Brief Reports

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Gravitational entropy of nonstationary black holes and spherical shells

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The problem of defining the gravitational entropy of a nonstationary black hole is considered in a simple model consisting of a spherical shell which collapses into a preexisting black hole. The second law of black-hole mechanics strongly suggests identifying one-quarter of the area of the event horizon as the gravitational entropy of the system. It is, however, impossible to accurately locate the position of the global event horizon using only local measurements. In order to maintain a local thermodynamics, it is suggested that the entropy of the black hole be identified with one-quarter the area of the apparent horizon. The difference between the event-horizon entropy (to the extent it can be determined) and the apparent-horizon entropy may then be interpreted as the gravitational entropy of the collapsing shell. The total (event-horizon) gravitational entropy evolves in a smooth (C^0) fashion, even in the presence of δ-functional shells of matter.

The first substantial progress in identifying a geometric quantity with a measure of gravitational entropy was made when Hawking[1] discovered that black holes radiate particles as if they were blackbodies with temperatures given by

$$T=\kappa/2\pi \ , \tag{1}$$

where κ is the surface gravity of the black hole (units are chosen so that $G=c=\hbar=k=1$). This discovery confirmed the hypothesis of Bekenstein[2] that the laws of black-hole mechanics are actually the laws of thermodynamics expressed for black holes. The second law of black-hole mechanics,[3] which states that the area of an event horizon does not decrease into the future (assuming the weak-energy condition), is thus identified with the second law of thermodynamics. The entropy of a stationary[4] black hole is found to be

$$S_{\rm BH}=\tfrac{1}{4}A \ , \tag{2}$$

where A is the area of the intersection of the horizon with a spacelike hypersurface. The second law of thermodynamics may then be generalized to describe systems including black holes and matter, by requiring that the sum of the black-hole entropy and the conventional matter entropy be a nondecreasing function into the future.

Since the recognition that black holes possess an entropy which can be quantified, there have been many efforts aimed at trying to identify some geometrical quantity associated with non-black-hole spacetimes which can be identified with the gravitational entropy. Penrose[5] has suggested that some appropriate measure associated with the Weyl curvature of the spacetime should be identified with the entropy. Hawking and Gibbons[6] have extended the notion of black-hole entropy to spacetimes containing cosmological event horizons, such as de Sitter space. Davies, Ford, and Page[7] attempted to establish the existence of a gravitational entropy for a stationary spherical shell of matter concentric around an interior black hole. They examined the changes in the black-hole thermodynamics caused by the presence of the spherical shell and concluded that no gravitational entropy could be associated with the shell.

Most previous studies have examined stationary states. Nonstationary states (and, in particular, the irreversible behavior associated with gravitational collapse) correspond to nonequilibrium thermodynamics. An essential step in setting up a theory of nonequilibrium thermodynamics is deciding on a set of rules for identifying a given nonequilibrium state with a fiducial equilibrium state[8] (so that the laws of thermodynamics may be applied to the nonequilibrium state). The purpose of this paper is to consider how to define the black-hole entropy in the nonstationary case. Two possibilities are obvious: that it be set equal to one-quarter the area of the global event horizon, or one-quarter the area of the apparent horizon. In the stationary limit these surfaces coincide. These definitions are explored in this paper in the context of a simple model in which a spherical shell collapses into a preexisting black hole.[9] The freedom of choice in defining the black-hole entropy here is somewhat similar to a situation encountered in setting up theories of relativistic dissipative fluids: in that case there are several ways to define the nonequilibrium four-velocity of the fluid, all of which agree in equilibrium,[8] but which lead to dynamically different theories of nonequilibrium relativistic hydrodynamics

The most obvious way to extend the result of Hawking[1] to the case of a nonstationary black hole is to take as the entropy of the black hole

$$S_1=\tfrac{1}{4}A_{\mathcal{H}}\ , \tag{3}$$

where $A_{\mathcal{H}}$ is the area of the intersection of the event horizon $\mathcal{H}$ with a spacelike hypersurface Σ. The event horizon is defined as the boundary of the past of future null infinity;[10] it is the actual "edge" of the black hole.

The gravitational entropy defined by Eq. (3) has the advantage that it is guaranteed to not decrease into the future (so long as the weak-energy condition holds; i.e., for classical matter interactions) according to the second law of black-hole mechanics. The entropy defined by Eq. (3) will also evolve smoothly (C^0) in time, even when δ-functional shells of matter are present (as is shown explicitly below). Based on these properties, it seems certain that one-quarter the area of the event horizon should be identified with the gravitational entropy on Σ. It is, however, not clear that S_1 should be identified as the black-hole entropy.

