Radiative corrections and renormalization at finite temperature: A real-time approach

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We analyze the one-loop radiative corrections to finite-temperature decay and scattering rates, using techniques based on the Niemi-Semenoff time-path formulation of finite-temperature field theory. Previous work is examined in the context of our general framework. We find that the finite-temperature part of the self-energy corrections cannot be absorbed into mass and wave-function renormalization counterterms and argue that finite-temperature renormalization is not a meaningful concept. We give an explicit algorithm for the calculation of the finite-temperature self-energy corrections and discuss applications in cosmology and astrophysics.

I. INTRODUCTION

Quantum field theory at finite temperature and density has become an important tool in the study of elementary particles in hot and/or dense environments, such as the early Universe, neutron stars, and heavy-ion collisions. A comprehensive review of the subject, in particular of the different approaches to finite-temperature field theory—the older Euclidean (imaginary-time) Matsubara formalism, the more recent complex-time-path method by Niemi and Semenoff, and the operator-based thermo field dynamics of Umezawa and co-workers. An extensive bibliography can be found in Ref. 1.

The time-path method by Niemi and Semenoff offers certain advantages. Starting from a functional integral in the complex time plane one arrives at a perturbation theory for Green's functions in Minkowski space, using Feynman rules and diagrams very much like the conventional zero-temperature formalism. The characteristic feature of this real-time approach (and thermo field dynamics) is the doubling of the degrees of freedom: that is, to each physical field a conjugate ghost field has to be introduced. Consequently the propagator assumes a 2×2 matrix structure and for each physical vertex there exists now a complex-conjugate ghost counterpart. This doubling ensures the cancellation of ill-defined distributions (pinch singularities) arising from self-energy insertions on internal lines in multiloop diagrams, but makes perturbative calculations very cumbersome.

For many practical calculations at the one-loop level an earlier version of the real-time formalism,[2] which involves only the physical fields, is used instead; this assumes of course that the full matrix structure of the theory is necessary only in higher orders in perturbation theory. In particular, self-energy corrections on external fermion lines are treated as at zero temperature and absorbed into a temperature-dependent mass shift and a finite-temperature wave-function renormalization constant, obtained from the interacting finite-temperature fermion propagator (inverse Dirac operator)

$$i\mathcal{S}(p)=\frac{i}{\not p-m-\Sigma^{\beta}(p)+i\epsilon}\ , \tag{1}$$

where Σ^{β} is the temperature-dependent part of the fermion self-energy: $\Sigma=\Sigma^{0}+\Sigma^{\beta}$. However, the lack of Lorentz invariance at finite temperature obscures the identification of the renormalization constants from (1) and leads to such unusual features as momentum-dependent counterterms, a finite-temperature Dirac equation, and finite-temperature spinors.[3,4] In Ref. 5 we have already pointed out some of the problems arising from this approach.

The problem of absorbing self-energy corrections into suitable renormalization factors has also been studied in the full matrix formalism.[6,7] However, the matrix structure of the theory together with the lack of Lorentz invariance leads to considerable complications and makes the results unsuitable for practical applications.

In this paper we will address these problems and present a comprehensive analysis of finite-temperature decay rates with radiative one-loop QED corrections, using techniques based on the Niemi-Semenoff formalism. Our approach is based on the method of calculating decay rates from the imaginary part of the self-energy Π of the decaying particle. At $T=0$ this is completely equivalent to the standard approach of integrating squares of transition matrix elements over the available phase space. At $T\neq0$ the rates derived from Π are a nontrivial combination of the partial decay and inverse decay processes in the heat bath and are interpreted as the physical rate at which a nonequilibrium distribution of unstable particles approaches thermal equilibrium with its surroundings.[8] For definiteness and simplicity we examine the decay of a scalar (Higgs) boson into two fermions: $H\rightarrow e^{+}e^{-}$. Although currently of no physical interest, this system has the advantage of being computationally rather simple and has already been treated extensively in the literature.[3,4,9] The techniques and results, however, are quite general and can be applied to physically more important reactions.

Our results can be summarized as follows. We find that, at the one-loop level, the ghost vertices and propagators do not give finite contributions to the decay rate. They are, however, necessary to cancel the pinch singularities arising from the self-energy insertion diagrams. For the radiative corrections and their interpretation in

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terms of renormalization constants we find that the vertex correction and the photon emission and absorption contributions are identical to previous results. The fermion self-energy insertion, however, turns out to be problematic. If we follow the generally accepted philosophy for renormalizable field theories that renormalization counterterms should be of the same form as the corresponding terms in the bare Lagrangian, then the (ultraviolet-finite) temperature-dependent part of the self-energy contribution does *not* admit an interpretation in terms of mass and wave-function renormalization counterterms, due to the lack of Lorentz invariance. For this case we give a general algorithm how to compute the self-energy contribution for a general decay process. For the special case of two-body decay, viz., Higgs-boson decay, we are able to define operational analogs of on-shell renormalization counterterms, but there are still significant differences to the familiar zero-temperature procedure. We conclude that finite-temperature renormalization is not a useful concept for decay and scattering rate calculations. Previous work is discussed in our more general framework. Our results have immediate applications in cosmology, in particular for radiatively corrected neutron β decay which determines the abundances of light elements in primordial nucleosynthesis.

The paper is organized as follows. In Sec. II we give a brief, self-contained summary of the techniques and results that we are going to use. In particular, we will discuss the connection between finite-temperature decay rates and the imaginary part of the self-energy, and summarize the real-time Feynman and Cutkosky rules needed to calculate the latter. In Sec. III we apply these techniques and calculate the finite-temperature decay rate for a scalar particle decaying into two fermions, both at the tree level and with radiative corrections. These results are used to derive a general framework for finite-temperature decay rates at the one-loop level, with special emphasis on how to absorb fermion self-energy corrections into appropriate mass and wave-function renormalization counterterms. In Sec. IV we extend these results to general decay processes and discuss the possible consequences of our results for reactions of cosmological and astrophysical interest, in particular for neutron β decay at finite temperature. Section V contains a summary of our results and some of the calculational details can be found in the Appendix.

II. DECAY RATES AT FINITE TEMPERATURE

We will begin our discussion by giving a brief summary of the connection between finite-temperature decay rates and self-energies. We will also give a brief review of the Niemi-Semenoff time-path formalism, in particular the Feynman and Cutkosky rules needed for perturbative calculations.

The conventional method for the perturbative calculation of the decay rate Γ of a particle in vacuum field theory can be summarized as follows.[10]

Consider a particle with mass m and four-momentum $p=(\omega,\mathbf{p})$, decaying into n different particles with four-momenta $k_i=(\omega_i,\mathbf{k}_i)$, $i=1,\ldots,n$. The decay rate $\Gamma(\omega)$ is then given by

$$\Gamma(\omega)=\frac{1}{2\omega}\int\frac{d^3k_1}{2\omega_1(2\pi)^3}\cdots\frac{d^3k_n}{2\omega_n(2\pi)^3}|\mathcal{M}|^2 \times(2\pi)^4\delta^4\left[p-\sum_{i=1}^{n}k_i\right].$$

The transition amplitude $\mathcal{M}$ is calculated to any order in perturbation theory from the relevant Feynman diagrams and rules. The above formula is valid for bosons and Dirac fermions, provided we normalize the fermion spinors to $2m$.

Alternatively we can use the imaginary part $\mathrm{Im}\Pi(\omega)$ of the particle self-energy $\Pi(\omega)$, that is, the discontinuity of Π across the real axis in the complex energy plane

$$\mathrm{Disc}\Pi(\omega)\equiv\lim_{\epsilon\to 0}\Pi(\omega+i\epsilon)-\Pi(\omega-i\epsilon)=2i\,\mathrm{Im}\Pi(\omega)\ .$$

$\mathrm{Im}\Pi$ and Γ are related to the optical theorem and we find

$$\mathrm{Im}\Pi(\omega)=-\omega\Gamma(\omega)\ . \tag{2}$$

The imaginary part of the self-energy graph, or any Feynman graph, can be calculated with the standard Cutkosky or cutting rules[11,12] which we will discuss later in a more general form.

At zero temperature these two approaches are completely equivalent. At finite temperature, however, this connection is more involved.

It was pointed out by Weldon[8] that (2) has to be modified at finite temperature. Using the Euclidean (imaginary-time) Matsubara formalism he calculated, to lowest order, the imaginary part of the self-energy for a boson and a fermion undergoing two-body decay in a heat bath. The results can be summarized as follows.

For a boson field Φ at temperature T with analytically continued Euclidean self-energy $\Pi(\omega)$, Eq. (2) is replaced by

$$\begin{aligned}&\mathrm{Im}\Pi(\omega)=-\omega\cdot\Gamma(\omega)\ ,\\&\Gamma(\omega)=\Gamma_d(\omega)-\Gamma_i(\omega)\ .\end{aligned} \tag{3}$$

Γ_d denotes the sum of the thermally suppressed tree-level decay and scattering rates $\Phi\to\phi_1\phi_2$, $\Phi\bar{\phi}_1\to\phi_2$, etc., that decrease the number of Φ bosons, and Γ_i is the sum of all the inverse rates such as $\phi_1\phi_2\to\Phi$, etc., that increase the number of Φ's.

This agrees with physical intuition. The Φ boson decays, but the decay products ϕ_1 and ϕ_2 in the heat bath have a thermal probability to recombine into Φ, and likewise for the scattering processes. For a fermion ψ with self-energy Σ, however, this result has to be modified.

First, we contract the matrix Σ with the free Dirac spinors $\bar{u}$ and u (normalized to $2m$) to form the scalar

$$\Pi(\omega)=\bar{u}(\omega)\Sigma(\omega)u(\omega) \tag{4}$$

and obtain

$$\mathrm{Im}\Pi(\omega)=-\omega\cdot\Gamma(\omega),\quad \Gamma=\Gamma_d+\Gamma_i\ . \tag{5}$$

This shows that Γ is not simply the naive "net decay rate." Rather (3) and (5) have to be interpreted as the rate which a nonequilibrium distribution $f(\omega,t)$ of unstable particles approaches its equilibrium value $f_0(\omega)$. More specifically, consider the master equation for $f(\omega,t)$:

$$\frac{\partial}{\partial t}f(\omega,t)=-\Gamma_d(\omega)f(\omega,t)+\Gamma_i(\omega)[1+\sigma f(\omega,t)] \ , \qquad (6)$$

where $\sigma=1$ for bosons and $\sigma=-1$ for fermions as a consequence of the Pauli exclusion principle.

The solution to (6) is given by

$$f(\omega,t)=f_0(\omega)+c(\omega)e^{-\Gamma(\omega)t} \ . \qquad (7)$$

where

$$\Gamma(\omega)=\Gamma_d(\omega)-\sigma\cdot\Gamma_i(\omega), \quad f_0(\omega)=\frac{\Gamma_i(\omega)}{\Gamma(\omega)} \ ,$$

so that $f(\omega,t\to\infty)\to f_0(\omega)$, as stated.

We emphasize that Γ, and not the partial rates Γ_i and Γ_d, represents the physically measurable decay rate. Even for $t\to\infty$ the initial distribution will not go to zero, but approach thermal equilibrium with its environment.

Strictly speaking, the whole concept of asymptotic states and S-matrix elements becomes meaningless at finite temperature. Instead, the appropriate notion is that of a quasiparticle excitation in a plasma that is "Landau damping," i.e., thermalizing with its surrounding.

As an aside, we note that this analysis can be generalized to other Green's functions as well. For example, the discontinuity in the four-point function can be interpreted as the thermal scattering rates with two particles in the initial state. However, in this case Eq. (6) has to be generalized; for the time being we will restrict ourselves to two-point functions.

The above results were confirmed, up to the one-loop level, in the real-time formalism by Kobes and Semenoff,[6,7] who developed a finite-temperature and -density generalization of the standard Cutkosky rules for the computation of the imaginary part of a Feynman graph.

We will now turn to the real-time formulation of finite-temperature field theory and give a brief summary of the techniques and results, viz., Feynman and Cutkosky rules, that we are going to use. For the derivation and a discussion we refer to the original papers.[6,7,13–15]

As we already mentioned, the calculation of real-time Green's functions at finite temperature and density via Feynman diagrams proceeds as in zero-temperature quantum field theory, but the degrees of freedom are now doubled. For each physical "type-1" propagator and vertex there exists a complex conjugate "type-2" ghost counterpart. Consequently the free thermal propagator assumes the form of a 2×2 matrix. The 1-1 component connects the physical type-1 vertices, the 2-2 component the type-2 vertices and the off-diagonal elements mix the two types. We will now list the propagator matrices in momentum space for the three generic cases, that is, scalar, fermion, and gauge-boson fields.

(1) For a free scalar field ϕ at temperature $T\equiv\beta^{-1}$ with Lagrange density

$$L=\tfrac{1}{2}(\partial\phi^2-m^2\phi^2)$$

the matrix propagator is given by

$$iD_{ab}(p)=U(\beta,p)\begin{bmatrix} i\Delta(p) & 0 \\ 0 & -i\Delta^*(p)\end{bmatrix}U(\beta,p) \ , \qquad (8)$$

where

$$U(\beta,p)=\begin{bmatrix}\cosh\theta(p) & \sinh\theta(p) \\ \sinh\theta(p) & \cosh\theta(p)\end{bmatrix} ,$$

$$\cosh\theta(p)=\frac{1}{\sqrt{1-e^{-\beta|p^0|}}} \ ,$$

$$\sinh\theta(p)=\frac{e^{-\beta|p^0|/2}}{\sqrt{1-e^{-\beta|p^0|}}} \ ,$$

$$i\Delta(p)=\frac{i}{p^2-m^2+i\epsilon} \ ,$$

or, in component form,

$$iD_{11}(p)=-iD_{22}^*(p)$$
$$=\frac{i}{p^2-m^2+i\epsilon}+2\pi n_B(p)\delta(p^2-m^2) \ , \qquad (9)$$

$$iD_{12}(p)=iD_{21}(p)$$
$$=2\pi n_B(p)e^{\beta|p^0|/2}\delta(p^2-m^2) \ , \qquad (10)$$

where $n_B(p)=1/(e^{\beta|p^0|}-1)$.

(2) For a free, massive Dirac fermion ψ with Lagrange density

$$L=\bar{\psi}(i\not\partial-m)\psi$$

we have

$$iS_{ab}(p)=V(\beta,p)\begin{bmatrix} iS(p) & 0 \\ 0 & -iS^*(p)\end{bmatrix}V(\beta,p) \ , \qquad (11)$$

where

$$V(\beta,p)=\begin{bmatrix}\cos\phi(p) & \epsilon(p^0)\sin\phi(p) \\ \epsilon(p^0)\sin\phi(p) & \cos\phi(p)\end{bmatrix} ,$$

$$\cos\phi(p)=\frac{1}{\sqrt{1+e^{-\beta|p^0|}}} \ ,$$

$$\sin\phi(p)=\frac{e^{-\beta|p^0|/2}}{\sqrt{1+e^{-\beta|p^0|}}} \ ,$$

$$iS(p)=\frac{i}{\not p-m+i\epsilon} \ .$$

In component form

$$iS_{11}(p)=-iS_{22}^*(p)$$
$$=\frac{i}{\not p-m+i\epsilon}-2\pi(\not p+m)n_F(p)\delta(p^2-m^2) \ , \qquad (12)$$

$$iS_{12}(p)=-iS_{21}(p)$$
$$=-2\pi(\not p+m)\epsilon(p^0)n_F(p^0)e^{\beta|p^0|/2}\delta(p^2-m^2) \ , \qquad (13)$$

where $n_F(p)=1/(e^{\beta|p^0|}+1)$ and $\epsilon(p^0)\equiv\theta(p^0)-\theta(-p^0)$ is the sign function.

