Effective field theory from a β function for the p-adic string

Hitoshi Nishino, Yasuhiro Okada,* and Marcelo R. Ubriaco
Department of Physics and Astronomy, University of North Carolina, Chapel Hill, North Carolina 27599-3255
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We describe the calculation of a β function for the p-adic string σ model in a tachyonic background, and show that the vanishing condition of the β function reproduces the tachyonic field equation obtained from p-adic amplitude calculations, up to the fifth order in the tachyonic field. By examining this σ model in the presence of a soliton background, we show that a previously obtained soliton solution of the all-order tachyonic field equation follows from the requirement of discrete scale invariance.

I. INTRODUCTION

It has been known in string theory that the requirement of the world-sheet conformal invariance reproduces the field equations of the background fields in which the string is propagating. Initially it was shown that the vanishing condition of β functions for conformal transformations results in the background-field equations.[1] Subsequently, the above statement was also checked by various methods,[2–6] for example, the nilpotency of the Becchi-Rouet-Stora-Tyutin (BRST) charge,[2] and the unitarity requirement of string propagation[3] for the tachyonic and massive modes,[3,4] and also at loop orders.[5]

Recently a different approach to string theory has been originated by considering the world-sheet variables to take values on a p-adic field.[7] The observation that the Veneziano amplitude can be written as an infinite product of p-adic amplitudes[8] suggested that number theory could play an important role in string theory. One of the interesting features of p-adic string theory is that N-point amplitudes can be explicitly evaluated and a space-time effective action for the tachyonic field was obtained to all orders in the string tension parameter α' (Ref. 9). Similarly to the ordinary string case the p-adic string can be formulated by using functional integration and the world-sheet action given by a field theory on a p-adic field.[10–12] In particular, in Ref. 11, the O(N) nonlinear σ model on a p-adic field was formulated and it was pointed out that a discrete scale invariance of the action plays a similar role to the scale invariance in the ordinary two-dimensional nonlinear σ model. Other aspects of the p-adic string and p-adic quantum mechanics have also been investigated.[13]

In a previous paper,[12] we have defined a β function associated with a discrete scale invariance of a p-adic string theory with a tachyonic background field. We have checked that the vanishing condition of the β function, i.e., the discrete scale invariance of the system at the quantum level, indeed reproduces the tachyonic field equation (up to the third order in the tachyonic field), which had been obtained from amplitude calculations (Ref. 9).

In this paper, we will go to higher orders in the tachyonic field. We first give a general form of our effective action at an arbitrary order, and next show that the vanishing β-function condition results in a tachyonic background-field equation, which had been previously obtained from the amplitude calculations. We perform essentially a perturbative calculation in the tachyonic field and the rigorous comparison between the vanishing β-function condition and the tachyonic field equation is accomplished up to fifth order. We also consider in the σ model a soliton background, which is an exact solution of the *all*-order tachyonic field equation, and we see that this solution follows from the requirement of the scaling property of a propagator in our σ model, and of the scale invariance of the world-sheet effective action. It is remarkable that we can get the exact nontrivial solution by a discrete scale-invariance requirement of our σ model, instead of solving the classical field equation directly.

This paper is organized as follows. In Sec. II we give the general structure of our system with relevant notational arrangements, and give a general formula for the Nth-order effective action with the vanishing β-function condition as a formal relationship at an arbitrary order. In Sec. III we apply these general formulas to the cases $N=1$, 2, 3, and 4, in order to show that the vanishing condition of our β function is in agreement with the tachyonic field equation up to fifth order obtained from amplitude calculations. In Sec. IV we put our σ model in a special soliton background, and consider the discrete scale invariance of the world-sheet effective action with the right scaling properties of our propagators, and show how a previously derived soliton solution can be understood from these conditions. We regard this as good evidence of the agreement between the vanishing β-function condition and the *full* tachyonic field equation. Section V is devoted to the conclusion of this paper. Some details of the calculation to obtain the fourth-order effective action are given in the Appendix.

II. GENERAL FORMALISM

In this section we give the basic and general structure of our model.

Our p-adic string is described by a "world-sheet" σ-model Lagrangian coupled to a tachyonic background field $\Phi(X(x))$, as we gave in our previous paper:[12]

$$S=-\frac{1}{2}\int_{Q_p}dx\,X^\mu(x)\Delta X_\mu(x)-\int_{Q_p}dx\,g\Phi(X(x))$$
$$=-\frac{1}{2}\frac{p(p-1)}{p+1}\frac{1}{\ln p}\int_{Q_p}dx\,dy\frac{X^\mu(x)X_\mu(y)}{|x-y|_p^2}$$
$$-\int_{Q_p}dx\,g\Phi(X(x))\ , \tag{2.1}$$

where $x,y,\ldots$ are elements of a p-adic field Q_p, while $\mu,\nu,\ldots$ are for the target space-time world indices, and g is a coupling constant. (For an introduction to the p-adic analysis, see Ref. 14.) The integral $\int_{Q_p}$ is defined over the p-adic field Q_p. The operator Δ is a p-adic analogue of the Laplacian operator, which is an integral operator.[10,11] From now on we omit the subscript Q_p for p-adic integrals, and also the lower indices p for p-adic norms, just for simplicity.

Notice the existence of scale invariance under

$$x\rightarrow px,\quad X^\mu(x)\rightarrow X^\mu(x)\ , \tag{2.2}$$

in the first term in Eq. (2.1), while the second term of (2.1) breaks this invariance by the canonical dimension of $\Phi(X)$, like the usual bosonic-string case.