Equation (3) is a difficult definition to use in practice, because the definition of the event horizon is global in nature. The entire future history of the spacetime must be known in order to find the boundary of the past of future null infinity, and hence the horizon radius, here and now on Σ. If S_1 is accepted as the black-hole entropy, then that entropy is essentially unknowable in a practical sense. It is not realistically possible to obtain data over a complete Cauchy surface. For example, in terms of the model considered here, we could be unaware of a very large mass shell which is currently outside our past cosmological light cone, and which will collapse into the black hole at some future time. The possible existence of such a shell renders the evaluation of S_1 using Eq. (3) impossible; it negates any hope of defining a local, "closed" system which could be studied thermodynamically. If S_1 is taken as the black-hole entropy, it is perhaps also intuitively displeasing that the entropy of the black hole contains a contribution, on Σ, from matter which is still outside the hole on Σ.

In view of the difficulty of calculating the area of a global event horizon, it is perhaps worthwhile to consider another possible geometrical quantity which might define the entropy of a nonstationary black hole, namely, one-quarter of the surface area of the apparent horizon $\mathcal{A}$:

$$S_2=\tfrac{1}{4}A_{\mathcal{A}}\ , \tag{4}$$

where this area is to be measured (as in S_1) on some spacelike hypersurface Σ. The apparent horizon is defined as the boundary of the trapped region, or as the outermost trapped surface.[11] The position of the apparent horizon, unlike the event horizon, can be determined using entirely local measurements. In the stationary limit, the apparent horizon and event horizon are identical, and so Eq. (4) agrees with the known result[1] in that limit.

Unlike the event horizon, the apparent horizon can sometimes evolve in a discontinuous manner, e.g., when a δ-functional shell of matter enters a black hole. There can even exist multiple apparent horizons[11] at some times; in this case it seems appropriate to define the black-hole entropy as one-quarter the area of the outermost apparent horizon.

As discussed above, the second law of black-hole mechanics strongly suggests that the actual event-horizon area should be identified with the gravitational entropy, regardless of how difficult it is to locate the event horizon. However, we can accept this and still obtain a local thermodynamic description (not involving the entire future history of the Universe) if we distinguish between the total gravitational entropy of the system (S_1) and the gravitational entropy of the black hole (S_2). We are then led to consider the possibility of identifying the difference, S_1-S_2, as the gravitational entropy of the collapsing shell. In this way it is possible to define, with only partial Cauchy data on a hypersurface Σ, gravitational entropies for all components of the system of which we are aware, viz., the black hole and the collapsing shell. It is intuitively attractive that the black-hole entropy thus defined does not contain contributions from matter which has not yet crossed the event horizon. If there should exist other shells (possibly currently outside of our past light cone) which might collapse into the black hole in the distant future, their influence on the global event horizon will not affect our ability to calculate useful entropies for the currently known black hole and shell.

In order to illustrate these ideas, consider the collapse of a thin spherical shell into a preexisting Schwarzschild black hole. The geometry inside the shell is described by the Schwarzschild metric with mass m:

$$ds^2=-\left[1-\frac{2m}{r}\right]dT^2+\left[1-\frac{2m}{r}\right]^{-1}dr^2+r^2d\Omega^2\ , \tag{5}$$

where $d\Omega^2$ is the metric of the two-sphere. Outside the shell the geometry is described by the Schwarzschild metric with mass M:

$$ds^2=-\left[1-\frac{2M}{r}\right]dt^2+\left[1-\frac{2M}{r}\right]^{-1}dr^2+r^2d\Omega^2\ . \tag{6}$$

The radial coordinates r in Eqs. (5) and (6) may be identified since they are defined geometrically using the area of the two-spheres. The equation of motion for the shell is then[12]

$$\left[1-\frac{2m}{R}+\left[\frac{dR}{d\tau}\right]^2\right]^{1/2}-\left[1-\frac{2M}{R}+\left[\frac{dR}{d\tau}\right]^2\right]^{1/2}=\frac{\mu}{R}\ , \tag{7}$$

where R is the radius of the shell, τ is proper time along the shell's trajectory, and μ is the proper mass of the shell (for dust, μ=const along the trajectory). For simplicity in integrating the motion, I shall consider a shell of dust which is marginally bound (i.e., $dR/d\tau\rightarrow 0$ as $R\rightarrow\infty$);

for such a shell, $\mu = M - m$.

It is most convenient to describe the evolution of the system as a function of advanced time. Surfaces of constant advanced time are null rather than spacelike, but have the advantage of providing a natural time coordinate along the horizons (unlike, say, the Schwarzschild time coordinates T and t). Advanced time coordinates may be defined in the Schwarzschild interior and exterior in the usual way:

$$V = T + r + 2m \ln \left[\frac{r}{2m} - 1 \right] - V_0 , \quad (8)$$

$$v = t + r + 2M \ln \left[\frac{r}{2M} - 1 \right] - v_0 , \quad (9)$$

where V_0 and v_0 are constants chosen so that $V = v = 0$ when $R = 2M$. The radius of the event horizon, r_H, as a function of interior advanced time is then given implicitly by

$$V = 2(r_H - 2M) + 4m \ln \left[\frac{r_H - 2m}{2(M-m)} \right] . \quad (10)$$

The shell radius may now be found as a function of interior and exterior advanced time by integrating Eq. (7) using Eqs. (5) and (6) and (8) and (9). For the case of marginally bound dust, all of these integrations may be performed analytically; however, several functions are encountered which cannot be inverted analytically [as in Eq. (10)]. The resulting expressions for $R(V)$ and $R(v)$ may then be combined with Eq. (10) to find $r_H(v)$.