(3) For a massless gauge boson with Lagrange density

$$L=-\tfrac{1}{4}(F_{\mu\nu})^2-\frac{\alpha}{2}(\partial A)^2$$

we obtain

$$iD_{ab}^{\mu\nu}(p)=\left[-g^{\mu\nu}+\frac{1-\alpha}{\alpha}p^\mu p^\nu\frac{\partial}{\partial p^2}\right]iD_{ab}(p) \tag{14}$$

with iD_{ab} given by (8) with $m^2=0$, and α denotes the gauge parameter. We will now turn to interacting theories. Again we consider the scalar-boson case first.

(1) The full propagator for a scalar field in the interacting theory can be written as

$$i\mathcal{D}_{ab}(p)=U(\beta,p)\begin{bmatrix} i\mathcal{D}(p) & 0\\ 0 & -i\mathcal{D}^*(p)\end{bmatrix}U(\beta,p) \tag{15}$$

with $\mathcal{D}$ some complex function or distribution. Dyson's equation in matrix form

$$\mathcal{D}_{ab}=\mathcal{D}_{ab}+\mathcal{D}_{ac}(-i\Pi_{cd})\mathcal{D}_{db}$$

leads to the self-energy matrix

$$-i\Pi_{ab}(p)=U^{-1}(\beta,p)\begin{bmatrix} -i\Pi(p) & 0\\ 0 & i\Pi^*(p)\end{bmatrix}U^{-1}(\beta,p)\ . \tag{16}$$

This gives, for $i\mathcal{D}(p)$ in (15),

$$i\mathcal{D}(p)=\frac{i}{p^2-m^2-\Pi(p)+i\epsilon} \tag{17}$$

and for the self-energy matrix one deduces immediately, from (16),

$$\begin{aligned}&\Pi_{11}(p)=-\Pi_{22}^*(p)\ ,\\ &\Pi_{12}(p)=\Pi_{21}(p)=-i\tanh2\theta(p)\mathrm{Im}\Pi_{11}(p)\ ,\end{aligned} \tag{18}$$

as well as

$$\begin{aligned}&\mathrm{Re}\Pi(p)=\mathrm{Re}\Pi_{11}(p)\ ,\\ &\mathrm{Im}\Pi(p)=\epsilon(p^0)\tanh(\beta p^0/2)\mathrm{Im}\Pi_1(p)\ .\end{aligned} \tag{19}$$

In terms of Feynman graphs Π_{11} is simply the self-energy diagram with type-1 external vertices, Π_{12} the diagram with one type-1 and one type-2 external vertex, and so on. We note that $i\mathcal{D}$ is the propagator for a quasiparticle but the full propagator matrix is more complicated. For example, the propagator for a physical excitation, $i\mathcal{D}_{11}$, is given by

$$i\mathcal{D}_{11}(p)=i\mathcal{D}(p)+n_B(p)[i\mathcal{D}(p)-i\mathcal{D}^*(p)]\ .$$

For decay rate calculations, however, the important quantity is the self-energy function Π: its imaginary part $\mathrm{Im}\Pi$ gives the thermal decay rate Γ discussed before.

(2) The same derivation holds for fermions as well and we obtain for the interacting propagator

$$i\mathcal{S}_{ab}(p)=V(\beta,p)\begin{bmatrix} i\mathcal{S}(p) & 0\\ 0 & -i\mathcal{S}^*(p)\end{bmatrix}V(\beta,p) \tag{20}$$

with

$$i\mathcal{S}(p)=\frac{i}{\not p-m-\Sigma(p)+i\epsilon} \tag{21}$$

and for the self-energy matrix

$$-i\Sigma_{ab}(p)=V^{-1}(\beta,p)\begin{bmatrix} -i\Sigma(p) & 0\\ 0 & i\Sigma^*(p)\end{bmatrix}V^{-1}(\beta,p)\ ,$$

where now

$$\begin{aligned}&\Sigma_{11}(p)=-\Sigma_{22}^*(p)\ ,\\ &\Sigma_{12}(p)=-\Sigma_{21}(p)=i\epsilon(p^0)\tan2\phi(p)\mathrm{Im}\Sigma_{11}(p)\ ,\end{aligned} \tag{22}$$

and

$$\begin{aligned}&\mathrm{Re}\Sigma(p)=\mathrm{Re}\Sigma_{11}(p)\ ,\\ &\mathrm{Im}\Sigma(p)=\epsilon(p^0)\coth(\beta p/2)\mathrm{Im}\Sigma_{11}(p)\ .\end{aligned} \tag{23}$$

(3) For the gauge-boson case the only difference to the scalar case is again the factor

$$-g_{\mu\nu}+\frac{1-\alpha}{\alpha}p_\mu p_\nu\frac{\partial}{\partial p^2}$$

for the propagators.

We will now summarize the rules for the computation of imaginary parts of Feynman diagrams at finite temperature in momentum space.

First, we define the "circled" propagators

$$\begin{aligned}&iD^{\pm}(p)=2\pi[\theta(\pm p^0)+n_B(p)]\delta(p^2-m^2)\quad\text{for bosons}\ ,\\ &iS^{\pm}(p)=2\pi(\not p+m)[\theta(\pm p^0)-n_F(p)]\delta(p^2-m^2)\quad\text{for fermions}\ ,\end{aligned} \tag{24}$$

which are related to the off-diagonal propagator matrix elements by

$$\begin{aligned}&iD^{\pm}(p)=e^{\pm\beta p^0/2}iD_{12}(p)\ ,\\ &iS^{\pm}(p)=\mp e^{\pm\beta p^0/2}iS_{12}(p)\ ,\end{aligned} \tag{25}$$

and hence

$$\begin{aligned}&iD^{\pm}(p)=e^{\pm\beta p^0}iD^{\mp}(p)\ ,\\ &iS^{\pm}(p)=-e^{\pm\beta p^0}iS^{\mp}(p)\ .\end{aligned} \tag{26}$$

To calculate the imaginary part of a generic Feynman diagram with complex vertices and propagators P_{ij} we use the following generalized Cutkosky rules: (i) draw all Feynman graphs with all possible combinations of physical and ghost vertices; (ii) in every diagram, circle the vertices such that the diagram contains both circled and uncircled vertices in all possible combinations; (iii) reverse the sign of a circled vertex; (iv) leave the P_{12}/P_{21} propagators unchanged; (v) for the P_{11}/P_{22} propagators connecting the type-1 and/or -2 vertices (a) leave P_{11}/P_{22} unchanged if both vertices are uncircled, (b) replace

P_{11}/P_{22} by P_{22}/P_{11} if both vertices are circled, (c) replace P_{11}/P_{22} by P^{+}/P^{-} if the momentum flows from an uncircled vertex towards a circled vertex, (d) replace P_{11}/P_{22} by P^{-}/P^{+} if the momentum flows from a circled vertex towards an uncircled vertex; (vi) the sum of all these circled diagrams yields minus 2 times the imaginary part of the original diagram.

It is not difficult to show that, in the zero-temperature limit, this prescription reduces to the standard Cutkosky or cutting rules. At $T=0$ the circled propagators reduce to forward and backward mass-shell δ functions. Hence every diagram that contains an isolated circled vertex, that is, a vertex not connected to at least another circled vertex, will vanish due to conflicting θ functions. Consequently the only contribution comes from diagrams for which the circled and uncircled vertices form simply connected regions; these areas are linked by circled propagators, i.e., mass-shell δ functions. This is, however, precisely the definition of a cut diagram (see, e.g., Ref. 12).

In general, all contributions of physical and ghost vertices have to be included in the Feynman diagrams. Needless to say, this leads to a proliferation of terms and makes actual calculations very cumbersome. A considerable simplification occurs if all external lines (vertices) are physical, as shown by Kobes and Semenoff.[6] They derived the following result: *For the calculation of the imaginary part of Feynman diagrams with physical external legs and/or vertices, it suffices to include only diagrams with physical internal vertices.*

This is precisely the result we need: to calculate decay rates for bosons or fermions we have to determine the imaginary part of the self-energy function Π or Σ, hence, by (19) or (23), of Π_{11} or Σ_{11}, which are the self-energy diagrams with physical external vertices.

Let us finally point out that it is not correct to conclude that one can dispense with the ghost vertices entirely. Their existence is crucial for the derivation of the above result. Furthermore, they are present implicitly in the finite-temperature Cutkosky rules. Circling a vertex is the same as replacing it with its complex conjugate, and they will appear explicitly in intermediate states of the calculation, as we will show later.

III. HIGGS-BOSON DECAY AT FINITE TEMPERATURE

We will apply this formalism and calculate the finite-temperature decay rate of a scalar boson decaying into two fermions. Physically this can be considered as the decay of a Higgs boson into an electron and a positron.

We will consider Higgs-boson decay at the tree level in the Higgs-fermion coupling, and with radiative QED corrections up to second order in the electromagnetic coupling.

First, let us recall some basic definitions and notation.

The Higgs-fermion sector in the standard electroweak model is described by the bare Lagrangian

$$\mathcal{L}=\bar{\psi}_0 i\partial\!\!\!/\psi_0-g_0\bar{\psi}_0\psi_0(v+\hat{h}_0)-e_0\bar{\psi}_0\gamma_\mu\psi_0A_0^\mu$$
$$\equiv\bar{\psi}_0(i\partial\!\!\!/-m_0)\psi_0-g_0\bar{\psi}_0\psi_0\hat{h}_0-e_0\bar{\psi}_0\gamma_\mu\psi_0A_0^\mu\ ,$$

where the bare fermion mass m_0 is generated by spontaneous symmetry breaking from the Yukawa coupling g_0 and the vacuum expectation value of the Higgs field, v. In standard notation ψ_0 denotes the fermion field, A_0 the photon field, and e_0 the electromagnetic coupling; $\hat{h}_0$ is the dynamical part of the Higgs field. Also in standard notation we have, for the renormalized fields and parameters,

$$\psi_0=\sqrt{Z_2}\psi,\quad m_0=m-\delta m\ ,$$
$$A_0=\sqrt{Z_3}A,\quad g_0=gZ_g/Z_2,\quad e_0=eZ_1/Z_2\sqrt{Z_3}\ ,$$

and the Higgs field remains unrenormalized to this order in perturbation theory. To $O(e^2)$ the bare Lagrangian is split into renormalized and counterterm parts:

$$\mathcal{L}=\mathcal{L}^{\text{ren}}+\mathcal{L}^{\text{ct}}$$
$$=\bar{\psi}(i\partial\!\!\!/-m)\psi-g\bar{\psi}\psi\hat{h}-e\bar{\psi}\gamma_\mu\psi A^\mu+\delta Z_2\bar{\psi}(i\partial\!\!\!/-m)\psi$$
$$+\delta m\bar{\psi}\psi-\delta Z_g g\bar{\psi}\psi\hat{h}-\delta Z_1 e\bar{\psi}\gamma_\mu\psi A^\mu\ ,$$

where $\delta Z\equiv Z-1\sim O(e^2)$ and we can read off the (zero-temperature) Feynman propagators and vertices for the decay matrix elements, shown in Fig. 1 for future reference. At nonzero temperature, this represents, of course, the propagators and vertices given in Sec. II.

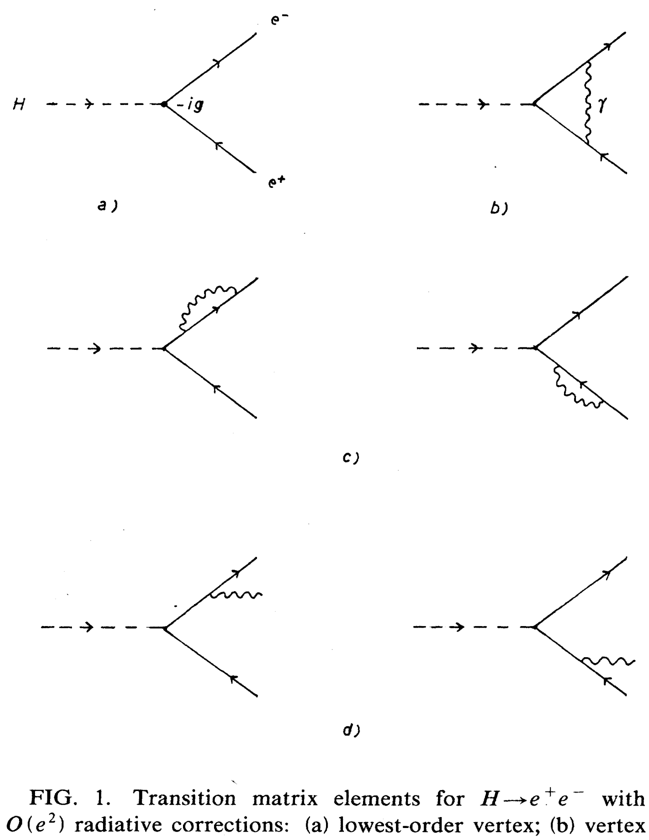

FIG. 1. Transition matrix elements for $H\rightarrow e^+e^-$ with $O(e^2)$ radiative corrections: (a) lowest-order vertex; (b) vertex correction; (c) self-energy correction; (d) photon emission and/or absorption processes. The counterterm diagrams are omitted.

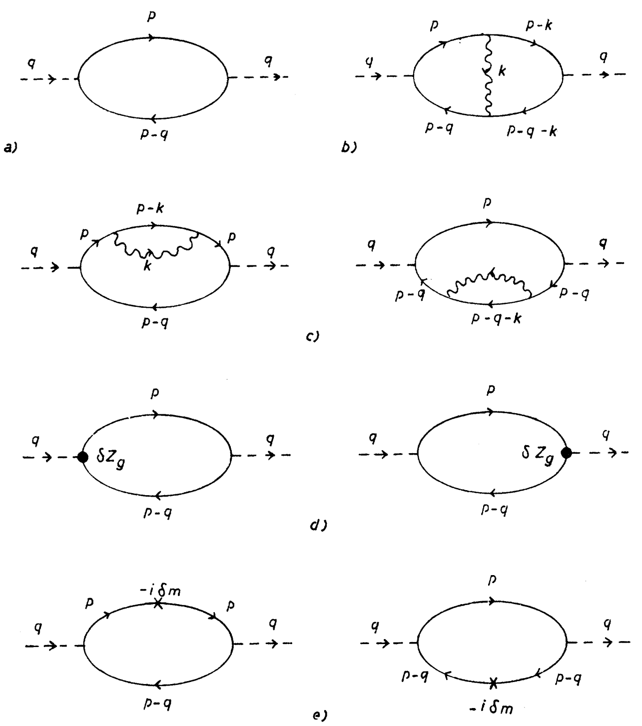

FIG. 2. The Higgs-boson self-energy $\Pi_{11}(q)$ with $O(e^2)$ radiative corrections: (a) lowest-order diagram; (b) "vertex correction" diagram; (c) "self-energy correction" diagram; (d) vertex counterterm diagram; (e) mass counterterm diagram.

A. The lowest-order decay rate

We will begin our discussion by calculating the Higgs-boson decay rate to lowest order in the Yukawa coupling. This will also illustrate the general formalism introduced in Sec. II with an explicit example.