We can adopt the background-field method for the $\Phi(X)$, starting with

$$X^\mu(x)=X_0^\mu(x)+\xi^\mu(x)\ , \tag{2.3}$$

where $X_0^\mu(x)$ is the background field and $\xi^\mu(x)$ is a quantum fluctuation. We can define our world-sheet effective action $W(\Phi)$ by

$$\exp\left[-\int dx\,W(\Phi)\right]=\int[\mathcal{D}\xi^\mu(x)]e^{-S}\ . \tag{2.4}$$

The $X_0^\mu(x)$ can be chosen to be x independent for the purpose of this paper. The ξ-dependent part of our action S is

$$S_\xi=-\frac{1}{2}\int dx\,\xi^\mu(x)\Delta\xi_\mu(x)$$
$$-\int dx\,g\sum_{n=1}^{\infty}\frac{1}{n!}\xi^{\mu_1}\cdots\xi^{\mu_n}\partial_{\mu_1}\cdots\partial_{\mu_n}\Phi(X_0^\mu)\ . \tag{2.5}$$

Our propagator for ξ is

$$\langle\xi^\mu(x)\xi^\nu(y)\rangle=g^{\mu\nu}G(x-y)\ , \tag{2.6}$$

where $G(x)$ is essentially proportional to $\ln|x|_p$ (Refs. 10 and 11). In actual calculation we need to regularize the ultraviolet (UV) and infrared (IR) divergences. For this purpose we use the regularization

$$G_K(x)=-\ln(m|x|_K)$$
$$\equiv\begin{cases}-\ln(m|x|) & \text{for } |x|\geq p^{-K}\ ,\\ -\ln(mp^{-K}) & \text{for } |x|<p^{-K}\ ,\end{cases} \tag{2.7}$$

for some large integer K and an IR cutoff m. Physically, the K is for the UV cutoff for very small distance of x's.

The effective action $W(\Phi)$ is now

$$\exp\left[-\int dx\,W(\Phi)\right]=\exp\left[\int dx\,g\Phi(X_0)\right]\int[\mathcal{D}\xi^\mu(x)]\exp\left[\frac{1}{2}\int dx\,\xi^\mu\Delta\xi_\mu\right]$$
$$\times\exp\left[g\int dx\sum_{n=1}^{\infty}\frac{1}{n!}\xi^{\mu_1}\cdots\xi^{\mu_n}\partial_{\mu_1}\cdots\partial_{\mu_n}\Phi(X_0)\right]$$
$$\equiv\exp\left[-\int dx\,(W^{(1)}+W^{(2)}+\cdots)\right]\ , \tag{2.8}$$

where $W^{(N)}$ denotes the Nth order in $\Phi(X_0)$, which is evaluated as

$$-\int dx\,W^{(N)}(\Phi)=\frac{1}{N!}g^N\left\langle\left[\int dx\sum_{n=1}^{\infty}\frac{1}{n!}\xi^{\mu_1}\cdots\xi^{\mu_n}\partial_{\mu_1}\cdots\partial_{\mu_n}\Phi(X_0)\right]^N\right\rangle\ . \tag{2.9}$$

The $W^{(1)}$ includes also $-g\Phi(X_0)$. In what follows we omit the suffix 0 for the background X_0^μ. Graphically, this corresponds to the evaluation of those Feynman diagrams in Fig. 1. Figure 1 contains all the connected graphs among external backgrounds $\Phi(X_1),\ldots,\Phi(X_N)$ which can be given as an integral over N p-adic variables $x_1,\ldots,x_N$. As an illustrative example for $W^{(3)}$, we evaluate the sum of a class of graphs in Fig. 2, with m, n, and p ξ propagators connecting three external backgrounds $\Phi(X_1)$, $\Phi(X_2)$, and $\Phi(X_3)$:

$$-\frac{1}{3!}g^3\int dx_1dx_2dx_3\sum_{m,n,p}{}'\frac{m!\binom{m+n}{m}n!\binom{n+p}{n}}{(m+n)!(n+p)!}\frac{p!\binom{p+m}{p}}{(p+m)!}$$
$$\times(\partial_1\cdot\partial_2)^m(\partial_2\cdot\partial_3)^n(\partial_3\cdot\partial_1)^pG_K^m(x_1-x_2)G_K^n(x_2-x_3)G_K^p(x_3-x_1)\Phi(X_1)\Phi(X_2)\Phi(X_3)$$
$$=-\frac{1}{6}g^3\int dx_1dx_2dx_3\sum_{m,n,p}{}'\frac{(\partial_1\cdot\partial_2)^m}{m!}\frac{(\partial_2\cdot\partial_3)^n}{n!}\frac{(\partial_3\cdot\partial_1)^p}{p!}G_K^m(x_1-x_2)G_K^n(x_2-x_3)G_K^p(x_3-x_1)\Phi(X_1)\Phi(X_2)\Phi(X_3)$$
$$=-\frac{1}{6}g^3m^{-\sum_{i<j}\partial_i\cdot\partial_j}\int dx_1dx_2dx_3(|x_{12}|_K^{-\partial_1\cdot\partial_2}|x_{23}|_K^{-\partial_2\cdot\partial_3}|x_{31}|_K^{-\partial_3\cdot\partial_1})\Phi(X_1)\Phi(X_2)\Phi(X_3)-(\text{n.c.g.})\ , \tag{2.10}$$

where (n.c.g.) stands for the contribution of nonconnected graphs and $\sum'_{m,n,p}$ denotes the summation over all nonnegative integers m, n, and p except for the cases when the three external backgrounds are *not* connected. The x_{ij} is for $x_i - x_j$. The factor $\partial_1 \cdot \partial_2$ arises from the contraction

$$\langle \xi^\mu(x_1)\xi^\nu(x_2)\rangle \partial_{1\mu}\partial_{2\nu} = G_K(x_1-x_2)\partial_1\cdot\partial_2 \equiv G_K(x_{12})\partial_1\cdot\partial_2 \tag{2.11}$$

for a pair of derivative operators $\xi_1^\mu \partial_\mu^1$ and $\xi_2^\mu \partial_\mu^2$. The factors such as $m!\binom{m+n}{m}/(m+n)!$ take care of all combinatorics of contractions of m, n, and p ξ propagators connecting external background fields. After the operations of ∂'s, we identify X's: $X_1 = X_2 = X_3 \equiv X$.

