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Fermions in an Aharonov-Bohm field and cosmic strings

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The scattering of fermions by an infinitely thin flux tube is governed by a Dirac Hamiltonian that requires specification of a one-parameter self-adjoint extension. This Aharonov-Bohm scenario arises generically for cosmic strings. In particular for some range of the extension parameter the string can bind fermions.

Recently there has been completed a study of gravitational scattering by particles off a spinning source in two spatial dimensions.[1] This is also the relevant setting for (spinning) infinite cosmic strings in three spatial dimensions. In Ref. 1 it was observed that for an energy eigenstate the equations for a particle in the field of a massless spinning source are equivalent to those in a background Aharonov-Bohm[2] gauge field of an infinitely thin flux tube. Moreover the above study[1] revealed that the Dirac Hamiltonian on this background requires a self-adjoint extension, or, in other words, that nontrivial boundary conditions on the spinor wave functions have to be imposed at the origin, relaxing the conventional regularity requirements. The extensions can be parametrized by these boundary conditions and different choices lead to inequivalent theories.

On the other hand, Alford and Wilczek[3] observed that in typical cosmic-string scenarios the fermionic charges can be noninteger multiples of the Higgs charge. As the flux is quantized with respect to the Higgs charge this will give rise to a nontrivial Aharonov-Bohm scattering of these fermions. This is a phenomenologically interesting result, since the cross section is then much larger than the one coming from gravitational scattering, which is usually considered. The authors originally did not remark on the need for a self-adjoint extension of the Dirac Hamiltonian, but implicitly picked one value of the parameter that labels the extension.

In the present paper we show how the results of Ref. 1 complete the analysis of Ref. 3. We discuss the solutions to the Dirac equation in the Aharonov-Bohm background field of an infinitely thin vortex configuration in the relevant two spatial dimensions. We establish the one-parameter family of self-adjoint extensions of the Dirac Hamiltonian. Although this study was motivated by the above-mentioned physical context it has broader applications (see, e.g., Ref. 1). Therefore, the interior of the vortex is strictly treated as a "black box." The extension parameter is a manifestation of the physics within this "box" and together with the flux Φ it determines the effective Hamiltonian outside the vortex. In particular we find that for half the parameter values there are fermionic bound states. They have the property that for fixed extension parameter the adiabatic decrease of flux Φ (in units of $2\pi/e$) by unity lifts a level from $E=-m$ to $E=m$. For half-integral flux Φ we point out an analogy to earlier work in monopole physics.[4,5]

The organization of the paper is as follows. First, we state the problem and introduce the one-parameter family of extensions by a simple partial-integration argument. We then derive the resulting energy eigenfunctions and give the wave function for the scattering problem. We conclude with some general remarks that encompass bound states and gravitational scattering.[6] The mathematically rigorous derivation of the self-adjoint extensions is given in the Appendix.

We study the massive Dirac equation in an Aharonov-Bohm background potential. We take the γ matrices to be $\gamma^0\equiv\beta=\sigma^3,\gamma^1=i\sigma^2,\gamma^2=-i\sigma^1$. The electromagnetic potential $\mathbf{A}=-(\Phi/r)\hat{\varphi}$, where $\Phi\equiv$magnetic flux$/(2\pi/e)$; for the cosmic strings considered in Ref. 3, $\Phi=e/Q_{\text{Higgs}}$. $\mathbf{A}$ has the well-known property of being locally a pure gauge.

The Dirac equation for this problem is

$$(i\partial\!\!\!/+A\!\!\!/-m)\Psi(t;r,\varphi)=0\ . \tag{1}$$

Rotational symmetry allows passing to an eigenstate of angular momentum $n+\frac{1}{2}$. By defining

$$\Psi_{E;n}(t;r,\varphi)=\begin{bmatrix}\chi^1(r)\\ \chi^2(r)e^{i\varphi}\end{bmatrix}e^{in\varphi}e^{-iEt} \tag{2}$$

the radial eigenvalue problem is

$$h\chi(r)=\begin{bmatrix} m & -i\left[\partial_r+\dfrac{\nu+1}{r}\right]\\ -i\left[\partial_r-\dfrac{\nu}{r}\right] & -m\end{bmatrix}\chi(r)=E\chi(r) \tag{3}$$