The decay rate of a Higgs boson with mass m_H and four-momentum $q=(\omega_q,\mathbf{q})$, $\omega_q=(\mathbf{q}^2+m_H^2)^{1/2}$, is given by

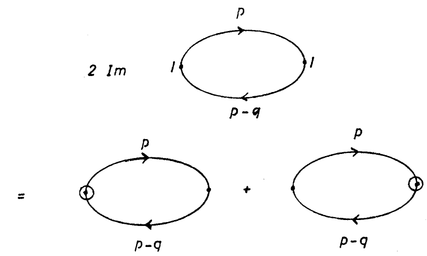

FIG. 3. Circled diagrams for the imaginary part of the lowest-order self-energy Π_{11}.

$$\Gamma(\omega_q)=-\frac{1}{\omega_q}\text{Im}\Pi(q)$$
$$=-\frac{1}{\omega_q}\epsilon(\omega_q)\tanh(\beta\omega_q/2)\text{Im}\Pi_{11}(q)\ .$$

The self-energy diagram Π_{11}, shown in Fig. 2(a), is given by

$$-i\Pi_{11}(q)=-(-ig)^2\int\frac{d^4p}{(2\pi)^4}\text{Tr}[iS_{11}(p)iS_{11}(p-q)]\ ,$$

where S_{11} is the free finite-temperature fermion propagator (12). Using our finite-temperature Cutkosky rules, shown in Fig. 3, we obtain, for the imaginary part,

$$-2\,\text{Im}\Pi_{11}(q)$$
$$=-(-ig)(+ig)\int\frac{d^4p}{(2\pi)^4}\text{Tr}[iS^-(p)iS^+(p-q)$$
$$+iS^+(p)iS^{-1}(p-q)]\ .$$

The two terms in the integrand are related by Eq. (26):

$$iS^{\pm}(p)=-iS^{\mp}(p)e^{\pm p^0}\ ;$$

hence, the decay rate simplifies to

$$\Gamma(\omega_q)=-\frac{\tanh(\beta\omega_q/2)}{\omega_q}(1+e^{-\beta q_0})\frac{g^2}{2}\int\frac{d^4p}{(2\pi)^4}\delta(p^2-m^2)\delta((p-q)^2-m^2)$$
$$\times[\theta(p^0)-n_F(p)][\theta(-p^0+q^0)-n_F(p-q)]\text{Tr}[(\not{p}+m)(\not{p}-\not{q}+m)]\ ,\qquad(27)$$

where we used the definition (24) of $S^{\pm}$. As discussed in Sec. II, this expression corresponds to the difference between thermal decay and inverse decay rates, that is, the lowest-order transition amplitudes squared and integrated over thermal phase space. We will now proceed and evaluate Γ explicitly. Note, however, that Lorentz invariance is lost at finite temperature; hence, the decay rate will no longer be invariant, but depend on the reference frame. In the following, we will choose the rest of the decaying particle, that is, we set $\mathbf{q}=0$.

First, consider the product of the mass-shell δ functions. With $q=(m_H,0,0,0)$ the compatible zeros are easily found to be

$$p^0=\omega_p=m_H/2\ .$$

Hence the δ functions reduce to

$$\delta(p^2-m^2)\delta((p-q)^2-m^2)=\frac{1}{4m_H\omega_p}\delta(p^0-\omega_p)\delta(\omega_p-m_H/2)$$

and fix the momentum dependence of the integrand completely. The trace is easily evaluated, and with the momenta on shell we obtain

$$\mathrm{Tr}[(\not p+m)(\not p-\not q+m)]=4(p^2-p\cdot q+m^2)=-2m_H^2\left[1-\frac{4m^2}{m_H^2}\right].$$

The thermal factors can be rewritten as hyperbolic functions which reduce on shell to

$$\tanh(\beta m_H/2)(1+e^{-\beta m_H})[\theta(p^0)-n_F(p^0)][\theta(q^0-p^0)-n_F(p^0-q^0)]$$
$$=(1-e^{-\beta m_H})[1-n_F(p^0)][1-n_F(p^0-q^0)]=\frac{\sinh(\beta m_H/2)}{2\cosh^2(\beta m_H/4)}=\tanh(\beta m_H/4).$$

Thus, we obtain, for the thermal decay rate,

$$\Gamma^{\text{tree}}(m_H)=\tanh(\beta m_H/4)\frac{g^2}{8\pi}m_H\left[1-\frac{4m^2}{m_H^2}\right]^{3/2}$$
$$\equiv\tanh(\beta m_H/4)\Gamma^0(m_H), \tag{28}$$

where $\Gamma^0(m_H)$ denotes the zero-temperature decay rate.

Thus, to lowest order, the temperature dependence is contained in a simple multiplicative factor. Let us consider the limiting cases of (28).

For $T=\beta^{-1}=0$ we have $\tanh(\beta m_H/4)=1$ and $\Gamma^{\text{tree}}=\Gamma^0$, as expected. In the low-temperature regime $\beta m_H \gg 1$ expanding the tanh yields

$$\Gamma^{\text{tree}}(m_H)\simeq(1-2e^{-\beta m_H/2})\Gamma^0(m_H),$$

that is, an exponentially small suppression of the $T=0$ decay rate which is usually neglected.[3,9]

At higher temperatures this suppression becomes more substantial. For $\beta m_H \simeq 1$ we have

$$\Gamma^{\text{tree}}(m_H)\simeq 0.25\Gamma^0(m_H)$$

and, in the high-temperature limit $\beta m_H \ll 1$,

$$\Gamma^{\text{tree}}(m_H)\to 0 .$$

This agrees with physical intuition. At higher temperatures the recombination rate of electrons in the heat bath will become more significant; hence, Γ, the difference between decay and inverse decay rates, will decrease. The high-temperature limit should not be taken too seriously, however, since perturbation theory for bosons breaks down at high temperatures.

As an aside, we note that, in order to obtain the correct result, we had to use the self-energy function $\mathrm{Im}\Pi$, and not simply $\mathrm{Im}\Pi_{11}$, which is of course a consequence of the underlying matrix structure of the theory.

B. Radiative correction at finite temperature

We will now turn to the main topic of the paper and analyze the radiative corrections to the decay rate, up to second order in the electromagnetic coupling e. The corrections to the transition amplitudes are shown in Figs. 1(b)–1(d) for future reference; for simplicity we have omitted the counterterm diagrams.

The radiative correction to the self-energy diagram Π_{11} are shown in Figs. 2(b) and 2(c). We note that, according to the discussion in Sec. II, all vertices are understood as type-1, physical vertices unless stated otherwise. We work in the Feynman gauge, that is, we choose $\alpha=1$ for the photon propagator (14).

We will now proceed and evaluate the imaginary part of these diagrams which corresponds to the decay rate with $O(e^2)$ radiative corrections. First, consider the "vertex-correction diagram" Fig. 2(b) which turns out to be structurally quite simple (the actual computation is of course quite involved).

1. The vertex-correction diagram

The vertex-correction diagram Fig. 2(b) at finite temperature is given by

$$-i\Pi_{11}^V(q)=(-ig)^2(-ie)^2$$
$$\times\int\frac{d^4p\,d^4k}{(2\pi)^8}\mathrm{Tr}[\,iS(p-k)\gamma_\mu iS(p)$$
$$\times iS(p-q)\gamma_\nu$$
$$\times iS(p-q-k)iD^{\mu\nu}(k)]\,,$$

where the propagators are understood as the physical 1-1 components of the finite-temperature propagator matrices (10) and (13). For notational simplicity we will from now on omit the subscript 11.

Applying our finite-temperature circling (cutting) rules, as shown in Fig. 4, we obtain, for the imaginary part,

$$-2\,\mathrm{Im}\Pi_{11}^{V}(q)=-(-ig)(+ig)(-ie)^2\int\frac{d^4p\,d^4k}{(2\pi)^8}$$
$$\times\mathrm{Tr}[iS(p-k)\gamma_\mu iS^-(p)iS^+(p-q)\gamma^\mu iS(p-q-k)iD(k)$$
$$-iS^-(p-k)\gamma_\mu iS^*(p)iS^*(p-q)\gamma^\mu iS^+(p-q-k)iD^*(k)$$
$$+iS^+(p-k)\gamma_\mu iS(p)iS(p-q)\gamma^\mu iS^-(p-q-k)iD(k)$$
$$-iS^*(p-k)\gamma_\mu iS^+(p)iS^-(p-q)\gamma^\mu iS^*(p-q-k)iD^*(k)$$
$$+iS^-(p-k)\gamma_\mu iS^*(p)iS^+(p-q)\gamma^\mu iS(p-q-k)iD^-(k)$$
$$+iS(p-k)\gamma_\mu iS^-(p)iS^*(p-q)\gamma^\mu iS^+(p-q-k)iD^+(k)$$
$$+iS^*(p-k)\gamma_\mu iS^+(p)iS(p-q)\gamma^\mu iS^-(p-q-k)iD^-(k)$$
$$+iS^+(p-k)\gamma_\mu iS(p)iS^-(p-q)\gamma^\mu iS^*(p-q-k)iD^+(k)]\ . \qquad (29)$$

Each of the circled, or cut, diagrams is equivalent to a product of transition matrix elements Fig. 1, integrated over thermal phase space. This correspondence has already been worked out[7] and is also shown in Fig. 4.

Using this relation we can now simplify the rather unwieldy looking terms in (29) to a more familar form that allows a direct interpretation. First, we will concentrate on the first four terms which contain the Yukawa vertex correction diagram:

$$(-ig)G(p,q)=-(-ig)(-ie)^2\int\frac{d^4k}{(2\pi)^4}\gamma_\mu iS(p-k)iS(p-q-k)\gamma^\mu iD(k)\ .$$

Using the cyclicity of the trace and shifting the integration variable we combine the first two terms into

$$\cdots=(-ig)(+ig)\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}[iS^-(p)2\,\mathrm{Re}G(p,q)iS^+(p-q)]\ .$$

Likewise we obtain, for the other two terms,

$$\cdots=(-ig)(+ig)\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}[iS^+(p)2\,\mathrm{Re}G(p,q)iS^-(p-q)]$$

and as before we can now use the relation (26) to combine the first four terms into

$$-2\,\mathrm{Im}\Pi_{11}^{G}(m_H)\equiv g^2(1+e^{-\beta m_H})\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}[iS^+(p)2\,\mathrm{Re}G(p,q)iS^-(p-q)]\ .$$

Thus we obtain a vertex-corrected decay rate

$$\Gamma^G(m_H)=-\frac{1}{m_H}\tanh(\beta m_H/2)\mathrm{Im}\Pi_{11}^{G}(m_H)$$
$$=\frac{g^2}{2m_H}(1-e^{-\beta m_H})\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}[(\not p+m)2\,\mathrm{Re}G(p,q)(\not p-\not q+m)]$$
$$\times[\theta(p^0)-n_F(p)][\theta(q^0-p^0)-n_F(p-q)]\delta(p^2-m^2)\delta((p-q)^2-m^2)\ . \qquad (30)$$

This is of course the standard zero-temperature results generalized to finite temperature. The vertex-correction diagram Fig. 1(b) modifies the Yukawa coupling to

$$g\to g\ \mathrm{Re}G(p,q)$$

and the correction to the decay rate is found by replacing g^2 in the tree rate (27) by the modified vertex and expanding to $O(e^2)$.

It remains to evaluate the vertex function $G(p,q)\equiv G^0(p,q)+G^\beta(p,q)$ with its external momenta on mass shell. This is of course more involved and we will simply quote the results from the literature.

The zero-temperature part $\mathrm{Re}G^0$ has been given in Ref. 16. For the finite-temperature part several approximations have been considered. The simplest one, including only the thermal photon distribution and neglecting the fermion contributions, leads to[3]

$$\mathrm{Re}G^\beta(m_H)=\frac{e^2}{4\pi^2}\frac{1+w^2}{2}\ln\frac{1+w}{1-w}\int_\epsilon^\infty d|\mathbf{k}|\frac{1}{|\mathbf{k}|}n_B(|\mathbf{k}|)\ , \qquad (31)$$

where $w=(1-4m^2/m_H^2)^{1/2}$.

We note that G^β is a scalar function with no Dirac indices. Also note that $1/|\mathbf{k}|^2$ infrared divergence and the logarithmic mass singularity for $m^2\to 0$, i.e., $w\to 1$ in (31).

More recently, the authors of Ref. 9 have improved

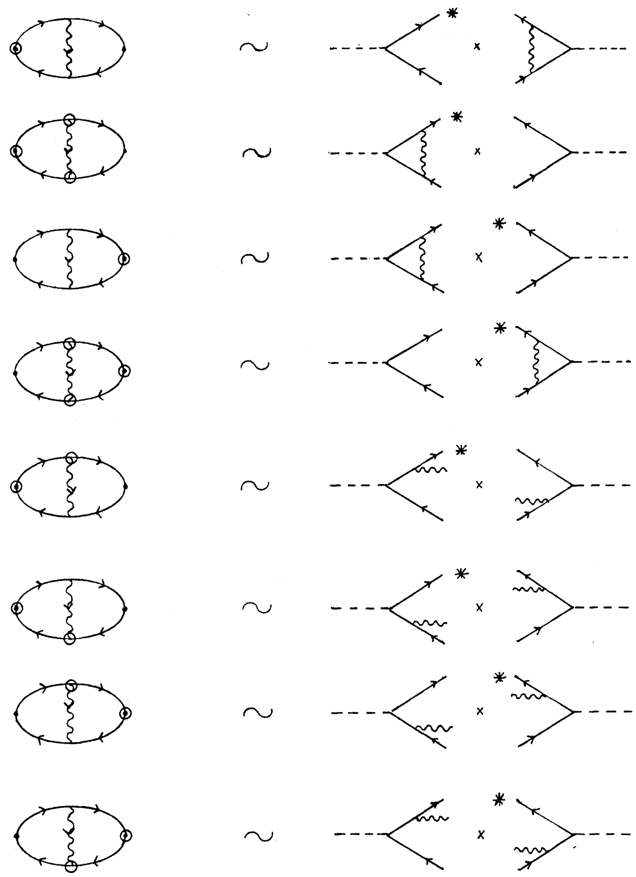

FIG. 4. Circled diagrams for the vertex-corrected Π_{11} and the equivalent products of transition matrix elements.

this result and taken the thermal fermion contributions into account. The resulting expressions and approximations are, however, quite complicated and we refer to Ref. 9 for details. We simply note that G^{β} remains a scalar and that no additional infrared divergences are introduced.

In summary, the vertex correction to the decay rate at finite temperature is given by

$$\Gamma^{G}(m_H)=2\,\mathrm{Re}G(m_H)\Gamma^{\mathrm{tree}}(m_H)\ . \tag{32}$$

This expression is of course both ultraviolet and infrared divergent. The ultraviolet divergence arises only from the zero-temperature part G^0 whereas the infrared divergences are contributed both by G^0 and G^{β}.