In general cases for $W^{(N)}$, we can repeat similar calculations to get the general formula

$$-\frac{1}{N!} g^N m^{-\sum_{i<j}\partial_i\cdot\partial_j} \int dx_1 \cdots dx_N \left[\prod_{i<j} |x_i - x_j|_K^{-\partial_i\cdot\partial_j} \right] \Phi(X_1)\cdots\Phi(X_N) - (\text{n.c.g.}) . \tag{2.12}$$

As in Fig. 3, the total $W^{(N)}$ contains also graphs of bubble-type graph insertions at each vertex $\Phi(X_1), \ldots, \Phi(X_N)$, which results in a multiplication of the ith vertex $\Phi(X_i)$ by a factor

$$\sum_{q=0}^{\infty} \frac{1}{q!}\left[\tfrac{1}{2}G_K(0)\Box_i\right]^q = \exp[-(\Box_i/2)\ln(mp^{-K})] = m^{-\Box_i/2} p^{K\Box_i/2} , \tag{2.13}$$

where the $q=0$ case corresponds to the case without any bubble insertion. Therefore the $W^{(N)}$ is

$$\int dx\, W^{(N)} = -\frac{1}{N!}\bar{g}^N m^{-\sum_{i=1}^N \Box_i/2} p^{K(\sum_{i=1}^N \Box_i/2 + N)} \times \int dx_1 \cdots dx_N \left[\prod_{i<j} (m|x_i - x_j|_K)^{-\partial_i\cdot\partial_j} \right] \Phi(X_1)\cdots\Phi(X_N) - (\text{n.c.g.}) , \tag{2.14}$$

where $\bar{g} \equiv p^{-K} g$ has a zero "p-adic dimension." In practical calculations, it is convenient to extract all the K dependence in the integral (2.14) by changing variables from x_i's to

$$y_i \equiv p^{-K} x_i . \tag{2.15}$$

Our integral (2.14) is

$$\int dx\, W^{(N)} = -\frac{1}{N!}\bar{g}^N m^{-\sum_{i=1}^N \Box_i/2} p^{K\sum_{i=1}^N \Box_i/2} \times \int dy_1 \cdots dy_N \prod_{i<j} (mp^{-K}|y_i - y_j|_0)^{-\partial_i\cdot\partial_j} \Phi(X_1)\cdots\Phi(X_N) - (\text{n.c.g.}) \quad (N = 2,3,\ldots) , \tag{2.16}$$

where $|y|_0$ is defined by

$$|y|_0 \equiv \begin{cases} |y| & \text{for } |y| \geq 1 , \\ 1 & \text{for } |y| < 1 . \end{cases} \tag{2.17}$$

In the case of $N=1$, it includes also the original action $-\int dx\, g\Phi(X)$ itself. By choosing y_N to be an overall integral variable, and changing the variables y_i to $z_i \equiv y_N - y_i$, we get

$$W^{(N)}(\Phi(X_N)) = -\frac{1}{N!}\bar{g}^N m^{-\Box/2} p^{K(\Box/2+1)} \times \int \left[\sum_{i=1}^{N-1} dz_i \right] \left[\left[\prod_{1\leq i<j\leq N-1} |z_i - z_j|_0^{-\partial_i\cdot\partial_j} \right] \left[\prod_{i=1}^{N-1} |z_i|_0^{-\partial_N\cdot\partial_i} \right] \right] \Phi(X_1)\cdots\Phi(X_N) . \tag{2.18}$$

The d'Alembertian $\Box$ without any indices denotes the "total momentum" d'Alembertian: $\Box \equiv (\partial_1 + \cdots + \partial_N)^2$. Here we have dropped the contribution of nonconnected graphs, because, as we will see later, it is always zero in our regularization scheme. [We have checked this explicitly up to $O(\Phi^5)$.] It is to be understood that the operator ∂_i acts on $\Phi(X_i)$ with respect to X_i, and afterwards we identify all the X's: $X_1 = X_2 = \cdots = X_N \equiv X$.

We can define the renormalization of the "coupling constant" $\Phi(X)$ by

$$W^{(1)} + W^{(2)} + \cdots + W^{(N)} + \cdots = -\bar{g}\Phi_R(X) . \tag{2.19}$$

Accordingly, all the Φ's before Eq. (2.19) are regarded as the "bare" coupling constant $\Phi_B(X)$. Our β function for this coupling constant $\Phi_B(X)$ associated with our discrete scale transformation (2.2) can be defined by a response of $\Phi_B(X)$ under the shift of our cutoff K:

$$\beta(\Phi_B) \equiv (\Phi_B^{(K+1)} - \Phi_B^{(K)})|_{\Phi_R \text{ fixed}} . \tag{2.20}$$

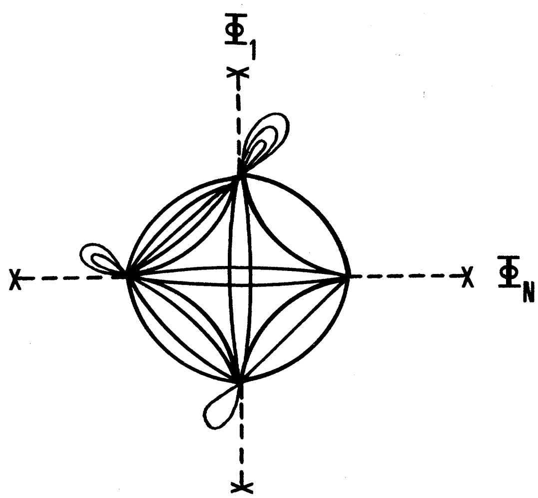

FIG. 1. A typical Feynman graph for a general effective action $W^{(N)}$. The solid lines are for ξ propagators, while the dashed lines are for N external background tachyonic fields.