The radius of the event horizon as a function of advanced exterior time v, is shown for several values of the ratio m/M in Fig. 1. The gravitational entropy S_1, however interpreted, is equal to $4\pi r_H^2$. The gravitational entropy S_2 is $4\pi m^2$ for $v<0$, and $4\pi M^2$ for $v>0$. If we choose to identify S_2 as the gravitational entropy of the black hole, then the gravitational entropy of the collapsing shell is $S_1 - S_2$. This difference is small, but nonzero, until close to the time of the shell's collapse into the black hole. The radius of the event horizon (and hence its area, and the total gravitational entropy) is a continuous function of advanced time, but only a C^0 function of advanced time at the point where the shell enters the black hole; this is a result of using a δ-functional shell: smooth matter distributions will yield smooth functions for the matter, hole, and total gravitational entropies.

For times long before the shell reaches the event horizon, the event horizon will nearly coincide with the apparent horizon, making the shell's gravitational entropy very small. For advanced times $v \ll 0$ the gravitational entropy of the shell is approximately given by

$$S_{\text{shell}} \simeq 4\pi m (m\mu)^{1/2} \exp(v/4m) , \quad (11)$$

where μ is the mass of the shell (recall $\mu = M - m$). Note that this entropy approaches zero as either $\mu \to 0$ or $m \to 0$.

If we adopt the viewpoint that S_2 is the gravitational entropy of the black hole, then some matter configurations will possess gravitational entropies. This

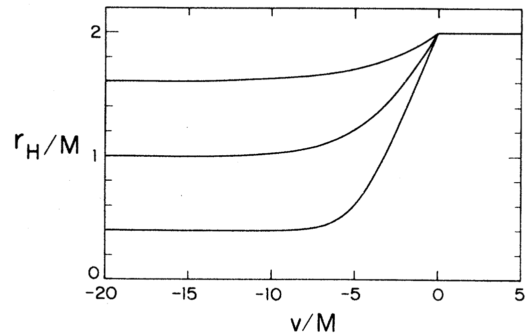

FIG. 1. The radius of the event horizon divided by the exterior mass, r_H/M, is plotted versus exterior advanced time divided by the exterior mass, v/M, for shells collapsing into preexisting black holes with masses $m = 0.2M$, $0.5M$, $0.8M$. The collapsing shell reaches radius $2M$ at $v=0$. The outermost apparent horizon is at $r_A = 2m$ for $v<0$ and at $r_A = 2M$ for $v>0$. The total gravitational entropy is πr_H^2; if πr_A^2 is identified as the black-hole entropy, then $\pi(r_H^2 - r_A^2)$ is the gravitational entropy of the collapsing shell.

is in contrast with previous studies,[6,7] which have found no entropy associated with the gravitational fields of matter outside horizons. Here, however, only those matter configurations which eventually enter black holes (and thus affect the horizon area) will possess a gravitational entropy. One might think that this is an uninteresting class of material configurations; surely most stars have low enough masses that they will not form black holes after leaving the main sequence. However, studies of the late evolution of our Universe[13,14] indicate that galaxies and clusters of galaxies will collapse to form supermassive black holes at $t \approx 10^{18}$ yr (assuming the Universe is open, $k=-1$; collapse of these structures to form black holes will probably take place sooner in a closed universe). Thus, much of the matter in the Universe today may eventually enter a black hole, and thus could possess a gravitational entropy even today. If we assume that our Sun will eventually join a supermassive black hole (the nucleus of which already may exist at our Galaxy's center), then we may use Eq. (11) to make an order-of-magnitude estimate of the Sun's gravitational entropy today. Taking μ to be the mass of the Sun, m to be the mass of supermassive hole, and v the cosmic time when our Sun enters the hole (10^{18} yr), then

$$S_\odot \approx 10^{-10^{19}} , \quad (12)$$

a very small number compared to the ordinary, material entropy of a solar mass star, about 10^{58} in natural units. On the other hand, the reason for the smallness of $S_\odot$ is the large amount of time (in units of m) between the present and the epoch when galactic collapse occurs. Again using Eq. (11), one finds that the gravitational entropy of a solar mass star will exceed its ordinary, material entropy about one year before it enters a galactic mass black hole.

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[3]See, e.g., S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973), Proposition 9.2.7.
[4]While the proof of the second law of black-hole mechanics does not depend on the black hole being stationary, the Hawking evaporation process does; the identification of thermodynamic quantities in the nonstationary case is then not certain *a priori.*
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