The ultraviolet renormalization is straightforward. The vertex counterterm $-ig\delta Z_g$ leads to the two counterterm diagrams Fig. 2(d) whose imaginary part is again determined by our circling rules. Proceeding as before we find immediately

$$-2\,\mathrm{Im}\Pi_{11}^{\mathrm{ct}}(m_H)=g^2(1+e^{-\beta m_H})\times\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}[iS^+(p)2\,\mathrm{Re}\delta Z_g\times iS^-(p-q)]\ .$$

Adding this counterterm contribution yields the renormalized decay rate

$$\Gamma_{\mathrm{ren}}^{G}=2\,\mathrm{Re}(G-\delta Z_g)\Gamma^{\mathrm{tree}}\equiv 2\,\mathrm{Re}G^{\mathrm{ren}}\Gamma^{\mathrm{tree}}\ ,$$

where δZ_g is chosen to subtract off the ultraviolet divergence in G^0 plus an finite part of G. Note that, if we choose to include part of G^{β} in δZ_g and make the coupling temperature dependent, Lorentz invariance of the Lagrangian will be lost. We will discuss this problem in more detail in the next section.

The infrared divergences (in the unrenormalized rate) and the mass singularity are canceled by contributions from the last four terms in (29) which represent part of the photon emission and/or absorption rate Γ^{γ_1}. As before we can combine these four seemingly different terms into a single expression, using a shift in integration variables and relation (26) between the circled propagators.

Thus, we obtain, for the (partial) photon emission and/or absorption rate,

$$\Gamma^{\gamma_1}(m_H)=-\frac{1}{m_H}\tanh(\beta m_H/2)\mathrm{Im}\Pi_{11}^{\gamma_1}(m_H)\ ,$$

where

$$\mathrm{Im}\Pi_{11}^{\gamma_1}(m_H)=g^2(-ie)^2(1+e^{-\beta m_H})\mathrm{Re}\int\frac{d^4p\,d^4k}{(2\pi)^8}\mathrm{Tr}[iS^+(p)\gamma_\mu iS(p-k)iS^-(p-k-q)\gamma^\mu iS^*(p-q)iD^-(k)]$$

and hence, for the transition rate,

$$\begin{aligned}\Gamma^{\gamma_1}(m_H)=-\frac{g^2e^2}{m_H}(1-e^{-\beta m_H})\mathrm{Re}\int&\frac{d^4p\,d^4k}{(2\pi)^5}\delta(p^2-m^2)\delta((p-k-q)^2-m^2)\delta(k^2)\\&\times\mathrm{Tr}[(\not p+m)\gamma_\mu(\not p-\not k+m)(\not p-\not k-\not q+m)\gamma^\mu(\not p-\not q+m)]\\&\times[\theta(p^0)-n_F(p)][\theta(q^0+k^0-p^0)-n_F(p-k-q)][\theta(-k^0)+n_B(k)]\\&\times\left[\frac{i}{(p-k)^2+m^2+i\epsilon}-2\pi n_F(p-k)\delta((p-k)^2-m^2)\right]\\&\times\left[\frac{i}{(p-q)^2-m^2-i\epsilon}-2\pi n_F(p-q)\delta((p-q)^2-m^2)\right]\ .\end{aligned}\tag{33}$$

This can be simplified further: splitting the complex Δ distributions into principal part and δ function and regulating the infrared divergence at $k=0$ with a small photon mass λ, it is easy to show that the product of the five four-momentum-conserving δ functions vanishes, and we are left with

$$\Gamma^{\gamma_1}(m_H)=-\frac{g^2e^2}{m_H}(1-e^{-\beta m_H})\mathrm{Re}\int\frac{d^4p\,d^4k}{(2\pi)^5}\delta(p^2-m^2)\delta((p-k-q)^2-m^2)\delta(k^2-\lambda^2)$$
$$\times\mathrm{Tr}[(\not p+m)\gamma_\mu(\not p-\not k+m)(\not p-\not k-\not q+m)\gamma^\mu(\not p-\not q+m)]$$
$$\times[\theta(p^0)-n_F(p)][\theta(q^0+k^0-p^0)-n_F(p-k-q)][\theta(-k^0)+n_B(k)]$$
$$\times\left[PP\frac{1}{(p-k)^2-m^2}\right]\left[PP\frac{1}{(p-q)^2-m^2}\right]. \tag{34}$$

Of course the actual evaluation of the integrals and the demonstration of the cancellation of the infrared divergences is quite involved and we refer to Refs. 16 and 3 for the calculational details.

So far our results are a straightforward generalization of the zero-temperature results: the vertex-correction matrix element renormalizes the Higgs-fermion coupling, and the infrared divergences are canceled by contributions from photon emission and/or absorption processes. This is not surprising since the circled (cut) diagrams correspond directly to the relevant transition matrix elements.

Things will become more interesting when we consider the fermion self-energy insertion diagram.

2. The self-energy correction diagram

The fermion self-energy correction Σ to the boson self-energy Π is shown in Fig. 2(c). Since the two diagrams are related by a simple shift in integration variables and a reversal of the external momentum, it suffices to consider only the diagram with the electron line corrected; the other one will contribute only a factor of 2. The graph is given by

$$-i\Pi_{11}^{\Sigma}(q)=-(-ig)^2(-ie)^2\int\frac{d^4p\,d^4k}{(2\pi)^8}\mathrm{Tr}[iS(p)\gamma_\mu iS(p-k)\gamma_\nu iS(p)iS(p-q)iD^{\mu\nu}(k)]$$

and applying our circling rules as shown in Fig. 5 we obtain, for the imaginary part,

$$-2\,\mathrm{Im}\Pi_{11}^{\Sigma}(q)=g^2(-ie)^2\int\frac{d^4p\,d^4k}{(2\pi)^8}$$
$$\times\mathrm{Tr}[iS^+(p)\gamma^\mu iS(p-k)\gamma_\mu iS(p)iS^-(p-q)iD(k)$$
$$-iS^*(p)\gamma^\mu iS^*(p-k)\gamma_\mu iS^+(p)iS^-(p-q)iD^*(k)$$
$$+iS^*(p)\gamma^\mu iS^+(p-k)\gamma_\mu iS(p)iS^-(p-q)iD^+(k)$$
$$+iS^+(p)\gamma^\mu iS^-(p-k)\gamma_\mu iS^+(p)iS^-(p-q)iD^-(k)$$
$$+iS(p)\gamma^\mu iS(p-k)\gamma_\mu iS^-(p)iS^+(p-q)iD(k)$$
$$-iS^*(p)\gamma^\mu iS^*(p-k)\gamma_\mu iS^-(p)iS^+(p-q)iD^*(k)$$
$$-iS(p)\gamma^\mu iS^-(p-k)\gamma_\mu iS^*(p)iS^+(p-q)iD^-(k)$$
$$-iS^-(p)\gamma^\mu iS^+(p-k)\gamma_\mu iS^-(p)iS^+(p-q)iD^+(k)]\,. \tag{35}$$

As before, these terms represent the product of transition matrix elements as shown in Fig. 5. Note the circled diagrams corresponding to a photon emission and/or absorption diagram with a type-2 vertex. Since these diagrams cannot be represented by a cut they have to vanish at zero temperature, but at finite temperature they will contribute and, as we shall see, are indispensible for the well-definedness of the self-energy.

Again not all terms in (35) are independent. Using the by now familiar relation (26) it is trivial to show that the last four terms are related to the first four by a factor of $e^{-\beta q^0}$. Thus (35) is reduced to

$$\mathrm{Im}\Pi_{11}^{\Sigma}(m_H)=-\frac{g^2(-ie)^2}{2}(1+e^{-\beta m_H})\int\frac{d^4p\,d^4k}{(2\pi)^8}\mathrm{Tr}\{iS^-(p-q)$$
$$\times[iS^+(p)\gamma^\mu iS(p-k)\gamma_\mu iS(p)iD(k)$$
$$-iS^*(p)\gamma^\mu iS^*(p-k)\gamma_\mu iS^+(p)iD^*(k)$$
$$-iS^*(p)\gamma^\mu iS_{12}(p-k)\gamma_\mu iS(p)iD_{12}(k)e^{\beta p^0/2}$$
$$-iS^+(p)\gamma^\mu iS_{12}(p-k)\gamma_\mu iS^+(p)iD_{12}(k)e^{-\beta p^0/2}]\}\,, \tag{36}$$

where we used (25) to rewrite the circled photon propagators in terms of the off-diagonal propagator matrix elements. Obviously each of the four terms in (36) contains ill-defined distributions (pinch singularities or "squares of δ functions") which are guaranteed to cancel on general grounds.[6] The actual cancellation procedure, however, requires some work and is rather instructive, so we will present it here in some detail.

First we use the definition of the fermion self-energy matrix elements Σ_{ab} and write

$$\mathrm{Im}\Pi_{11}^{\Sigma}(m_H)=-\frac{g^2}{2}(1+e^{-\beta m_H})\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\{iS^-(p-q)\times[iS^+(p)i\Sigma_{11}(p)iS(p)-iS^*(p)i\Sigma_{22}(p-k)iS^+(p)+iS^*(p)i\Sigma_{12}(p-k)iS(p)e^{\beta p^0/2}+iS^+(p)i\Sigma_{12}(p-k)iS^+(p)e^{-\beta p^0/2}]\}\ ,$$

which can be rewritten, using Eq. (22), as

$$\mathrm{Im}\Pi_{11}^{\Sigma}(m_H)=-\frac{g^2}{2}(1+e^{-\beta m_H})\int\frac{d^4}{(2\pi)^4}\mathrm{Tr}(iS^-(p-q)\times\{iS^+(p)i\Sigma_{11}(p)iS(p)+iS^*(p)i\Sigma_{11}^*(p)iS^+(p)-\epsilon(p^0)\tan 2\phi(p)[iS^*(p)\mathrm{Im}\Sigma_{11}(p)iS(p)e^{\beta p^0/2}+iS^+(p)\mathrm{Im}\Sigma_{11}(p)iS^+(p)e^{-\beta p^0/2}]\})\ . \quad (37)$$

We note that, in terms of matrix elements, the first two terms arise from the self-energy insertion on an external fermion line, whereas the last two correspond to squares of photon emission and/or absorption diagrams.

In order to show the cancellation of the ill-defined distributions in (37), we recall the basic definition (11) of the propagator matrix elements and rewrite the finite-temperature propagators as follows:

$$iS^{\pm}(p)=2\pi(\not p+m)[\theta(p^0)-n_F(p)]\delta(p^2-m^2)\ ,$$
$$iS(p)=(\not p+m)[\cos^2\phi\Delta(p)-\sin^2\phi\Delta^*(p)]\ ,$$
$$iS^*(p)=(\not p+m)[-\cos^2\phi\Delta(p)+\sin^2\phi\Delta^*(p)]\ ,$$

where

$$\Delta(p)=i/(p^2-m^2+i\epsilon)\ ,\quad \Delta^*(p)=-i/(p^2-m^2-i\epsilon)\ ,$$
$$\cos^2\phi=1/(e^{-\beta|p^0|}+1),\quad \sin^2\phi=1/(e^{\beta|p^0|}+1)\ .$$

For the Δ distribution we use the well-known relation

$$\lim_{\epsilon\to 0}\frac{i}{p^2-m^2\pm i\epsilon}=PP\frac{i}{p^2-m^2}\pm\pi\delta(p^2-m^2)$$

from which it is easy to derive the useful identity

$$2\pi\delta(p^2-m^2)\frac{i}{p^2-m^2\pm i\epsilon}=i\pi\frac{\partial}{\partial m^2}\delta(p^2-m^2)\pm 2\pi^2\delta^2(p^2-m^2)\ ,$$

where the δ functions are understood as convenient shorthand for the sums and squares of the proper ε-regularized Δ distributions.

Consider now the first two terms corresponding to self-energy insertions on the external fermion lines (from now on all distributions are understood as properly ε regularized). With the identities just introduced it is straightforward to show that

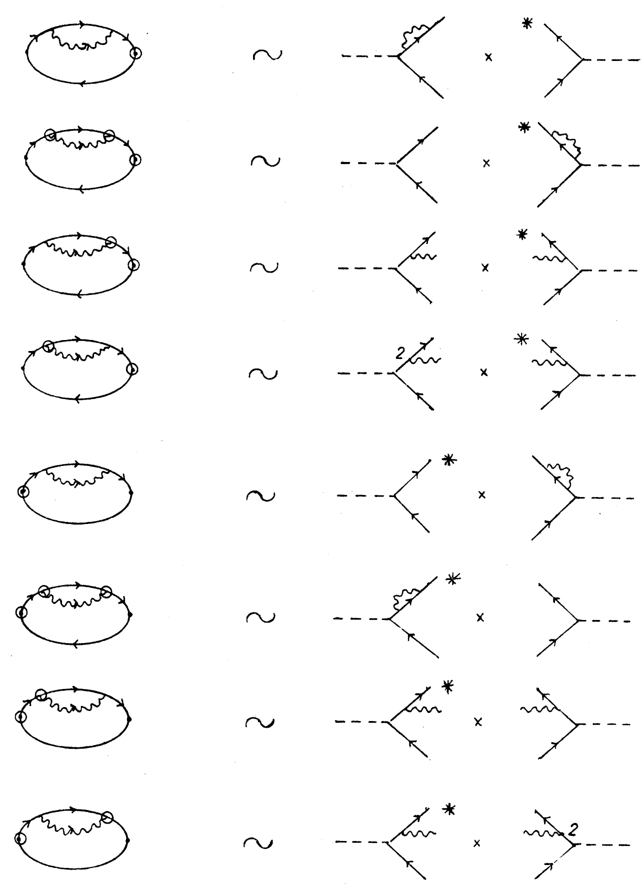

FIG. 5. Circled diagrams for the self-energy corrected Π_{11} and the equivalent products of transition matrix elements.

$$iS^+(p)iS(p)=(\not p+m)^2[\theta(p^0)-n_F(p)]\left[i\pi\frac{\partial}{\partial m^2}\delta(p^2-m^2)+(\cos^2\phi-\sin^2\phi)2\pi^2\delta^2(p^2-m^2)\right]$$

and likewise

$$iS^+(p)iS^*(p)=(\not p+m)^2[\theta(p^0)-n_F(p)]\left[i\pi\frac{\partial}{\partial m^2}\delta(p^2-m^2)-(\cos^2\phi-\sin^2\phi)2\pi^2\delta^2(p^2-m^2)\right].$$

Thus the first two terms in (37) can be combined into

$$iS^+(p)i\Sigma_{11}(p)iS(p)+iS^*(p)i\Sigma^*_{11}(p)iS^+(p)$$
$$=-2\pi(\not p+m)[\theta(p^0)-n_F(p)]\left[\mathrm{Re}\Sigma_{11}(p)\frac{\partial}{\partial m^2}\delta(p^2-m^2)+\mathrm{Im}\Sigma_{11}(p)\cos2\phi\,2\pi\delta^2(p^2-m^2)\right](\not p+m)\,, \quad (38)$$

that is, a well-defined mass derivative of the mass-shell δ function proportional to the real part of Σ, and a pinch singularity proportional to the imaginary part which has to be cancelled by contributions from the two remaining photon emission and/or absorption terms, that is, the third and fourth term in (37).