The requirement of discrete scale invariance of our system at the quantum level corresponds to the vanishing β-function condition

$$\beta(\Phi_B)=(\Phi_B^{(K+1)}-\Phi_B^{(K)})|_{\Phi_R \text{ fixed}}=0 \ . \tag{2.21}$$

In practical calculations at higher orders in N, the explicit form of $\beta(\Phi)$ itself becomes complicated. However, we can use a simpler condition equivalent to (2.21). Notice that since Φ_R is fixed in the variation (2.21), we can rewrite the condition (2.21) as

$$\begin{aligned}0&=(\Phi_R^{(K+1)}-\Phi_R^{(K)})|_{\Phi_B \text{ fixed}}\\ &=+\tilde{g}^{-1}m^{-\Box/2}p^{(K+1)(\Box/2+1)}(p^{-(\Box/2+1)}-1)\\ &\quad\times(\tilde{W}^{(1)}+\tilde{W}^{(2)}+\cdots+\tilde{W}^{(N)}+\cdots)\ ,\end{aligned} \tag{2.22}$$

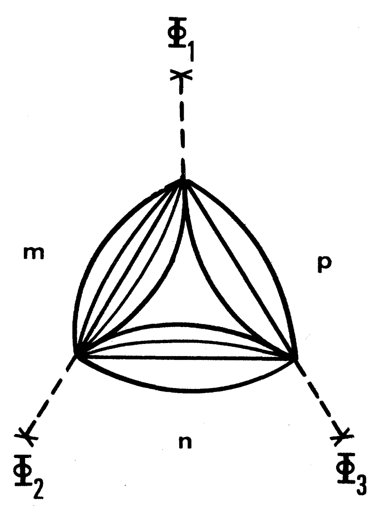

FIG. 2. A class of graphs for $W^{(3)}$ without bubble-type insertions to each vertex.

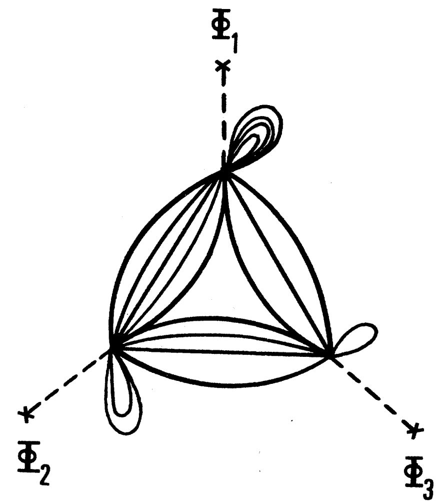

FIG. 3. A class of graphs for $W^{(3)}$ with bubble-type insertions to each vertex.

where

$$W^{(N)}(\Phi_B)\equiv m^{-\Box/2}p^{K(\Box/2+1)}\tilde{W}^{(N)}(\Phi_B)\ . \tag{2.23}$$

Our main purpose in this paper is to show that the condition (2.22) reproduces the field equation for the background tachyonic field ϕ (Ref. 9):

$$p^{-\Box/2}\left[1+\frac{\tilde{g}}{p}\phi\right]=\left[1+\frac{\tilde{g}}{p}\phi\right]^p \ . \tag{2.24}$$

In Ref. 9, the p-adic string N-point amplitudes were evaluated, and it was shown that these amplitudes can be reproduced by the field equation (2.24) of the effective theory. It was also noticed that besides the tachyonic vacuum $\phi=0$ there exists a shifted vacuum $1+\tilde{g}p^{-1}\phi=0$, where we have no particle excitation but a soliton solution. In order to show that our condition (2.22) reproduces the field equation (2.24), it is convenient to perform a perturbation expansion in $\tilde{g}$ of our tachyonic field ϕ:

$$\tilde{g}\phi\equiv\tilde{g}\phi^{(1)}+\tilde{g}^2\phi^{(2)}+\tilde{g}^3\phi^{(3)}+\tilde{g}^4\phi^{(4)}+\cdots\ . \tag{2.25}$$

Thus the field equation (2.24) is, up to $O(\tilde{g}^5)$,

$$(p^{-\Box/2-1}-1)\phi^{(1)}=0\ , \tag{2.26}$$

$$(p^{-\Box/2-1}-1)\phi^{(2)}=\tfrac{1}{2}(1-p^{-1})\phi^{(1)2}\ , \tag{2.27}$$

$$\begin{aligned}(p^{-\Box/2-1}-1)\phi^{(3)}&=(1-p^{-1})\phi^{(1)}\phi^{(2)}\\ &\quad+\tfrac{1}{6}(1-p^{-1})(1-2p^{-1})\phi^{(1)3}\ ,\end{aligned} \tag{2.28}$$

$$\begin{aligned}&(p^{-\Box/2-1}-1)\phi^{(4)}\\ &\quad=\tfrac{1}{2}(1-p^{-1})\phi^{(2)2}+\tfrac{1}{2}(1-p^{-1})(1-2p^{-1})\phi^{(1)2}\phi^{(2)}\\ &\quad\quad+\tfrac{1}{24}(1-p^{-1})(1-2p^{-1})(1-3p^{-1})\phi^{(1)4}\\ &\quad\quad+(1-p^{-1})\phi^{(1)}\phi^{(3)}\ .\end{aligned} \tag{2.29}$$

In Sec. III, we will examine if these equations are implied by our vanishing β-function condition (2.22).