with $\nu\equiv n+\Phi$. For $E^2>m^2$ it has the solutions

$$\chi_\nu(r)=\frac{1}{N}\begin{bmatrix}\sqrt{E+m}\,(\epsilon_n)^nJ_{\epsilon_n\nu}(kr)\\ i\sqrt{E-m}\,(\epsilon_n)^{n+1}J_{\epsilon_n(\nu+1)}(kr)\end{bmatrix}, \tag{4}$$

where N is a normalization factor, $k=\sqrt{E^2-m^2}$,

$\epsilon_n = \pm 1$, and J_λ denotes the Bessel functions. Their asymptotic behavior is given by

$$\lim_{x\to 0} J_\lambda(x) \sim \frac{x^\lambda}{2^\lambda \Gamma(1+\lambda)} ,$$
$$\lim_{x\to\infty} J_\lambda(x) \sim \left[\frac{2}{\pi x}\right]^{1/2} \cos(x - \tfrac{1}{2}\lambda\pi - \tfrac{1}{4}\pi) . \tag{5}$$

When Φ is nonintegral, the sign ϵ_n must be fixed. Square integrability at the origin does this except for the partial wave with

$$-1 < \nu < 0 \Longleftrightarrow n = -[\Phi] - 1 \tag{6}$$

($[x]$ denotes the largest integer $\leq x$). In that case both choices of sign lead to solutions that are square integrable, though singular in one component, at the origin. Henceforth we shall restrict ourselves to the study of the subspace of Eq. (6).

Insisting on regularity of both spinor components at the origin forces one to reject the two solutions in this eigenspace, entailing a loss of completeness in the angular basis. In mathematical terms a self-adjoint extension of the Dirac Hamiltonian is required, whose derivation by the standard theory of von Neumann deficiency indices[7] is given in the Appendix. Here we observe that the radial Hamiltonian is *symmetric* if, for arbitrary spinors $\varphi(r)$ and $\chi(r)$,

$$\int_0^\infty r\,dr\,\varphi^\dagger(r) h \chi(r) = \int_0^\infty r\,dr\,[h\varphi(r)]^\dagger \chi(r) . \tag{7}$$

This is easily established in our case as long as the endpoint contribution from the partial integration vanishes:

$$i \lim_{r\to 0} r\varphi^\dagger(r)\sigma^1\chi(r) = 0 . \tag{8}$$

A symmetric Hamiltonian is *self-adjoint,* if its domain coincides with that of its adjoint. Regularity of the spinor wave function at the origin is too strong a requirement, as the dual space then contains functions which are singular at the origin and the adjoint operator has a larger domain. One has to posit a boundary condition such that requiring (8) entails the *same* boundary condition in the dual space. An appropriate condition in our case is

$$\lim_{r\to 0} (mr)^\eta \chi^1(r) \cos\left[\frac{\pi}{4}+\frac{\theta}{2}\right] = i \lim_{r\to 0} (mr)^{1-\eta}\chi^2(r) \sin\left[\frac{\pi}{4}+\frac{\theta}{2}\right] \tag{9a}$$

or equivalently

$$\lim_{r\to 0} \chi(r) \sim \begin{bmatrix} i(mr)^{-\eta}\sin\left[\frac{\pi}{4}+\frac{\theta}{2}\right] \\ (mr)^{\eta-1}\cos\left[\frac{\pi}{4}+\frac{\theta}{2}\right] \end{bmatrix} , \tag{9b}$$

where $\eta = 1 - \{\Phi\}$, ($\{\Phi\} \equiv \Phi - [\Phi]$), and m is inserted to assure proper dimensionality. The angle $0 \leq \theta < 2\pi$ parametrizes the self-adjoint extensions. This result is derived formally in the Appendix.

With the boundary condition established, the energy eigenstates [for the critical case stated in Eq. (6): $\nu = -\eta$] are

$$\chi_\nu(r) = \frac{1}{\sqrt{2}}[1+(-1)^n \sin 2\mu \cos\nu\pi]^{-1/2} \begin{bmatrix} \sqrt{E+m}\,[\sin\mu J_\nu(kr) + (-1)^n \cos\mu J_{-\nu}(kr)] \\ i\sqrt{E-m}\,[\sin\mu J_{\nu+1}(kr) + (-1)^{n+1}\cos\mu J_{-(\nu+1)}(kr)] \end{bmatrix} \tag{10}$$

with μ related to θ by the relation

$$\tan\left[\frac{\pi}{4}+\frac{\theta}{2}\right] = (-1)^n \left[\frac{E+m}{E-m}\right]^{1/2} \left[\frac{k}{2m}\right]^{2\nu+1} \times \frac{\Gamma(-\nu)}{\Gamma(\nu+1)} \tan\mu . \tag{11}$$