For the third term we find

$$iS^*(p)iS(p)=(\not p+m)^2[(\sin^2\phi\cos^2\phi-\tfrac{1}{2})(\Delta+\Delta^*)^2+\tfrac{1}{2}(\Delta^2+\Delta^{*2})]$$
$$=(\not p+m)^2\left[(\tfrac{1}{4}\sin^22\phi-\tfrac{1}{2})[2\pi\delta(p^2-m^2)]^2+\frac{\partial}{\partial m^2}[i\Delta-i\pi\delta(p^2-m^2)]\right]$$

and the fourth term is simply a "pure" pinch singularity

$$[iS^+(p)]^2=(\not p+m)^2\{[\theta(p^0)-n_F(p)]2\pi\delta(p^2-m^2)\}^2\,.$$

The cancellation of the δ^2 terms is now easy to see if we rewrite them in terms of the off-diagonal propagator matrix element iS_{12}. Recalling that

$$iS^+(p)=-e^{\beta p^0/2}iS_{12}(p)\,,$$
$$iS_{12}(p)=-\epsilon(p^0)(\not p+m)e^{\beta|p^0|/2}n_F(p)2\pi\delta(p^2-m^2)=-\epsilon(p^0)(\not p+m)\sin2\phi\pi\delta(p^2-m^2)$$

we obtain, for the last two terms in (37),

$$\epsilon(p^0)\tan2\phi[iS(p)\mathrm{Im}\Sigma_{11}(p)iS^*(p)e^{\beta p^0/2}+iS^+(p)\mathrm{Im}\Sigma_{11}(p)iS^+(p)e^{-\beta p^0/2})]$$
$$=\epsilon(p^0)e^{\beta p^0/2}\left[(\not p+m)\mathrm{Im}\Sigma_{11}(p)(\not p+m)\tan2\phi\frac{\partial}{\partial m^2}[\Delta(p)-i\pi\delta(p^2-m^2)]\right.$$
$$\left.+2\left[\tan2\phi-\frac{1}{\sin2\phi\cos2\phi}\right]iS_{12}(p)\mathrm{Im}\Sigma_{11}(p)iS_{12}(p)\right] \quad (39)$$

and likewise we have, for the ill-defined term in (38),

$$(\not p+m)^2[\theta(p^0)-n_F(p)]\cos2\phi[2\pi\delta(p^2-m^2)]^2=2\epsilon(p^0)e^{\beta p^0/2}\cot2\phi[iS_{12}(p)]^2\,.$$

Combining (38) and (39) we obtain, for (37),

$$\mathrm{Im}\Pi^{\Sigma}_{11}(q)$$
$$=-\frac{g^2}{2}(1+e^{-\beta m_H})\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left\{iS^-(p-q)\right.$$
$$\times\left[(\not p+m)\left[-[\theta(p^0)-n_F(p)]\mathrm{Re}\Sigma_{11}(p)2\pi\frac{\partial}{\partial m^2}\delta(p^2-m^2)\right.\right.$$
$$\left.-\epsilon(p^0)\mathrm{Im}\Sigma_{11}(p)e^{\beta p^0/2}\tan2\phi\frac{\partial}{\partial m^2}[i\Delta(p)-i\pi\delta(p^2-m^2)]\right](\not p+m)$$
$$\left.\left.+\epsilon(p^0)\mathrm{Im}\Sigma_{11}(p)e^{\beta p^0/2}\left[-\cot2\phi+\tan2\phi-\frac{1}{\cos2\phi\sin2\phi}\right][iS_{12}(p)]^2\right]\right\}$$

and the ill-defined S^2_{12} term disappears, leaving only well-defined distributions. To simplify the remaining part we use Eq. (23) for $\mathrm{Im}\Sigma_{11}$ and obtain

$$\epsilon(p^0)e^{\beta p^0/2}\tan 2\phi \,\mathrm{Im}\Sigma_{11}(p)=2[\theta(p^0)-n_F(p)]\mathrm{Im}\Sigma(p)$$

and for the distribution

$$\frac{\partial}{\partial m^2}[i\Delta-i\pi\delta(p^2-m^2)]=-\frac{\partial}{\partial m^2}PP\frac{1}{p^2-m^2}\equiv-\frac{1}{(p^2-m^2)^2}$$

by definition of the principal part. Thus, the self-energy corrected decay rate is given by

$$\Gamma^{\Sigma}(m_H)=-\frac{1}{m_H}\tanh(\beta m_H/2)\mathrm{Im}\Pi^{\Sigma}_{11}(m_H)$$

$$=-\frac{g^2}{m_H}(1-e^{-\beta m_H})\int\frac{d^4p}{(2\pi)^3}\delta((p-q)^2-m^2)[\theta(p^0)-n_F(p)][\theta(q^0-p^0)-n_F(p-q)]$$

$$\times\mathrm{Tr}\left[(\not p-\not q+m)(\not p+m)\left[\mathrm{Re}\Sigma(p)2\pi\frac{\partial}{\partial m^2}\delta(p^2-m^2)-\frac{2\,\mathrm{Im}\Sigma(p)}{(p^2-m^2)^2}\right](\not p+m)\right]. \quad (40)$$

We note that, in order to achieve complete cancellation of the ill-defined δ^2 terms it was necessary to include the circled but noncuttable self-energy graph or, equivalently, the matrix element with a type-2 vertex in our calculation. Thus, strictly speaking, the ghost vertices are needed in perturbation theory even at the one-loop level to ensure the well-definedness of our results. They do, however play only a "minimal role" since they do not contribute to the finite part of the result. It is also obvious from our derivation that pinch singularities will occur (and cancel) even at zero temperature. In this case the type-2 contribution vanishes and the cancellation involves only the regular type-1 terms.

Furthermore, we emphasize that our derivation and the final result are quite general since we have to consider only products of distributions that are common to the radiative QED corrections for any decay and scattering process. Also note that our derivation involves only the properties of the distributions $1/p^2-m^2\pm i\epsilon$ and hence holds for boson self-energies as well.

In terms of transition matrix elements the part proportional to $\mathrm{Re}\Sigma$ in (40) arises from the self-energy insertions on the external fermion lines Fig. 1(c). The term containing the imaginary part $\mathrm{Im}\Sigma$ represents the remaining part Γ^{γ_2} of the photon emission and/or absorption processes Fig. 1(d). It can be written in a more familiar form analogous to Γ^{γ_1} if we use the explicit form of $\mathrm{Im}\Sigma$: by definition

$$-i\Sigma_{11}(p)=(-ie)^2\int\frac{d^4p}{(2\pi)^4}\gamma_\mu iS(p+k)\gamma_\nu iD^{\mu\nu}(k)$$

and, applying our circling rules, it is easy to show that

$$\mathrm{Im}\Sigma(p)=\epsilon(p^0)\coth(\beta p^0/2)\mathrm{Im}\Sigma_{11}(p)$$

$$=\epsilon(p^0)\frac{e^2}{2}(1+e^{-\beta p^0})\int\frac{d^4k}{(2\pi)^2}\gamma_\mu iS^+(p)\gamma^\mu \times iD^-(k)\,.$$

This yields, for Γ^{γ_2},

$$\Gamma^{\gamma_2}(m_H)=\frac{g^2e^2}{m_H}(1-e^{-\beta m_H})\int\frac{d^4p\,d^4k}{(2\pi)^5}\delta(p^2-m^2)\delta((p-k-q)^2-m^2)\delta(k^2)$$

$$\times\mathrm{Tr}[(\not p-\not k+m)\gamma_\mu(\not p+m)\gamma^\mu(\not p-\not k+m)(\not p-\not k-\not q+m)]$$

$$\times[\theta(p^0)-n_F(p)][\theta(q^0+k^0-p^0)-n_F(p-k-q)][\theta(-k^0)+n_B(k)]\left[\frac{1}{(p-k)^2-m^2}\right]^2. \quad (41)$$

As before, this expression contains both zero-temperature and temperature-dependent infrared and mass singularities which have to cancel against the ones arising from the real part of the electron self-energy. For the problem of mass and wave-function renormalization, however, Γ^{γ_2} is of no direct interest; hence, we will concentrate in the following on the $\mathrm{Re}\Sigma$ part.

3. Mass and wave-function renormalization at finite temperature

The mass derivative of the δ function is best evaluated as

$$\int d^4p\,F(p,m^2,\dots)\frac{\partial}{\partial m^2}\delta(p^2-m^2)\equiv\lim_{\hat m^2\to m^2}\frac{\partial}{\partial\hat m^2}\int d^4p\,F(p,m^2,\dots)\delta(p^2-\hat m^2)$$

and the contribution of the self-energy correction to the external fermion lines can be written as

$$\Gamma^{\rm SE}(m_H) = -\frac{g^2}{m_H}(1-e^{-\beta m_H}) \lim_{\hat{m}^2\to m^2}$$
$$\times \int \frac{\partial}{\partial \hat{m}^2}\left[\frac{d^4p}{(2\pi)^2}[\theta(p^0)-n_F(p)][\theta(q^0-p^0)-n_F(p-q)]\delta(p^2-\hat{m}^2)\delta((p-q)^2-m^2) \times \mathrm{Tr}(\not{p}-\not{q}+m)(\not{p}+m)\right] \mathrm{Re}\Sigma(p)(\not{p}+m)$$
$$+ \left[\frac{d^4p}{2\pi^2}[\theta(p^0)-n_F(p)][\theta(q^0-p^0)-n_F(p-q)]\delta(p^2-\hat{m}^2)\delta((p-q)^2-m^2) \times \mathrm{Tr}(\not{p}-\not{q}+m)(\not{p}+m)\right] \frac{\partial}{\partial \hat{m}^2}[\mathrm{Re}\Sigma(p)(\not{p}+m)] , \tag{42}$$

where the above notation is understood as integrating over the δ functions before taking the derivative. In the real-time formalism the fermion self-energy can be split into a zero-temperature and a finite-temperature part: $\mathrm{Re}\Sigma = \mathrm{Re}\Sigma^0 + \mathrm{Re}\Sigma^\beta$. First consider the $T=0$ part.

Usually $\mathrm{Re}\Sigma^0$ is expanded in a formal Taylor series around the mass-shell point "$\not{p}=m$" (which as a matrix equation is of course nonsensical):

$$\mathrm{Re}\Sigma^0(p) = \mathrm{Re}\Sigma^0_{\not{p}=m} + \frac{\partial}{\partial \not{p}}\mathrm{Re}\Sigma^0_{\not{p}=m}(\not{p}-m) + \cdots$$
$$\equiv \delta m + \delta Z_2(\not{p}-m) + \cdots , \tag{43}$$

where the first two coefficients are defined as the ultraviolet-divergent mass and wave-function renormalization counterterms (the finite higher-order terms can be set to zero by the mass-shell renormalization condition and will be neglected). More precisely, Lorentz invariance restricts $\mathrm{Re}\Sigma^0$ to the general form

$$\mathrm{Re}\Sigma^0(p) = a_1(p^2)\not{p} + a_2(p^2) \tag{44}$$

and, expanding the coefficients around the propagator pole $p^2=m^2$, we obtain, to $O(p^2-m^2)$,

$$\mathrm{Re}\Sigma^0(p) = [a_1'(m^2) + a_2'(m^2)(\not{p}+m)](\not{p}-m) + m a_1(m^2) + a_2(m^2) + \cdots$$
$$\simeq [a_1'(m^2) + 2m a_2'(m^2)](\not{p}-m) + m a_1(m^2) + a_2(m^2) ,$$

which justifies the formal Taylor series (43). To renormalize the bare mass m_0 in the free fermion Lagrangian to its physical on-shell value $m_{\rm phys}$,

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m_0)\psi \to \bar{\psi}(i\not{\partial} - m_0 - \delta m)\psi$$
$$\equiv \bar{\psi}(i\not{\partial} - m_{\rm phys})\psi ,$$

we have to include the mass counterterm diagrams Fig. 2(e) in our set of self-energy diagrams. Their imaginary part is easily determined with our circling rules, and adding it to the unrenormalized rate replaces $\mathrm{Re}\Sigma^0$ by $\mathrm{Re}\tilde{\Sigma}^0 \equiv \mathrm{Re}\Sigma^0 - \delta m$ in the decay rate (40). With this mass-subtracted self-energy we find immediately, for (42),

$$\lim_{\hat{m}^2\to m^2} \mathrm{Re}\tilde{\Sigma}^0(\not{p}+m) = 0 ,$$

$$\lim_{\hat{m}^2\to m^2} \frac{\partial}{\partial \hat{m}^2}[\mathrm{Re}\tilde{\Sigma}^0(\not{p}+m)] = \delta Z_2 ,$$

and thus the $\mathrm{Re}\Sigma^0$ contribution to the decay rate reduces to

$$\Gamma_0^{\rm SE}(m_H) = \delta Z_2 \frac{-g^2}{m_H}(1-e^{-\beta m_H}) \lim_{\hat{m}^2\to m^2} \int \frac{d^4p}{(2\pi)^2}\delta(p^2-\hat{m}^2)\delta((p-q)^2-m^2)$$
$$\times [\theta(p^0)-n_F(p)][\theta(q^0-p^0)-n_F(p-q)]\mathrm{Tr}[(\not{p}-\not{q}+m)(\not{p}+m)]$$
$$= 2\delta Z_2 \Gamma^{\rm tree}(m_H) , \tag{45}$$

which is precisely the result one obtains from multiplicative on-shell renormalization and the Lehmann-Symanzik-Zimmermann (LSZ) theorem for the transition matrix elements. This is usually expressed as the following Feynman rule:[10] self-energy corrections on external (fermion) lines are replaced by a factor of $\sqrt{Z_2}$ for each line, provided the mass counterterm δm has been included.

The remaining ultraviolet divergence in δZ_2 can of course be eliminated by adding a suitable counterterm vertex $\delta Z_2^{\rm ct}(\not{p}-m)$ to δm in the diagram Fig. 2(e).

The crucial property in the derivation of (45) is Lorentz invariance which permits the formal Taylor expansion (43) of $\mathrm{Re}\Sigma^0$. For the finite-temperature part $\mathrm{Re}\Sigma^\beta$, however, Lorentz invariance is lost, but we can still maintain Lorentz *covariance* of the theory by introducing the four-velocity u of the heat bath.[17,18] The noncovariant approach described so far corresponds to the choice $u=(1,0,0,0)$; that is, the heat bath is taken to be at rest with respect to the laboratory frame. In the following $u=(1,0,0,0)$ is always understood, unless stated otherwise. The requirement of Lorentz covariance restricts $\mathrm{Re}\Sigma^{\beta}$ to the general form

$$\begin{aligned}\mathrm{Re}\Sigma^{\beta}(p)&=a_1\not{p}+a_2+a_3\not{u}+a_4\not{u}\not{p}\\&\equiv a_1\not{p}+a_2+a_3\gamma^0+a_4\gamma^0\not{p}\ ,\end{aligned}$$

where the four coefficient functions a_i will now depend on the two available Lorentz scalars p^2 and $p\cdot u\equiv p^0$: $a_i=a_i(p^0,p^2)$. Thus (44) is now generalized to

$$\mathrm{Re}\Sigma^{\beta}(p)=\mathcal{A}_1(p^0,p^2)(\not{p}-m)+\mathcal{A}_2(p^0,p^2) \tag{46}$$

with matrix-valued coefficient functions

$$\mathcal{A}_1=a_1+a_4\gamma^0\ ,$$

$$\mathcal{A}_2=ma_1+a_2+(a_3+a_4)\gamma_0\ .$$

We could now proceed as in the $T=0$ case and, expanding the coefficients $\mathcal{A}_i$ around p^2-m^2, introduce the matrix-valued mass counterterm

$$\delta m^{\beta}=\mathcal{A}^2_{p^2=m^2}\ .$$

Thus the physical mass is now defined as the pole in the finite-temperature propagator

$$i\mathcal{S}(p)=\frac{i}{\not{p}-m-\mathrm{Re}\Sigma^{\beta}(p)}$$

and the finite-temperature wave-function renormalization factor Z_2^{β} might, e.g., be taken as the residue of $\mathcal{S}$.