III. VANISHING β FUNCTION UP TO $O(\Phi^5)$

In this section, we give explicit forms for $W^{(1)}, \ldots, W^{(4)}$, using our general formulas (2.14)–(2.18). We next derive the vanishing β-function condition (2.22) up to $O(\Phi^5)$ terms. In Ref. 12, we have given $W^{(1)}$ and $W^{(2)}$, ignoring $O(\Phi^3)$ terms which should be kept in this paper.

A. $N=1$ and $N=2$ cases

We first derive $W^{(1)}$ and $W^{(2)}$, and next derive the vanishing β-function conditions up to $O(\Phi^3)$. The $W^{(1)}$ is given by

$$W^{(1)}=-\tilde{g}m^{-\Box/2}p^{K(\Box/2+1)}\Phi_B \ , \tag{3.1}$$

while the $N=2$ case is given by (2.14) as

$$\int dx\, W^{(2)}=-\tfrac{1}{2}\tilde{g}^2 m^{-(\Box_1+\Box_2)/2}p^{K(\Box_1+\Box_2)/2+2K} \times\int dx_1 dx_2[(m|x_1-x_2|_K)^{-\partial_1\cdot\partial_2}-1] \times\Phi_B(X_1)\Phi_B(X_2) \ . \tag{3.2}$$

The subtraction of 1 is to exclude the nonconnected graph contribution. As we did in (2.14)–(2.18), after changing variables, we have

$$W^{(2)}=-\tfrac{1}{2}\tilde{g}^2 m^{-(\Box_1+\Box_2)/2}p^{K(\Box_1+\Box_2)/2+K} \times\int dx[(mp^{-K}|x|_0)^{-\partial_1\cdot\partial_2}-1]\Phi_B(X_1)\Phi_B(X_2) \ . \tag{3.3}$$

The integral here is easy to evaluate:

$$\begin{aligned} I^{(2)}(\alpha)&\equiv\int dx(mp^{-K}|x|_0)^{\alpha} \\ &=m^{\alpha}p^{-K\alpha}(1-p^{-1})\left[\sum_{k=-\infty}^{-1}p^k+\sum_{k=0}^{\infty}p^{(\alpha+1)k}\right] \\ &=m^{\alpha}p^{-K\alpha}\left[p^{-1}+\frac{1-p^{-1}}{1-p^{\alpha+1}}\right] \ . \end{aligned} \tag{3.4}$$

Equation (3.2) is now proportional to $I^{(2)}(-\partial_1\cdot\partial_2)-I^{(2)}(0)$, where the $I^{(2)}(0)$ term is for the subtraction of nonconnected graph contributions. Actually we easily see that this $I^{(2)}(0)$ is just zero, and

$$W^{(2)}=-\tfrac{1}{2}\tilde{g}^2 m^{-\Box/2}p^{K(\Box/2+1)}\left[\frac{1-p^{-1}}{1-p^{1-\partial_1\cdot\partial_2}}+p^{-1}\right]\Phi_B^2 \ , \tag{3.5}$$

where, in general, $\Phi_B^N\equiv\Phi_B(X_1)\cdots\Phi_B(X_N)$. The vanishing β-function condition up to $O(\Phi^3)$ is now, from (2.22),

$$0=(p^{-\Box/2-1}-1)\left[-\tilde{g}\Phi_B -\tfrac{1}{2}\tilde{g}^2\left[\frac{1-p^{-1}}{1-p^{1-\partial_1\cdot\partial_2}}+p^{-1}\right]\Phi_B^2\right] \ . \tag{3.6}$$

It is advantageous to perform the field redefinition

$$\tilde{g}\Phi_B'\equiv\tilde{g}\Phi_B+\tfrac{1}{2}\tilde{g}^2p^{-1}\Phi_B^2 \ . \tag{3.7}$$

This is because, as we see later at $O(\Phi^3)$ and $O(\Phi^4)$, it simplifies considerable number of terms in the effective action, and it has a general form to all orders, as we clarify in the Appendix. After this field redefinition, our condition (3.6) is

$$0=(p^{-\Box/2-1}-1)\left[\tilde{g}\Phi_B'+\frac{1}{2}\tilde{g}^2\frac{1-p^{-1}}{1-p^{1-\partial_1\cdot\partial_2}}\Phi_B'^2\right] \ . \tag{3.8}$$