In addition for $\pi/2 < \theta < 3\pi/2$ there is a bound state

$$B_\nu(r) = \left[\frac{2\sin(-\nu\pi)}{\pi} \frac{\kappa^2}{m - E(1+2\nu)}\right]^{1/2} \times \begin{bmatrix} \sqrt{m+E}\,K_\nu(\kappa r) \\ i\sqrt{m-E}\,K_{\nu+1}(\kappa r) \end{bmatrix} , \tag{12}$$

where $\kappa = -ik = \sqrt{m^2 - E^2}$ and $K_\nu(x)$ are modified Bessel functions. The bound-state energy is implicitly determined by the equation

$$\frac{(1+E/m)^{\nu+1}}{(1-E/m)^{-\nu}} = -2^{2\nu+1} \frac{\Gamma(\nu+1)}{\Gamma(-\nu)} \tan\left[\frac{\pi}{4}+\frac{\theta}{2}\right] . \tag{13}$$

This strongly suggests that this range of θ in the effective Hamiltonian parametrizes nontrivial physics in the core.

To complete the presentation, we state the result for the asymptotic wave function of the scattering problem, which can be obtained from Ref. 1:

$$\lim_{r\to\infty} \Psi_E(t;r,\varphi) \sim \begin{bmatrix} \sqrt{E+m} \\ \sqrt{E-m} \end{bmatrix} e^{ikr\cos\varphi - i\Phi(\varphi-\pi)} e^{-iEt} + \left[\frac{i}{r}\right]^{1/2} f(\varphi) \begin{bmatrix} \sqrt{E+m} \\ \sqrt{E-m}\, e^{i\varphi} \end{bmatrix} e^{i(kr-Et)} \tag{14}$$

with $0 \leq \varphi < 2\pi$ and the scattering amplitude

$$f(\varphi) = \frac{i}{\sqrt{2\pi k}} e^{-i([\Phi]+1)\varphi} \times \left[e^{i\pi\Phi}\frac{1}{1-e^{i\varphi}} + e^{-i\pi\Phi}\frac{1}{1-e^{-i\varphi}} - e^{2i\delta} \right] . \tag{15}$$

Here δ is related to μ of Eqs. (10) and (11) by

$$\tan\delta=\frac{1-\tan\mu}{1+\tan\mu}\tan\frac{\Phi\pi}{2}\ . \qquad (16)$$

The two terms on the right-hand side of Eq. (14) represent the incoming and the scattered wave, though we observe that the dominant topological interference (conventionally called the "Aharonov-Bohm effect") comes from the factor $e^{-i\Phi\varphi}$ in the first term of Eq. (14) (Ref. 2). It is independent of r, k, and the precise boundary condition at the origin (θ). Including it in the scattered wave would, however, lead to a scattering amplitude containing δ-function singularities. On the other hand, it is misleading to call $|f(\varphi)|^2$ the complete cross section for the Aharonov-Bohm effect.[3] For a more detailed discussion of these issues see Refs. 1 and 6.

We conclude with three comments.

The correct boundary condition has to be found for each individual case by an analysis of the specific physical situation. $\theta=-\pi/2$ ($\pi/2$) is equivalent to insisting that the upper (lower) component stay regular at the origin. The wave function given by Alford and Wilczek[3] corresponds to $\theta=-\pi/2$. An analysis of the zero-radius limit of extended, radially symmetric magnetic flux configurations without additional physics in the core exhibits indeed $\theta=\text{sgn}(\Phi)\pi/2$ as the correct choice:[8] the sign of the flux determines which component of the Dirac spinor diverges at the origin.

We would like to draw attention to the point that generically the angular dependence at the origin of the partial wave in question is nontrivial ($e^{-i[\Phi]\varphi}$ or $e^{-i[\Phi+1]\varphi}$ depending on the component), so the complete wave function diverges at the origin in a direction-dependent way.