However, this straightforward and popular procedure presents several problems if we accept the general philosophy that normalization counterterms should be of the same form as the unrenormalized quantities, that is, the renormalized Lagrangian should remain Lorentz invariant and of the same functional form as the bare Lagrangian.

Lorentz invariance is obviously lost for any finite-temperature counterterm defined from $\mathrm{Re}\Sigma^{\beta}$ and the renormalized Lagrangian, a dynamical quantity, will now be temperature dependent. Although one might accept this, together with the more general γ-matrix structure, as a generalization necessary for finite-temperature field theory there remains the problem that the renormalization point $p^2=m^2$ is not sufficient to eliminate the momentum dependence of the covariant coefficient functions in $\mathrm{Re}\Sigma^{\beta}$. This momentum dependence is nonpolynomial and rather complicated (see, e.g., Ref. 18 for an example); thus any counterterm constructed from the a_i's will introduce new, nonlocal and nonrenormalizable interactions in the renormalized finite-temperature Lagrangian.

These problems will not arise if we take the conservative approach and introduce only the zero-temperature counterterms necessary for the removal of the ultraviolet divergences. This is legitimate since $\mathrm{Re}\Sigma^{\beta}$ is ultraviolet finite (the thermal distributions act as regulators). Thus the Lagrangian remains Lorentz invariant and dynamical. The $\mathrm{Re}\Sigma^{\beta}$ contribution will of course not reduce to a simple multiplicative Z_2 factor but requires the evaluation of the integral

$$\begin{aligned}\Gamma_{\beta}^{\mathrm{SE}}(m_H)=&-\frac{g^2}{m_H}(1-e^{-\beta m_H})\lim_{\hat{m}^2\to m^2}\frac{\partial}{\partial\hat{m}^2}\int\frac{d^4p}{(2\pi)^2}\delta(p^2-\hat{m}^2)\delta((p-q)^2-m^2)[\theta(p^0)-n_F(p)][\theta(q^0-p^0)-n_F(p-q)]\\&\times\mathrm{Tr}[(\not{p}-\not{q}+m)(\not{p}+m)\mathrm{Re}\Sigma^{\beta}(p)(\not{p}+m)]\\\equiv&-\frac{g^2}{m_H}(1-e^{-\beta m_H})\int d^4p\ \delta(p^2-m^2)\cdots\mathrm{Tr}[(\not{p}-\not{q}+m)(\not{p}+m)\mathcal{A}_1(p)]\\&-\frac{g^2}{m_H}(1-e^{-\beta m_H})\lim_{\hat{m}^2\to m^2}\frac{\partial}{\partial\hat{m}^2}\int d^4p\cdots\mathrm{Tr}[(\not{p}-\not{q}+m)(\not{p}+m)\mathcal{A}_2(p)(\not{p}+m)]\end{aligned} \tag{47}$$

and the masses are the physical (renormalized) zero-temperature parameters.

This approach is the most general one: the decay rates are evaluated as a function of the known (and measurable) zero-temperature parameters and there is no conflict with the basic requirements of locality and renormalizability.

In the case of Higgs-boson decay, or any decay and/or scattering process with two-body phase space, the problem of momentum-dependent counterterms does not arise since the mass-shell δ functions fix p^0 and p^2,

$$\delta(p^2-m^2)\delta((p-q)^2-m^2)\to p^0=\omega_p=m_H/2\ ,$$

and give a natural on-shell renormalization point. Thus we can proceed and define operational renormalization constants. Lorentz invariance is of course still lacking, so the following prescription is heuristic and "natural" only insofar as it uses the covariant generalization of the familiar counterterms and, as we shall see, reduces the self-energy contribution to a scalar Z_2^{β} constant analogous to the $T=0$ renormalization procedure. Moreover, the covariant finite-temperature counterterms will have features that make an interpretation as QED renormalization constants extremely problematic. Thus the following covariant on-shell renormalization scheme should be regarded only as a convenient procedure to deal with the momentum dependence of $\mathrm{Re}\Sigma^\beta$. Indeed, our discussion will demonstrate that, even without the problem of momentum dependence, finite-temperature counterterms are not a physically meaningful concept; thermal transition rates are hence best evaluated in terms of the (well-defined) zero-temperature parameters.

From (46) we define the matrix-valued finite-temperature mass counterterm

$$\delta m^\beta \equiv \mathcal{A}_2(p^0=\omega_p=m_H/2)\ , \tag{48}$$

which replaces $\mathrm{Re}\Sigma^\beta$ by $\mathrm{Re}\tilde{\Sigma}^\beta=\mathrm{Re}\Sigma^\beta-\delta m^\beta$ as before. The matrix structure of δm^β, however, has some unusual consequences which we will discuss later on.

The first term in (42) is eliminated by δm^β, as at zero temperature, since

$$\lim_{\hat{m}^2\to m^2} \mathrm{Re}\tilde{\Sigma}^\beta(\not{p}+m)=0$$

and we define the remaining term as a matrix-valued "$\mathcal{Z}_2^\beta$ function:"

$$\begin{aligned}\delta\mathcal{Z}_2^\beta &= \lim_{\hat{m}^2\to m^2}\frac{\partial}{\partial\hat{m}^2}[\mathrm{Re}\tilde{\Sigma}^\beta(p)(\not{p}+m)]\\ &=\mathcal{A}_1+\left[\frac{\partial}{\partial\hat{m}^2}\mathcal{A}_2\right](\not{p}+m)\\ &=a_1+a_4\gamma^0+\left[\frac{\partial}{\partial\hat{m}^2}(ma_1+a_2)\right](\not{p}+m)\\ &\quad+\left[\frac{\partial}{\partial\hat{m}^2}(a_3+ma_4)\gamma^0\right](\not{p}+m)\end{aligned}$$

at $p^0=\omega_p=m_H/2$, $\hat{m}^2\to m^2$, which is of course still momentum dependent. Evaluating the trace with $\mathcal{Z}_2^\beta$ we find

$$\begin{aligned}&\mathrm{Tr}[(\not{p}-\not{q}+m)(\not{p}+m)\delta\mathcal{Z}_2^\beta]\\ &\quad=\mathrm{Tr}[(\not{p}-\not{q}+m)(\not{p}+m)]\left[a_1+2p^0\frac{\partial}{\partial\hat{m}^2}(a_3+ma_4)\right.\\ &\qquad\left.+2m\frac{\partial}{\partial\hat{m}^2}(ma_1+a_2)\right]\ .\end{aligned}$$

Note that we used

$$\mathrm{Tr}[(\not{p}-\not{q}+m)(\not{p}+m)a_4\gamma^0]=(2p^0-q^0)ma_4=0$$

at $p^0=\omega_p=m_H/2$.

Thus we have, for our present system, an effective scalar finite-temperature wave-function renormalization constant Z_2^β given by

$$\begin{aligned}\delta Z_2^\beta &= Z_2^\beta-1\\ &=a_1+2p^0\frac{\partial}{\partial\hat{m}^2}(a_3+ma_4)+2m\frac{\partial}{\partial\hat{m}^2}(ma_1+a_2)\end{aligned} \tag{49}$$

at the renormalization point $p^0=\omega_p=m_H/2$, which is nothing but the covariant generalization of the zero temperature Z_2. The self-energy correction to the external fermion lines is thus reduced to an effective scalar renormalization constant

$$\Gamma_\beta^{\mathrm{SE}}(m_H)=2\delta Z_2^\beta\Gamma^{\mathrm{tree}}(m_H)$$

as desired for on-shell renormalization.

Let us now return to the finite-temperature mass counterterm. Using the γ-matrix-valued δm^β as a counterterm in the free fermion Lagrangian

$$\mathcal{L}=\bar{\psi}(i\not{\partial}-m_{\mathrm{phys}})\psi\to\bar{\psi}(i\not{\partial}-m_{\mathrm{phys}}-\delta m^\beta)\psi$$

leads to a temperature dependence of the Dirac operator as can be easily seen in momentum space. Writing a general matrix-valued mass counterterm as $\delta m^\beta\equiv\Delta\not{p}^\beta+\Delta m^\beta$ we have, in momentum space, for the Lagrangian,

$$\begin{aligned}\mathcal{L}&=\bar{\psi}(i\not{p}-m_{\mathrm{phys}}-\delta m^\beta)\psi\\ &=\bar{\psi}[(p-\Delta p^\beta)\cdot\gamma-(m_{\mathrm{phys}}+\Delta m^\beta)]\psi\\ &\equiv\bar{\psi}(\tilde{\not{p}}-\tilde{m})\psi\ ,\end{aligned}$$

where we used the tilde notation of Ref. 3. Thus the mass counterterm δm^β shifts not only the mass but also the momentum operator. Rederiving the Feynman rules for the shifted finite-temperature Dirac operator is straightforward and amounts to replacing p and m by $\tilde{p}$ and $\tilde{m}$ in the propagators. Thus all decay rates have to be reevaluated with tilde quantities. For the tree-level decay rate we obtain

$$\begin{aligned}\Gamma^{\mathrm{tree}}\to-\frac{g^2}{2m_H}(1-e^{-\beta m_H})\int\frac{d^4\tilde{p}}{(2\pi)^4}\delta(\tilde{p}^2-\tilde{m}^2)\delta((\tilde{p}-q)^2-\tilde{m}^2)\\ \times[\theta(\tilde{p}^0)-n_F(\tilde{p})][\theta(q^0-\tilde{p}^0)-n_F(\tilde{p}-q)]\mathrm{Tr}[(\tilde{\not{p}}-\not{q}+\tilde{m})(\tilde{\not{p}}+\tilde{m})]\ .\end{aligned}$$

As an aside we note that the terms in the trace can be regarded as products of spinors on the external fermion legs of the transition matrix elements. Here these spinors would be solutions to the "finite-temperature Dirac equation" $(\not{p}-m-\delta m^{\beta})u=0$ and are a special case of the finite-temperature spinors introduced in Ref. 3. It can be easily shown that the integral is invariant under any translation by a constant four-vector: $p\to\tilde{p}=p-\Delta p^{\beta}$. Hence we have to replace only m by $\tilde{m}=m+\Delta m^{\beta}$ in (28) and we have for the mass-renormalized tree rate

$$\Gamma^{\text{tree}}\to\frac{g^2}{8\pi}m_H\left[1-\frac{4\tilde{m}^2}{m_H^2}\right]^{3/2}\tanh(\beta m_H/4)$$
$$\simeq\Gamma^{\text{tree}}(m_H)\left[1-\frac{12m}{m_H^2w^2}\Delta m^{\beta}\right]$$
$$\equiv\Gamma^{\text{tree}}+\Gamma^{\Delta m} \qquad (50)$$

to $O(e^2)$, where $w=(1-4m^2/m_H^2)^{1/2}$. The other contributions—vertex correction, etc.—are already of $O(e^2)$ and do not have to be reevaluated in tilde variables. Thus the net effect of the mass counterterm δm^{β} is indeed a mass renormalization $m_{\text{phys}}\to m_{\text{phys}}+\Delta m^{\beta}$.

In summary the covariant renormalization prescription gives indeed the operational analog of the zero-temperature renormalization procedure. The fermion self-energy correction is absorbed into a multiplicative Z factor and the physical mass in the tree rate (28) is shifted by a constant. There are, however, important differences to the zero-temperature case.

The zero-temperature QED renormalization constants are well-defined and "universal" in the sense that they are fixed by the fermion mass-shell condition alone and involve only QED parameters, that is, the fermion mass and the fermion-photon coupling. The finite-temperature counterterms δm^{β} and δZ_2^{β}, however, depend not only on the fermion mass but also on the mass of the Higgs boson or, more generally, the specific kinematics of the reaction, due to the choice of the renormalization point [it is of course possible to choose an arbitrary point (p^0,p^2) without reference to the Higgs boson, but this would not eliminate the δm^{β} term in the decay rate and make the concept of thermal on-shell renormalization ambiguous]. Moreover, consider the $\mathcal{Z}_2^{\beta}$ function from which we derived the covariant wave-function renormalization constant Z_2^{β}. If we replace the simple Yukawa Higgs-fermion vertex by a general coupling (vector, axial, etc.) the trace in the decay rate will be of the general form $\text{Tr}[\cdots(\not{p}+m)\text{Re}\Sigma^{\beta}(\not{p}+m)]$ and hence we have to consider the trace $\text{Tr}[\cdots(\not{p}+m)\delta\mathcal{Z}_2^{\beta}]$. With $\mathcal{Z}_2^{\beta}$ of the general form

$$\delta\mathcal{Z}_2^{\beta}=a_1+a_4\gamma^0+\left[\frac{\partial}{\partial\hat{m}^2}(ma_1+a_2)\right](\not{p}+m)$$
$$+\left[\frac{\partial}{\partial\hat{m}^2}(a_3+ma_4)\gamma^0\right](\not{p}+m)$$
$$\equiv A_1+A_2+A_3(\not{p}+m)+A_4(\not{p}+m)$$

and the fermion mass-shell δ functions this is easily reduced to

$$\text{Tr}[\cdots(\not{p}+m)\delta\mathcal{Z}_2^{\beta}]$$
$$=\text{Tr}[\cdots(\not{p}+m)](A_1+2p\cdot A_4+2mA_3)$$
$$+\text{Tr}[\cdots(\not{p}+m)A_2]\ .$$

Obviously the first factor is our Z_2^{β} factor (49) but the second term will in general depend on the specifics of the trace (in the case of Higgs-boson decay it vanishes). In general, the wave-function renormalization factor derived from $\mathcal{Z}_2^{\beta}$ will depend not only on the fermion-photon coupling but on the other fermion couplings as well.

Thus thermal on-shell renormalization leads to the somewhat paradoxical situation that QED counterterms depends also on *non*-QED interactions.

Furthermore, the mass-renormalized finite-temperature Dirac operator

$$\not{D}=\not{\partial}-m-\delta m^{\beta}$$

has poles in momentum space given in the dispersion relation

$$\tilde{p}^2-\tilde{m}^2\simeq p^2-m^2-2p\cdot\Delta p^{\beta}-2m\,\Delta m^{\beta}=0\ ,$$

which has solutions

$$p^0=\Delta p^{\beta}\pm(\omega_p^2+2m\,\Delta m^{\beta})^{1/2}$$

by definition (48) of δm^{β}. If we define a finite-temperature mass as the solution p^0 at some fixed three-momentum $\mathbf{p}$ (see, e.g., Refs. 19 and 20) it is obvious that this "dispersion relation mass" will depend on all components of δm^{β} whereas the "effective decay rate mass" in (50) involves only a shift by Δm^{β}.

It is also obvious that the finite-temperature Dirac-operator $\not{D}$ is not identical to the inverse finite-temperature propagator $i\mathcal{S}^{-1}=\not{\partial}-m-\text{Re}\Sigma^{\beta}(p^2=m^2)$; the latter would correspond to a Dirac operator with momentum-dependent mass counterterm which we rejected as incompatible with a local and renormalizable Lagrangian.

These considerations show that the familiar zero-temperature concepts of mass and wave-function renormalization cannot be extended to finite temperature in a generic way, not even for the case of two-body decay. The heuristic covariant finite-temperature counterterms will in general depend on the specific kinematics and couplings of all particles in a particular decay process and do not have the physical and model-independent interpretation of their zero-temperature counterparts. We conclude that the notions of finite-temperature parameters in the Lagrangian, Dirac operators, and spinors are problematic concepts and of no use for the study of decay processes at finite temperature.