We are now ready to check if our condition (3.8) implies the expected tachyonic field Eqs. (2.26) and (2.27) at $O(\Phi)$ and $O(\Phi^2)$. For a systematic calculation, we expand Φ_B' in terms of $\tilde{g}$,

$$\tilde{g}\Phi_B'=\tilde{g}\Phi^{(1)}+\tilde{g}^2\Phi^{(2)}+\tilde{g}^3\Phi^{(3)}+\tilde{g}^4\Phi^{(4)}+\cdots \ , \tag{3.9}$$

and insert this into (3.8) in order to compare it with Eqs. (2.26) and (2.27). We thus get

$$\begin{aligned} 0&=\tilde{g}(p^{-\Box/2-1}-1)\Phi^{(1)}+\tilde{g}^2\left[(p^{-\Box/2-1}-1)\Phi^{(2)}-\frac{1}{2}\frac{(1-p^{-1})(1-p^{-\Box/2-1})}{1-p^{1-\partial_1\cdot\partial_2}}\Phi_1^{(1)}\Phi_2^{(1)}\right] \\ &=\tilde{g}(p^{-\Box/2-1}-1)\Phi^{(1)}+\tilde{g}^2[(p^{-\Box/2-1}-1)\Phi^{(2)}-\tfrac{1}{2}(1-p^{-1})\Phi^{(1)2}] \\ &\quad+\tfrac{1}{2}\tilde{g}^2\frac{p^{-\Box/2-1}}{1-p^{1-\partial_1\cdot\partial_2}}[(p^{\Box_1/2-1}-1)\Phi_1^{(1)}(p^{\Box_2/2-1}-1)\Phi_2^{(1)}+\Phi_1^{(1)}(p^{\Box_2/2-1}-1)\Phi_2^{(1)}+\Phi_2^{(1)}(p^{\Box_1/2-1}-1)\Phi_1^{(1)}] \ , \end{aligned} \tag{3.10}$$

where $\Phi_i^{(N)}\equiv\Phi^{(N)}(X_i)$, and as usual we identify $X_1=X_2\equiv X$, *after* the action of ∂_i on $\Phi_i^{(N)}$. In (3.10) we used relations such as

$$\begin{aligned} (1-p^{-\Box/2-1})\Phi_1^{(1)}\Phi_2^{(1)}&=(1-p^{1-\partial_1\cdot\partial_2}+p^{1-\partial_1\cdot\partial_2}-p^{-\Box/2-1})\Phi_1^{(1)}\Phi_2^{(1)} \\ &=[1-p^{1-\partial_1\cdot\partial_2}+(p^{\Box_1/2+1}p^{\Box_2/2+1}-1)p^{-\Box/2-1}]\Phi_1^{(1)}\Phi_2^{(1)} \ . \end{aligned} \tag{3.11}$$

The $O(\Phi^N)$ corresponds to $O(\tilde{g}^N)$ in the above perturbative calculation. At $O(\Phi)$, our condition (3.10) implies

$$(p^{-\Box/2-1}-1)\Phi^{(1)}=0 \ , \tag{3.12}$$

which is nothing else than (2.26) with the identification $\phi^{(1)}=\Phi^{(1)}$. The $O(\Phi^2)$ terms in (3.10) imply

$$(p^{-\Box/2-1}-1)\Phi^{(2)}=\tfrac{1}{2}(1-p^{-1})\Phi^{(1)2}\ , \tag{3.13}$$

which means that we can identify $\phi^{(2)}=\Phi^{(2)}$, i.e., up to $O(\Phi^3)$, we get the expected tachyonic field Eqs. (2.26) and (2.27) with $\phi=\Phi'_B$. Notice that all terms with poles, e.g., $1/(1-p^{1-\partial_1\cdot\partial_2})$ at $O(\Phi^2)$ in (3.10) disappear, if we use the $O(\Phi)$ Eq. (3.12), and only polynomial terms corresponding to our expected field equation at $O(\Phi^2)$ remain. We will perform a similar calculation when we check the vanishing β-function condition at $O(\Phi^3)$ in Sec. III C.

B. Effective action at $O(\Phi^3)$

In this subsection we describe the derivation of $W^{(3)}$. According to the general formula (2.18) for $N=3$, the integral we have to evaluate is

$$I^{(3)}\equiv\int dx\, dy|x|_0^{\alpha}\,|y|_0^{\beta}\,|x-y|_0^{\gamma}\ , \tag{3.14}$$

where $\alpha\equiv-\partial_1\cdot\partial_2$, $\beta\equiv-\partial_2\cdot\partial_3$, $\gamma\equiv-\partial_3\cdot\partial_1$. The integration region will be divided into the following subregions:

$$\begin{aligned}
&\text{(i)}\ \ |x|\ge1,\ \ |y|\ge1,\ \ |x-y|\ge1,\ \ \text{(a)}\ \ |x|>|y|\ge1,\ \ \text{(b)}\ \ |y|>|x|\ge1,\ \ \text{(c)}\ \ |x|=|y|\ge1,\ \ |x-y|\ge1\ ;\\
&\text{(ii)}\ \ |x|\ge1,\ \ |y|<1,\ \ |x-y|\ge1\ \ (\text{or } x\leftrightarrow y \text{ interchanged})\ ;\\
&\text{(iii)}\ \ |x|=|y|\ge1,\ \ |x-y|<1\ ;\\
&\text{(iv)}\ \ |x|<1,\ \ |y|<1,\ \ |x-y|<1\ .
\end{aligned} \tag{3.15}$$

For each of these subregions, the integrals can be carried out as follows.

Subregion (ia):

$$\begin{aligned}
I^{(3)}(\text{ia})&=\int_{|x|>|y|\ge1}dx\,dy|x|_0^{\alpha+\gamma}\,|y|_0^{\beta}\,|x-y|_0^{\gamma}=\int_{|x|>|y|\ge1}dx\,dy|x|^{\alpha+\gamma}\,|y|^{\beta}\\
&=\frac{1-p^{-1}}{1-p^{\alpha+\beta+\gamma+2}}\,\frac{(1-p^{-1})p^{\alpha+\gamma+1}}{1-p^{\alpha+\gamma+1}}\ .
\end{aligned} \tag{3.16}$$

Subregion (ib):

$$I^{(3)}(\text{ib})=I^{(3)}(\text{ia})\ |_{\text{with } \alpha \text{ and } \beta \text{ interchanged}}\ . \tag{3.17}$$

