/dw=\partial/\partial\eta-\partial/\partial\chi$ ("ingoing"). Applying Eqs. (51) and (55) of Ref. 16 and Eq. (2.29) of Ref. 17 to "outgoing" null geodesics of the metric (3), leads to the integral $\vartheta=\int\Phi(v,w)dv$, which must be evaluated keeping w fixed. From the metric (3), this integral clearly converges as $v\to\pi$, so that the locus $v=\eta+\chi=\pi$ does mark a null singularity of the "finite density" type.

Since $R\to 0$ as null geodesics approach the finite density singularity, the latter is indeed a singular point (singular center of symmetry) following a null world line rather than a "surface." Were it not for this strange singularity, the three-surfaces of constant coordinate time would be homeomorphic to three-spheres with $\chi=0$ and $\chi=\pi$ marking two regular centers of symmetry. The spacetime would have then the $\mathbf{S}^3\times\mathbf{R}$ topology of a "closed" Friedmann universe.[4] However, the existence of this singulari-

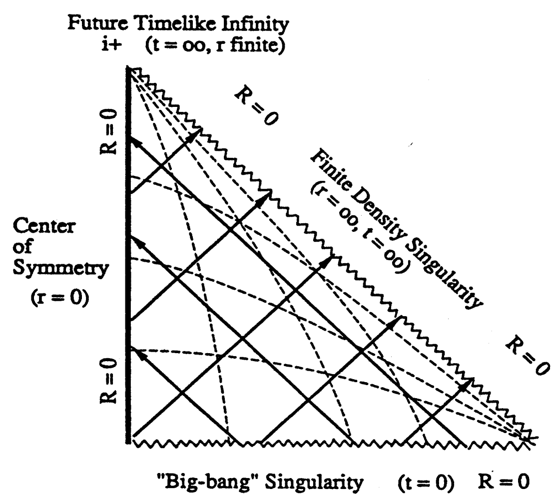

FIG. 1. Penrose diagram of the solution described by the metrics (1) and (3). This diagram is the (η,χ) representation of the conformal compactification of the metric (1) which leads to the metric (3). Each point represents a two-sphere of surface area $4\pi R^2(\eta,\chi)$, where $R^2=-g_{\theta\theta}$. "Ingoing" and "outgoing" radial null geodesics appear as solid straight lines with slopes ± 1. World lines of observers comoving with the fluid ($r=$const) appear as vertical dashed curves converging towards a future timelike infinity i_+. Radial curves along the three-surfaces ($t=$const) are depicted as horizontal dashed curves. The spacelike "big bang" and null "finite density" singularities are displayed as jagged lines.

ty makes these three-surfaces homeomorphic to $\mathbf{R}^3$, looking (if imbedded in $\mathbf{R}^4$) as spheroidal shapes punctured at $\chi=\pi$. Therefore, the full spacetime has an $\mathbf{R}^4$ topology (see Secs. II and III of Ref. 4), even if it bears more resemblance to a sort of "closed" Friedmann universe with a singular point than to an Einstein–de Sitter spacetime.

Even if (from Fig.1) the world lines of observers comoving with the fluid (world lines of galaxies) do not hit the finite density singularity, there must be other timelike world lines which terminate there. Besides these facts, near this singularity one has $dp/d\rho \rightarrow \infty$, and so one must agree with Wesson and Ponce de Leon's suggestion that this solution should be truncated and somehow matched to a suitable cosmological background. However, conformally flat solutions with metric (2), and this includes the one studied by Wesson and Ponce de Leon, cannot be matched smoothly (in the sense of Sec. XII of Ref. 4) to a Friedmann-Robertson-Walker background, though nonsmooth matchings (with discontinuous derivatives of the metric) with such a background can be achieved either along surfaces of constant r, or by means of the "thin-shell formalism."[18] Another possibility is to fix either one of the functions H_0 and L in (2) in order to match these solutions with a Schwarzschild spacetime,[4] leading then to models of collapsing spheres in an asymptotically flat background.

The analysis presented here can also be applied if $\alpha > 0$ and/or $t^{2/3}$ in (1) is replaced by an arbitrary function $H_0(t)$. If $\alpha < 0$ the results would be qualitatively similar, provided $H_0(t)$ has a zero. However, if $\alpha > 0$ the conformal factor in (3) would not be bounded everywhere, and so the global structure of the spacetime would change significantly.

Finally, it is worthwhile remarking that Wesson and Ponce de Leon found one conformal Killing field associated with the metric (1) (also Henriksen, Emslie, and Wesson found such a field in Ref. 5). However, being conformally flat, the solutions characterized by the metric (2) [and hence the particular case (1) and Eq. (60) of Ref. 5] admit the same G_{15} of conformal motions as Minkowski spacetime.[19] In particular, in the "radial direction" [the Lorentzian two-space parametrized by the coordinates t and r and orthogonal to the orbits of SO(3)], one can show[20] that these solutions admit three conformal Killing fields. An interesting feature of the metric (1) which passed unnoticed by Wesson and Ponce de Leon is the fact that it admits a conformal Killing vector parallel to the four-velocity of the fluid (see Ref. 20 for details).

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