Let us now consider an explicit example. The covariant expansion (46) was well suited to discuss the general momentum dependence of $\text{Re}\Sigma^{\beta}$, but the actual computation of the coefficients a_i for the full self-energy is rather cumbersome (see, e.g., Ref. 21 for a zero-temperature, finite-density example and[18] for the massless case). Instead we will consider the familiar low-temperature case treated in Ref. 3.

In the low-temperature regime $T \ll m$, $\mathrm{Re}\Sigma^{\beta}$ is approximated on mass shell by[3,22]

$$\mathrm{Re}\Sigma^{\beta}(p) = \frac{e^2}{8\pi^3}\left[I_A(\not{p}-m) + \frac{1}{2\omega_p}\frac{\partial \not{I}(\omega_p,\mathbf{p})}{\partial \omega_p}(p^2-m^2) + \not{I}(\omega_p,\mathbf{p})\right], \quad (51)$$

where

$$I_A = 4\pi\int_{\epsilon}^{\infty}\frac{d|\mathbf{k}|}{|\mathbf{k}|}n_B(|\mathbf{k}|),$$

$$I^{\mu}(\omega_p,\mathbf{p}) = \int \frac{d^3k}{|\mathbf{k}|}\frac{k^{\mu}}{\omega_p k^0 - \mathbf{pk}}n_B(|\mathbf{k}|),$$

where $\omega_p = \sqrt{\mathbf{p}^2+m^2}$, $k^0 = |\mathbf{k}|$, and $n_B(|\mathbf{k}|) = 1/(e^{\beta|\mathbf{k}|}-1)$. The explicit form of $I(\omega_p,\mathbf{p})$ is given in the Appendix.

First we observe that (51) is not a covariant expansion of the type (46) discussed so far, but an expansion of the finite-temperature photon contribution in $\mathrm{Re}\Sigma^{\beta}$ at the mass-shell point $p^2=m^2$. The renormalization constants given in Refs. 3 and 22 are

$$\delta m^{\mathrm{DH}} = \frac{e^2}{8\pi^3}\not{I}_{\text{``}\not{p}=m\text{''}}, \quad (52)$$

$$\delta Z_2^{\mathrm{DH}} = \frac{e^2}{8\pi^3}\left[I_A - \frac{1}{\omega_p}I_0\right] = \frac{e^2}{8\pi^3}\left[I_A - \frac{\pi^3 T^2}{3v\omega_p^2}\ln\frac{1+v}{1-v}\right], \quad (53)$$

where $v = |\mathbf{p}|/\omega_p$. From their explicit form it is obvious that these noncovariant counterterms have the same unacceptable momentum dependence that we discussed for the covariant expansion (46). Moreover, the momentum dependence in δm^{DH} is not even fixed by the mass-shell δ functions; thus, δm^{DH} cannot be used as a heuristic mass counterterm. We will now calculate directly the decay rate correction due to $\mathrm{Re}\Sigma^{\beta}$ without introducing any finite-temperature counterterms (the physical value of the decay rate is of course independent of the renormalization scheme used). Thus we have to evaluate Eq. (47):

$$\Gamma_{\beta}^{\mathrm{SE}}(m_H) = -\frac{g^2}{m_H}(1-e^{-\beta m_H})\lim_{\hat{m}^2\to m^2}\frac{\partial}{\partial \hat{m}^2}\int\frac{d^4p}{(2\pi)^2}\delta(p^2-\hat{m}^2)\delta((p-q)^2-m^2)[\theta(p^0)-n_F(p)][\theta(q^0-p^0)-n_F(p-q)] \times \mathrm{Tr}[(\not{p}-\not{q}+m)(\not{p}+m)\mathrm{Re}\Sigma^{\beta}(\not{p}+m)]\,.$$

Let us split $\mathrm{Re}\Sigma^{\beta}$ into

$$\mathcal{B}_1 = \frac{e^2}{8\pi^3}\left[I_A + \frac{1}{2\omega_p}\frac{\partial \not{I}(\omega_p,\mathbf{p})}{\partial\omega_p}(\not{p}+m)\right](\not{p}-m),$$

$$\mathcal{B}_2 = \frac{e^2}{8\pi^3}\not{I}(\omega_p,\mathbf{p}),$$

and hence the decay rate into

$$\Gamma_{\beta}^{\mathrm{SE}} = \Gamma^{\mathcal{B}_1} + \Gamma^{\mathcal{B}_2}.$$

A straightforward calculation, given in the Appendix, yields

$$\Gamma^{\mathcal{B}_1} = 2\delta Z_2^{\mathrm{DH}}\Gamma^{\mathrm{tree}}(m_H) = \left[\frac{e^2}{4\pi^3}I_A - \frac{e^2T^2}{3m_H^2 w}\ln\frac{1+w}{1-w}\right]\Gamma^{\mathrm{tree}}(m_H), \quad (54)$$

$$\Gamma^{\mathcal{B}_2} = -\frac{e^2T^2}{3m_H^2w^2}\left[1+\frac{1}{w}(1-w^2)\ln\frac{1+w}{1-w}\right]\Gamma^{\mathrm{tree}}(m_H), \quad (55)$$

where $w = (1-4m^2/m_H^2)^{1/2}$.

Thus the $\mathcal{B}_1$ part of $\mathrm{Re}\Sigma^{\beta}$ reduces to the wave-function renormalization factor δZ_2^{DH} which is not surprising since Z_2^{DH} corresponds to the part proportional to $(\not{p}-m)$ in $\mathrm{Re}\Sigma^{\beta}$. Note that δZ_2^{DH} contains an infrared divergence in I_A and a mass singularity in I_0 for $m^2\to 0$, that is, for $w\to 1$. However, they both cancel against similar terms in Γ^{γ_2} as shown explicitly in Ref. 3.

The $\mathcal{B}_2$ contribution to Γ^{SE} can be regarded as a noncovariant mass correction since it corresponds to δm^{DH}. We note that $\Gamma^{\mathcal{B}_2}$ is infrared finite, and does not contain a mass singularity but remains finite for $w\to 1$. Thus $\mathcal{B}_2$ does not introduce any new mass singularities in the decay rate, and there is no need to introduce a momentum-dependent finite-temperature counterterm in the Lagrangian to eliminate the (potentially troublesome) δm^{DH} contribution. This is important since, according to our discussion, there is no generic way to define finite-temperature renormalization constants. By comparison, at zero temperature the Kinoshita-Lee-Nauenberg theorem guarantees the absence of mass singularities in the unrenormalized decay rate, but only after mass counterterms have been included (cf. Ref. 16).

In Ref. 5 we pointed out that, since the self-energy (51) shifts the pole in the fermion propagator by a constant

$$m_{\mathrm{phys}}^2 \to m_{\mathrm{phys}}^2 + \frac{e^2T^2}{6} \equiv m_{\mathrm{phys}}^2(\beta),$$

we can define $m_{\mathrm{phys}}(\beta)$ as the physical "finite-temperature mass" and describe the shift by the constant mass counterterm

$$\delta m^{\beta} = \frac{e^2}{8\pi^3}\frac{p\cdot I}{m} \equiv \frac{e^2T^2}{12m}, \quad (56)$$

which is independent of the reaction kinematics. If one insists on a finite-temperature mass for the fermion, then δm^{β} provides a simple and physically transparent alternative to the covariant counterterms discussed before. On the other hand, this finite renormalization of the physical mass of course neither changes the value of the decay rate nor simplifies the calculation: unlike δm^{DH} the counterterm δm^{β} does *not* eliminate the "mass" contribution $\Gamma^{\mathcal{B}_2}$ completely, but instead replaces $\mathcal{B}_2$ by

$$\mathcal{B}_2' = \mathcal{B}_2 - \delta m^{\beta} = \frac{e^2}{8\pi^3}\left[I - \frac{2\pi^3 T^2}{3m} \right] .$$

The corresponding decay rate is then evaluated as (see the Appendix)

$$\Gamma^{\mathcal{B}_2'} = \frac{e^2 T^2}{3m_H^2 w^2}\left[2 - \frac{1}{w}(1-w^2)\ln\frac{1+w}{1-w} \right]\Gamma^{\mathrm{tree}}(m_H) . \quad (57)$$

The mass-shifted tree rate (50) for the mass shift (56) is easily found to be

$$\Gamma^{\Delta m} = -\frac{e^2 T^2}{m_H^2 w^2}\Gamma^{\mathrm{tree}} \quad (58)$$

and we have

$$\Gamma^{\mathcal{B}_2} = \Gamma^{\mathcal{B}_2'} + \Gamma^{\Delta m}$$

as it should be.

We note that the shift in the propagator pole that defines δm^{β} is momentum independent only for the approximation (51) of $\mathrm{Re}\Sigma^{\beta}$, which neglects the thermal fermion contributions. Taking the fermion corrections into account leads again to a nontrivial momentum dependence of the propagator pole (cf. Ref. 3) and we are faced with the same problems as before.

Also, our example implies that the definition of on-shell counterterms is ambiguous in the sense that it depends on the type of expansion (covariant or noncovariant) used for $\mathrm{Re}\Sigma^{\beta}$. This supports our conclusion that on-shell finite-temperature renormalization cannot be defined in a generic or unique way, which is of course a consequence of the lack of Lorentz invariance.

We can now compare our results for the finite-temperature radiative corrections to the Higgs-boson decay rate in the low-temperature limit to the one given in Ref. 3. The virtual-photon correction is given by

$$\Gamma_{\beta}^{\mathrm{virtual}} \equiv \Gamma_{\beta}^{G} + \Gamma_{\beta}^{\mathrm{SE}} = \Gamma_{\beta}^{G} + \Gamma^{\mathcal{B}_1} + \Gamma^{\Delta m} + \Gamma^{\mathcal{B}_2'} \quad (59)$$

with Γ_{β}^{G} given in (31) and (32), and with $\Gamma^{\mathcal{B}_1}$, $\Gamma^{\Delta m}$, $\Gamma^{\mathcal{B}_2'}$ given by (54), (58), and (57). The real-photon correction

$$\Gamma^{\mathrm{real}} \equiv \Gamma^{\gamma_1} + \Gamma^{\gamma_2} \quad (60)$$

was given in general form in (34) and (41). A complete evaluation is rather difficult; however, in the low-temperature approximation it suffices to expand (60) in a power series in T/m_H and consider only the terms up to $O(T^2)$. This was done in Ref. 3 and the authors find the following cancellation between the thermal part of (60) and (59), to $O(T^2)$:

$$\Gamma^{\Delta m} + \Gamma^{\mathcal{B}_1} + \Gamma_{\beta}^{G} + \Gamma_{\beta}^{\gamma_1} + \Gamma_{\beta}^{\gamma_2} = 0 + O(T^4) ,$$

Thus we obtain, to $O(T^2)$, for the total finite-temperature Higgs-boson decay rate,

$$\Gamma_{\beta}^{\mathrm{total}} = \Gamma^{\mathrm{tree}} + \Gamma^{\mathcal{B}_2'} = \left[1 + \frac{e^2 T^2}{3m_H^2 w^2}\left[2 - \frac{1-w^2}{w}\ln\frac{1+w}{1-w} \right] \right]\Gamma^{\mathrm{tree}}(m_H) . \quad (61)$$

The finite-temperature correction is nonzero, contrary to the result of Ref. 3 who eliminated $\Gamma^{\mathcal{B}_2'}$ with the momentum-dependent counterterm δm^{DH}. This example shows clearly that, in addition to the consistency problems discussed before, finite-temperature counterterms do not even give the correct decay rate at the one-loop level. Finally, we note that the temperature correction in (61) is positive, that is, the "thermal mass" correction actually enhances the decay rate. In particular, in the massless limit $w \to 1$, Eq. (61) reduces to

$$\Gamma_{\beta}^{\mathrm{total}} \to \left[1 + \frac{2e^2 T^2}{3m_H^2} \right]\Gamma^{\mathrm{tree}} . \quad (62)$$

However, in this case the thermal fermion sector can no longer be neglected and there will be additional contributions to (62).

IV. GENERALIZATION AND APPLICATIONS

It is straightforward to generalize our results to a general thermal decay process with n-body phase space $\phi \to \phi_1 \cdots \phi_n$ and $n > 2$. An important physical example is neutron β-decay $n \to p e^- \bar{\nu}$ with radiative corrections. The fermion self-energy correction to the self-energy for such a process is shown in Fig. 6. As already mentioned, the technique for the cancellation of the pinch singularities can be applied to any such diagram; hence, the $\mathrm{Re}\Sigma$ contribution will be of the generic form

$$\Gamma^{\mathrm{SE}} \sim \int d \text{ phase space } \frac{\partial}{\partial m^2}\delta(p^2 - m^2)\mathrm{Tr}[\cdots(\not{p}+m)\mathrm{Re}\Sigma(\not{p}+m)]$$

$$\equiv \lim_{\hat{m}^2 \to m^2}\frac{\partial}{\partial \hat{m}^2}\int d \text{ phase space } \delta(p^2 - \hat{m}^2)\mathrm{Tr}[\cdots(\not{p}+m)\mathrm{Re}\Sigma(p)(\not{p}+m)] . \quad (63)$$

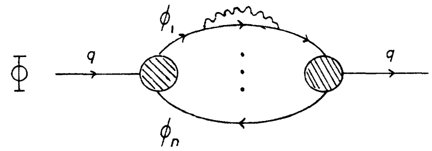

FIG. 6. Self-energy correction to $\Pi_{11}(q)$ for a general decay process $\phi \to \phi_1 \cdots \phi_n$.

The Lorentz-invariant zero-temperature part $\mathrm{Re}\Sigma^0$ can of course always be reduced to a δZ_2 factor by including the zero-temperature mass counterterm. For the finite-temperature part $\mathrm{Re}\Sigma^\beta$ we could use either the covariant expression (46), expanding around $p^2=m^2$, or a direct, noncovariant mass-shell expansion of the type (51). Evaluating the integral (63) we might call the contribution arising from the coefficient function proportional to $(\not{p}-m)$ a "wave-function renormalization" correction, the remainder a generalized "finite-temperature mass" contribution. Notice, however, that for a decay process with n-body phase space, $n \geq 3$, the kinematical constraints (mass-shell δ functions) are not sufficient to fix both energy and momentum for the external fermion; for example, in neutron β decay the electron is emitted with a continuous energy and momentum spectrum. Consequently the expansion coefficients of $\mathrm{Re}\Sigma^\beta$ in the integral will always be momentum dependent and do no longer admit an interpretation as mass and wave-function renormalization counterterms δm^β and δZ_2^β, not even in the heuristic sense discussed in the previous section. Thus it is also no longer possible to eliminate the "mass contribution" in the integral with an operational constant mass counterterm, and (63) is indeed the generic expression for the finite-temperature self-energy correction $\Gamma_\beta^{\mathrm{SE}}$ to an n-body decay process. Needless to say, this also confirms our previous conclusion that finite-temperature renormalization is not a meaningful concept for decay and/or scattering rate calculations.

If we are willing to give up Lorentz invariance of the Lagrangian we are of course free to fix some arbitrary four-momentum and define operational finite-temperature counterterms with respect to this renormalization point (as we did for two-body decay), but these constant counterterms will obviously not eliminate the momentum-dependent "mass" term in (63) and merely complicate the interpretation of the rate in terms of physical parameters. It is also a perfectly well-defined problem to analyze the quasiparticle propagators (17) for bosons and (21) for fermions and extract physical information such as correlation lengths, dispersion relations, etc. However, as our analysis showed, these quantities will have no direct relation to the parameters in our decay rates, which is of course a consequence of the lack of Lorentz invariance.