Subregion (ic):

$$\begin{aligned}
I^{(3)}(\text{ic})&=\int_{|x|=|y|\ge1,\ |x-y|\ge1}dx\,dy|x|_0^{\alpha}\,|y|_0^{\beta}\,|x-y|_0^{\gamma}\\
&=(1-p^{-1})\sum_{k=0}^{\infty}p^k\int_{|y|=p^k=|x|\ge|x-y|\ge1}dy\,p^{k(\alpha+\beta)}|x-y|_0^{\gamma}\\
&=(1-p^{-1})\sum_{k=0}^{\infty}p^{(\alpha+\beta+\gamma+2)k}\int_{\substack{|\zeta|=1\\ 1\ge|1-\zeta|\ge p^{-k}}}d\zeta|1-\zeta|^{\gamma}\ ,
\end{aligned} \tag{3.18}$$

where $y=x\zeta$, $|x-y|=p^k|1-\zeta|\ge1$. Hence

$$I^{(3)}(\text{ic})=(1-p^{-1})\sum_{k=0}^{\infty}p^{(\alpha+\beta+\gamma+2)k}\left[\int_{\substack{|\zeta|=1\\ |1-\zeta|=1}}d\zeta|1-\zeta|^{\gamma}+\int_{\substack{|\zeta|=1\\ 1>|1-\zeta|\ge p^{-k}}}d\zeta|1-\zeta|^{\gamma}\right]\ . \tag{3.19}$$

The first integration is performed by putting $\zeta\equiv a+py$ ($|y|\le1, a=2,\ldots,p-1$):

$$\int_{\substack{|\zeta|=1\\ |1-\zeta|=1}}d\zeta|1-\zeta|^{\gamma}=\sum_{a=2}^{p-1}\int_{\substack{\zeta=a+py\\ |y|\le1}}d\zeta=p^{-1}(p-2)\ , \tag{3.20}$$

while the second integral is, by $\zeta\equiv1+p\eta$, $p^{-k+1}\le|\eta|\le1$,

$$\begin{aligned}
\int_{\substack{|\zeta|=1\\ 1>|1-\zeta|\ge p^{-k}}}d\zeta|1-\zeta|^{\gamma}&=\int_{p^{-k+1}\le|\eta|\le1}d\eta\,p^{-1-\gamma}|\eta|^{\gamma}\\
&=(1-p^{-1})p^{-\gamma-1}\sum_{l=-k+1}^{0}p^{(\gamma+1)l}=(1-p^{-1})\frac{p^{-\gamma-1}}{1-p^{-\gamma-1}}(1-p^{-(\gamma+1)k})\ .
\end{aligned} \tag{3.21}$$

Therefore (3.18) is

$$I^{(3)}(\text{ic})=(1-2p^{-1})\frac{1-p^{-1}}{1-p^{\alpha+\beta+\gamma+2}}+\frac{1-p^{-1}}{1-p^{\alpha+\beta+\gamma+2}}\frac{(1-p^{-1})p^{\alpha+\beta+1}}{1-p^{\alpha+\beta+1}} . \tag{3.22}$$

Subregion (ii):

$$I^{(3)}(\text{ii})=\int_{|x|\geq 1,\ |y|<1} dx\, dy |x|^{\alpha+\gamma}=(1-p^{-1})\sum_{k=0}^{\infty} p^{k+(\alpha+\gamma)k-1}$$
$$=p^{-1}\frac{1-p^{-1}}{1-p^{\alpha+\gamma+1}} . \tag{3.23}$$

Subregion (iii):

$$I^{(3)}(\text{iii})=\int_{|x|=|y|\geq 1,\ |x-y|<1} dx\, dy |x|^{\alpha}|y|^{\beta} . \tag{3.24}$$

Notice that $|x-y|<1$ implies that $|x|=|y|$. Therefore,

$$I^{(3)}(\text{iii})=(1-p^{-1})\sum_{k=0}^{\infty} p^{(\alpha+\beta+1)k}\int_{|\eta|<1} d\eta , \tag{3.25}$$

where $y=x(1+p^k\eta)$ and $|x|=p^k$. Hence,

$$I^{(3)}(\text{iii})=(1-p^{-1})\sum_{k=0}^{\infty} p^{(\alpha+\beta+1)k-1}=p^{-1}\frac{1-p^{-1}}{1-p^{\alpha+\beta+1}} . \tag{3.26}$$

Subregion (iv):

$$I^{(3)}(\text{iv})=\int_{|x|<1,\ |y|<1} dx\, dy =p^{-2} . \tag{3.27}$$

It is easy to see that the contribution of nonconnected graphs to $W^{(3)}$ is just zero, by the observation that $I^{(3)}(\alpha,\beta,\gamma)=0$, if at least two of α,β,γ are zero. Thus adding up the subregions (i), . . . , (iv), we obtain

$$\tilde{W}^{(3)}=-\tfrac{1}{6}\tilde{g}^3\left[\frac{1-p^{-1}}{1-p^{2-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3-\partial_3\cdot\partial_1}}\left[\frac{(1-p^{-1})p^{1-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3}}{1-p^{1-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3}}+(2\text{ perms})+1-2p^{-1}\right]\right.$$
$$\left.+p^{-1}\left[\frac{1-p^{-1}}{1-p^{1-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3}}+(2\text{ perms})\right]+p^{-2}\right]\Phi_B^3 , \tag{3.28}$$

where we have rearranged some terms, and (2 perms) denotes two other terms obtained by the permutations with respect to ∂_1, ∂_2, and ∂_3 in the preceding term. The Φ_B^N always means $\Phi_B(X_1)\cdots\Phi_B(X_N)$. Since Φ_B^3 is symmetric in the three tachyonic fields, instead of adding (2 perms) we can multiply the preceding term by a factor of 3.