The next remark concerns the bound states, when θ does not depend on a continuous variation of the flux Φ. Then an adiabatic decrease of Φ (in units of the flux quantum $2\pi/e$) between the integers $N\rightarrow N-1$ lifts an energy level $E=-m\rightarrow E=m$, as seen from Eq. (13). This scenario is plotted in Fig. 1 for $\theta=\pi$.

The third point addresses the inclusion of gravity[6] in the analysis of the Aharonov-Bohm effect for infinite cosmic strings, as suggested in Ref. 3. This has in fact already been performed in Ref. 1, where the equivalent problem of (gravitational) scattering off a *massive* spinning source in two spatial dimensions was studied.

We conclude by mentioning that Eq. (3) for $\nu=-\frac{1}{2}$ has previously arisen in the context of monopole physics.[4,5] There θ is a fundamental parameter of the theory and was shown to coincide with the vacuum angle.[5]

Note added. After this work was completed we received an unpublished chapter of the Ph.D. thesis of R. Rohm, Princeton University, 1985, that discusses the same issues as Ref. 3. Despite the very careful analysis, the author also overlooked the arbitrariness in his choice of boundary conditions. We are grateful to E. Witten for drawing our attention to this work and to R. Rohm for communicating his results to us.

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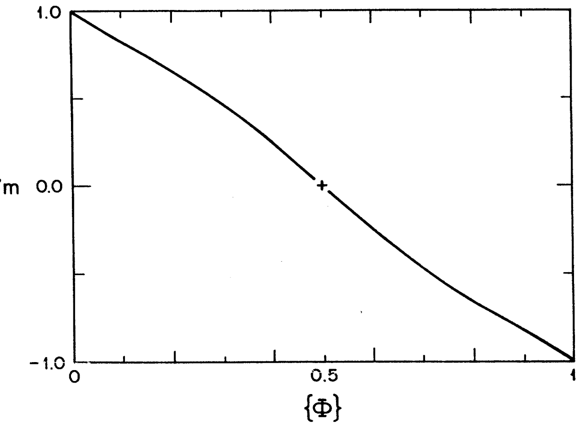

FIG. 1. Spectral flow of the bound-state energy upon adiabatic variation of the flux Φ between two integers [see Eq. (13)]. In the plot $\theta=\pi$ and the curve is symmetric upon reflection with respect to the point $\{\Phi\}=\frac{1}{2}$, $E=0$.

APPENDIX

We construct the self-adjoint extension of the Dirac Hamiltonian by the method of deficiency indices developed by von Neumann.[7] Let h be the radial Hamiltonian of Eq. (3) with domain $D=\{\phi(r)$: ϕ absolutely continuous, square integrable on the half-line with measure $r\,dr$ and regular at the origin$\}$. The theory requires constructing the eigenspaces $D^{\pm}$ of $h^{\dagger}$ with eigenvalue $\pm im$ ($m\neq 0$ is inserted for dimensional reasons). In our case they are spanned by the spinors

$$\phi^{\pm}(r)=\frac{1}{N}\begin{bmatrix} K_{\nu}(\sqrt{2}mr) \\ \pm e^{\pm i\pi/4}K_{\nu+1}(\sqrt{2}mr)\end{bmatrix}\ . \qquad (\text{A1})$$

[We are in the critical eigenspace of Eq. (6): $\nu=-1+\{\Phi\}$.] These belong to the dual space of D and the existence of complex eigenvalues for $h^{\dagger}$ emphasizes the lack of self-adjointness. The self-adjoint extensions of h are labeled by the isometries $D^{+}\rightarrow D^{-}$, which can be parametrized by

$$\phi^{+}(r)\rightarrow e^{i\omega}\phi^{-}(r)\ . \qquad (\text{A2})$$

The correct domain for the self-adjoint extension h^{ω} of h is then given by

$$D^{\omega}=\{\chi(r)=\phi(r)+\beta(\phi^{+}(r)+e^{i\omega}\phi^{-}(r)):\phi(r)\in D,\beta\in\mathbb{C}\}\ . \qquad (\text{A3})$$

This can be restated in terms of a boundary condition in the form of Eq. (9) yielding the relation

$$\tan\left[\frac{\pi}{4}+\frac{\theta}{2}\right]=\frac{\Gamma(-\nu)}{2^{\nu+1}\Gamma(1+\nu)}\frac{1}{\tan\frac{\omega}{2}-1}\ . \qquad (\text{A4})$$

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