For a direct application of these results consider neutron β decay with radiative corrections at finite temperature. This reaction is important in cosmology since it is a central ingredient for the nucleosynthesis rates of the light elements in the early Universe. These rates, in turn, are measurable and provide an excellent probe of the conditions in the early Universe. Previous calculations[23,24] treated the self-energy corrections to the electron in standard fashion as temperature-dependent mass shift and wave-function renormalization correction. The mass shift was taken into account by replacing the mass-shell δ function for the electron by the dispersion relation $\det(\not{p}-m-\mathrm{Re}\Sigma^\beta)$—which corresponds to a momentum-dependent mass counterterm in the Lagrangian—and the wave-function renormalization function was defined by $\delta Z_2^\beta = (\partial/\partial\not{p})\mathrm{Re}\Sigma^\beta_{\text{"}\not{p}=m\text{"}}$. A rigorous treatment requires the computation of the phase-space integral (63) for β decay. Since neutron β decay is extremely phase-space sensitive it would be interesting to see how this rigorous result differs numerically from the one in Ref. 23 and if there are any corrections for the nucleosynthesis rates.[25] The previous calculations found the corrections to the abundances to be only 0.1–0.2%. However, recently a debate has arisen over the reliability of the standard model of primordial nucleosynthesis and several modifications and alternatives have been proposed; a rigorous result for the temperature corrections is thus clearly important.

Finally, let us emphasize that our results apply also to thermal scattering rates, that is, to transition rates for initial distributions of two or more particle species. These thermal cross sections are again related to the discontinuity (imaginary part) of the relevant n-point Green's functions[8] which in turn can be determined by the finite-temperature Cutkosky rules, and the self-energy corrections to extract fermions lines will again contribute a phase-space integral of the form (63). Since the kinematics of these scattering processes does not fix both energy and momentum, all conclusions for decay processes with n-body phase space apply as well. Potential applications for our results are cooling rates for neutron stars, which are in part determined by neutrino–gauge-boson scattering at high densities;[26] the techniques described here for fermion self-energies can of course be extended to gauge theories.

V. SUMMARY AND CONCLUSIONS

We have analyzed the problem of radiative corrections to finite-temperature decay rates to first order in perturbation theory, using the decay of a scalar boson into two fermions as an explicit example. Our treatment was based on the Niemi-Semenoff real-time formalism of finite-temperature field theory. The following results were obtained.

Ghost vertices are necessary even to first order in perturbation theory to ensure a well-defined theory (cancel pinch singularities), but do not contribute to the finite part of the rate.

For the radiative corrections we found the vertex-correction and the photon emission and/or absorption processes to be essentially identical to previous results. In particular, the vertex diagram renormalizes the coupling constant. However, the finite-temperature part of the fermion self-energy correction does not, in general, admit an interpretation in terms of local and renormalizable mass and wave-function renormalization counterterms, due to the lack of Lorentz invariance.

We argued that finite-temperature renormalization is inherently ambiguous and concluded that it is not a meaningful concept for decay and/or scattering rate calculations. Instead we derived an explicit algorithm for the direct computation of the finite-temperature self-energy corrections to a decay process and generalized it for processes of cosmological and astrophysical interest.

An important point which we treated only briefly, is the cancellation of the infrared divergences and mass singularities. To our knowledge a finite-temperature version of the Kinoshita-Lee-Nauenberg theorem, which guarantees the absence of these singularities at zero temperature, is still lacking. In the low-temperature limit, with only the thermal photon distribution taken into account, the cancellation of the thermal singularities was shown for the vertex correction, emission and/or absorption rates, and the "wave-function renormalization part" of the self-energy.[3,22] Here we extended these results and showed explicitly that the "finite-temperature mass" does not introduce any new infrared or mass singularities.

After this work was completed we became aware of two papers[27,28] on a similar problem (dilepton production rates in a QCD plasma). The authors employ the same technique, finite-temperature Cutkosky rules, used in this paper, but the emphasis is on the cancellation of infrared and mass singularities, and their analysis includes also the thermal fermion distributions. Both groups find complete cancellation of the thermal divergences, a result that supports the infrared reliability of finite-temperature perturbation theory. We note that the authors of Ref. 27 employ the finite-temperature mass counterterms of Ref. 3 whereas the authors of Ref. 28 show the cancellation of the singularities also directly without these counterterms. This supports our assertion that the thermal "mass" correction is well defined and momentum-dependent counterterms are not needed to deal with infrared singularities.

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APPENDIX

We write the noncovariant low-temperature expansion of $\mathrm{Re}\Sigma^{\beta}$ as

$$\mathrm{Re}\Sigma^{\beta}(p)=\mathcal{B}_1(p)+\mathcal{B}_2(\omega_p,\mathbf{p}) ,$$

where

$$\mathcal{B}_1(p)=\frac{e^2}{8\pi^3}\left[I_A(\not p-m)+\frac{1}{2\omega_p}\frac{\partial \boldsymbol{I}(\omega_p,\mathbf{p})}{\partial\omega_p}(p^2-m^2)\right] ,$$

$$\mathcal{B}_2(\omega_p,\mathbf{p})=\frac{e^2}{8\pi^3}\boldsymbol{I}(\omega_p,\mathbf{p}) ,$$

and

$$I_A=4\pi\int_{\epsilon}^{\infty}\frac{d|\mathbf{k}|}{|\mathbf{k}|}n_B(|\mathbf{k}|) ,$$

$$I^{\mu}(\omega_p,\mathbf{p})=\int\frac{d^3k}{|\mathbf{k}|}\frac{k^{\mu}}{\omega_p k^0-\mathbf{pk}}\frac{1}{e^{\beta|\mathbf{k}|}-1} ,$$

where $k^0=|\mathbf{k}|$. I^{μ} can be calculated explicitly (cf. Ref. 3).

For I^0 we obtain by direct integration

$$I^0(\omega_p,\mathbf{p})=\frac{\pi^3T^2}{3v\omega_p}\ln\frac{1+v}{1-v} ,$$

where $v=|\mathbf{p}|/\omega_p$.

For p on the mass shell, that is, $p^0=\omega_p$, we have

$$p\cdot I\equiv\omega_p I^0-\mathbf{pI}=\int\frac{d^3k}{|\mathbf{k}|}\frac{1}{e^{\beta|\mathbf{k}|}-1}$$
$$=\frac{2\pi^3T^2}{3}=\text{const}$$

and hence, for $\mathbf{I}$,

$$\mathbf{I}=\frac{\mathbf{p}}{|\mathbf{p}|^2}(\omega_p I^0-p\cdot I)$$
$$=\frac{\mathbf{p}}{|\mathbf{p}|^2}\frac{\pi^3T^2}{3}\left[\frac{1}{v}\ln\frac{1+v}{1-v}-2\right] .$$

We calculate the decay rates

$$\Gamma^{\mathcal{B}_i}(m_H)=-\frac{g^2}{m_H}(1-e^{-\beta m_H})\lim_{\hat m^2\to m^2}\frac{\partial}{\partial\hat m^2}\int\frac{d^4p}{(2\pi)^2}\delta(p^2-\hat m^2)\delta((p-q)^2-m^2)[\theta(p^0)-n_F(p)][\theta(q^0-p^0)-n_F(p-q)]$$
$$\times\mathrm{Tr}[(\not p-\not q+m)(\not p+m)\mathcal{B}_i(\not p+m)] .$$

Let us first consider $\Gamma^{\mathcal{B}_1}$. The trace is easily reduced to

$$\mathrm{Tr}[(\not p-\not q+m)(\not p+m)\mathcal{B}_1(\not p+m)]$$
$$=\frac{e^2}{8\pi^3}\left[\mathrm{Tr}[(\not p-\not q+m)(\not p+m)]\left[I_A+\frac{1}{\omega_p}p\cdot\frac{\partial I}{\partial\omega_p}\right](p^2-m^2)-\mathrm{Tr}\left[(\not p-\not q+m)\frac{\partial \boldsymbol{I}}{\partial\omega_p}\right]\frac{1}{2\omega_p}(p^2-m^2)^2\right] .$$

Taking the derivative and limit are now trivial and $\Gamma^{\mathcal{B}_1}$ reduces to

$$\Gamma^{\mathcal{B}_1}(m_H) = -\frac{e^2}{4\pi^3}\frac{g^2}{2m_H}(1-e^{-\beta m_H})\int \frac{d^4p}{(2\pi)^2}\delta(p^2-m^2)\delta((p-q)^2)[\theta(p^0)-n_F(p)][\theta(p^0-q^0)-n_F(p-q)]$$
$$\times \mathrm{Tr}[(\not p-\not q+m)(\not p+m)]\left[I_A+\frac{1}{\omega_p}p\cdot\frac{\partial I}{\partial \omega_p}\right]$$
$$=\frac{e^2}{4\pi^3}\left[I_A-\frac{1}{\omega_p}I^0\right]\Gamma^{\mathrm{tree}}(m_H)$$
$$=\frac{e^2}{4\pi^3}\left[I_A-\frac{4\pi^3T^2}{3wm_H^2}\ln\frac{1+w}{1-w}\right]\Gamma^{\mathrm{tree}}(m_H)\ ,$$

where we used

$$p\cdot\frac{\partial I}{\partial\omega_p}=-I^0 \quad\text{and}\quad v=\left[1-\frac{4m^2}{m_H^2}\right]^{1/2}\equiv w \quad \text{for } p^0=\omega_p=m_H/2\ .$$

$\Gamma^{\mathcal{B}_2}$ requires the explicit evaluation of the integral. For the mass-shell δ functions we have

$$\delta(p^2-\hat m^2)\delta((p-q)^2-m^2)=\frac{1}{4m_H\omega_p}\delta(p^0-p^0(\hat m^2))\delta(\omega_p-\omega_p(\hat m^2))\ ,$$

where

$$p^0(\hat m^2)=\frac{m_H}{2}+\frac{\hat m^2-m^2}{2m_H},\quad \omega_p(\hat m^2)=\frac{m_H}{2}-\frac{\hat m^2-m^2}{2m_H}$$

and we have to evaluate

$$\Gamma^{\mathcal{B}_2}(m_H)=-\frac{e^2}{4\pi^3}\frac{g^2}{8\pi m_H^2}(1-e^{-\beta m_H})\lim_{\hat m^2\to m^2}\frac{\partial}{\partial\hat m^2}\int dp^0 d\omega_p|\mathbf{p}|\delta(p^0-p_0(\hat m^2))\delta(\omega_p-\omega_p(\hat m^2))$$
$$\times[\theta(p^0)-n_F(p)][\theta(q^0-p^0)-n_F(p-q)]$$
$$\times\mathrm{Tr}[(\not p-\not q+m)(\not p+m)\not I(\not p+m)]\ .$$

It is easy to show that for $\hat m^2=m^2$, that is, for $p^0=q^0-p^0=m_H/2$,

$$\frac{\partial}{\partial\hat m^2}[\theta(p^0)-n_F(p^0)][\theta(q^0-p^0)-n_F(p^0-q^0)]_{\hat m^2=m^2}=0$$

and therefore

$$\Gamma^{\mathcal{B}_2}(m_H)=-\frac{e^2}{4\pi^3}\frac{g^2}{8\pi m_H^2}\tanh(\beta m_H/4)\lim_{\hat m^2\to m^2}\left[|\mathbf{p}|\frac{\partial}{\partial\hat m^2}\mathrm{Tr}(\cdots\not I)+\frac{\partial|\mathbf{p}|}{\partial\hat m^2}\mathrm{Tr}(\cdots\not I)\right]\ .$$

We have for $\hat m^2=m^2$, that is, for $\omega_p=m_H/2$,

$$|\mathbf{p}|=(\omega_p^2-m^2)^{1/2}=\frac{m_H}{2}w \quad\text{and}\quad \frac{\partial}{\partial\hat m^2}|\mathbf{p}|=-\frac{1}{2m_Hv}=-\frac{1}{2m_Hw}\ .$$

For the trace we find

$$\lim_{\hat m\to m^2}\mathrm{Tr}[(\not p-\not q+m)(\not p+m)\not I(\not p+m)]=4[m_HI^0(p^2-m^2)+p(\hat m^2)\cdot I(\hat m^2)(p^2+3m^2-2m_Hp^0)]_{\hat m^2=m^2}\equiv-4m_H^2w^2(p\cdot I),$$

where we recall that $p\cdot I=\frac{2}{3}\pi^3T^2=\text{const}$. For the derivative of the trace we obtain

$$\frac{\partial}{\partial\hat m^2}\mathrm{Tr}[(\not p-\not q+m)(\not q+m)\not I(\not p+m)]_{\hat m^2=m^2}=4\left[m_HI^0-m_H^2w^2\frac{\partial}{\partial\hat m^2}[p(\hat m^2)\cdot I(\hat m^2)]\right]_{\hat m^2=m^2}$$
$$=4m_H(1-w^2)I^0\ ,$$

where we used

$$p(\hat{m}^2)\cdot I(\hat{m}^2)=\left[\omega_p(\hat{m}^2)+\frac{\hat{m}^2-m^2}{m_H}\right]I^0(\hat{m}^2)-\mathbf{p}(\hat{m}^2)\mathbf{I}(\hat{m}^2)=p\cdot I+\frac{\hat{m}^2-m^2}{m_H}I^0(\hat{m}^2)\ .$$

Thus the rate $\Gamma^{\mathcal{B}_2}$ is given by

$$\Gamma^{\mathcal{B}_2}(m_H)=-\frac{e^2}{4\pi^3}\frac{g^2}{8\pi m_H^2}\tanh\frac{\beta m_H}{4}[2m_H^2 w(1-w^2)I^0+2m_H wp\cdot I]=-\frac{e^2T^2}{3m_H^2w^2}\left[1+\frac{1-w^2}{w}\ln\frac{1+w}{1-w}\right]\Gamma^{\mathrm{tree}}(m_H)\ ,$$

where we used the explicit form of I^0. Note that

$$\lim_{w\to 1}(1-w^2)\ln\frac{1+w}{1-w}=0\ .$$

To calculate the rate $\Gamma^{\mathcal{B}_2'}$ where

$$\mathcal{B}_2'=\mathcal{B}-\delta m^{\beta}=\frac{e^2}{8\pi^3}\left[I-\frac{p\cdot I}{m}\right]$$

it remains to compute

$$\frac{\partial}{\partial\hat{m}^2}\left[|\mathbf{p}|\mathrm{Tr}\left[(\not{p}-\not{q}+m)(\not{p}+m)\frac{p\cdot I}{m}(\not{p}+m)\right]\right]\Bigg|_{\hat{m}^2=m^2}=4p\cdot I\frac{\partial}{\partial\hat{m}^2}[|\mathbf{p}|(3p^2+m^2-2m_Hp^0)]_{\hat{m}^2=m^2}=6m_H wp\cdot I\ .$$

Together with our result for $\Gamma^{\mathcal{B}_2}$ this yields

$$\Gamma^{\mathcal{B}_2'}(m_H)=-\frac{e^2}{4\pi^3}\frac{g^2}{8\pi m_H^2}\tanh\frac{\beta m_H}{4}[2m_H^2w(1-w^2)I^0-4m_Hwp\cdot I]$$

$$=-\frac{e^2T^2}{3m_H^2w^2}\left[-2+\frac{1-w^2}{w}\ln\frac{1+w}{1-w}\right]\Gamma^{\mathrm{tree}}(m_H)\ ,$$

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