C. Vanishing β-function condition at $O(\Phi^3)$

As we did for $\tilde{W}^{(2)}$ in Eq. (3.7), we can perform a field redefinition of Φ_B, in order to absorb some terms in Eq. (3.28). In fact, we can arrange terms in $\tilde{W}^{(1)}(\Phi_B)+\cdots+\tilde{W}^{(3)}(\Phi_B)$ as

$$\tilde{W}^{(1)}(\Phi_B)+\tilde{W}^{(2)}(\Phi_B)+\tilde{W}^{(3)}(\Phi_B)=-\tilde{g}\Phi_B-\tfrac{1}{2}\tilde{g}^2\left[\frac{1-p^{-1}}{1-p^{1-\partial_1\cdot\partial_2}}+p^{-1}\right]\Phi_B^2$$
$$-\frac{1}{3!}\tilde{g}^3\left[\frac{1-p^{-1}}{1-p^{2-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3-\partial_3\cdot\partial_1}}\left[3\frac{(1-p^{-1})p^{1-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3}}{1-p^{1-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3}}+(1-2p^{-1})\right]\Phi_B^3\right.$$
$$\left.+p^{-1}\left[3\frac{1-p^{-1}}{1-p^{1-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3}}\right]\Phi_B^3+p^{-2}\Phi_B^3\right]$$
$$=-\tilde{g}\left[\Phi_B+\tfrac{1}{2}\tilde{g}p^{-1}\Phi_B^2+\frac{1}{3!}\tilde{g}^3p^{-2}\Phi_B^3\right]-\tfrac{1}{2}\tilde{g}^2\left[\frac{1-p^{-1}}{1-p^{1-\partial_1\cdot\partial_2}}\right](\Phi_B+\tfrac{1}{2}\tilde{g}p^{-1}\Phi_B^2)^2$$
$$-\frac{1}{3!}\tilde{g}^3\frac{1-p^{-1}}{1-p^{2-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3-\partial_3\cdot\partial_1}}\left[3\frac{(1-p^{-1})p^{1-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3}}{1-p^{1-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3}}+1-2p^{-1}\right]\Phi_B^3$$
$$\equiv J^{(1)}\cdot(\Phi'_B)+J^{(2)}\cdot(\Phi'^2_B)+J^{(3)}\cdot(\Phi'^3_B) , \tag{3.29}$$

neglecting $O(\Phi^4)$ terms. Here our field redefinition is

$$\tilde{g}\Phi'_B \equiv \tilde{g}\Phi_B + \tfrac{1}{2}\tilde{g}^2 p^{-1}\Phi_B^2 + \frac{1}{3!}\tilde{g}^3 p^{-2}\Phi_B^3 \tag{3.30}$$

and the operators $J^{(i)}$ are defined by

$$J^{(1)}\cdot(\Phi'_B) \equiv -\tilde{g}\Phi'_B, \quad J^{(2)}\cdot(\Phi'^2_B) \equiv -\tfrac{1}{2}\tilde{g}^2 \frac{1-p^{-1}}{1-p^{1-\partial_1\cdot\partial_2}}\Phi'^2_B ,$$
$$J^{(3)}\cdot(\Phi'^3_B) \equiv -\tfrac{1}{6}\tilde{g}^3 \left[\frac{1-p^{-1}}{1-p^{2-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3-\partial_3\cdot\partial_1}} \left[3\frac{(1-p^{-1})p^{1-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3}}{1-p^{1-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3}} + 1 - 2p^{-1} \right] \right] \Phi'^3_B . \tag{3.31}$$

It is worthwhile to note that all the terms in $\tilde{W}^{(3)}$ which are absorbed into the field redefinitions come from the integral subregions (ii), (iii), and (iv) where at least one of $|x_i|$ or $|x_i - x_j|$ is less than unity.

In terms of J's, our vanishing β-function condition (2.22), ignoring $O(\Phi^4)$ terms, is

$$0 = (p^{-\Box/2-1}-1)[J^{(1)}\cdot(\Phi'_B) + J^{(2)}\cdot(\Phi'^2_B) + J^{(3)}\cdot(\Phi'^3_B)]$$
$$= (p^{-\Box/2-1}-1)\Bigg[-\tilde{g}\Phi'_B - \tfrac{1}{2}\tilde{g}^2 \frac{1-p^{-1}}{1-p^{1-\partial_1\cdot\partial_2}}\Phi'^2_B$$
$$-\tfrac{1}{6}\tilde{g}^3 \frac{1-p^{-1}}{1-p^{2-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3-\partial_3\cdot\partial_1}} \left[3\frac{(1-p^{-1})p^{1-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3}}{1-p^{1-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3}} + 1 - 2p^{-1} \right] \Phi'^3_B \Bigg] . \tag{3.32}$$

We now use the expansion (3.9), and arrange terms in (3.32) in a similar way as we did in Eq. (3.10). The $O(\Phi)$ and $O(\Phi^2)$ terms are already checked in Sec. III A, so that we examine only purely $O(\Phi^3)$ terms here. To avoid complication, we use at least the $O(\Phi)$ field equation (3.12). Now our condition (3.32) implies, at $O(\Phi^3)$,

$$0 = (p^{-\Box/2-1}-1)\Bigg[\tilde{g}^3\Phi^{(3)} + \tilde{g}^3 \frac{1-p^{-1}}{1-p^{1-\partial_1\cdot\partial_2}}\Phi_1^{(1)}\Phi_2^{(2)}$$
$$+\tfrac{1}{6}\tilde{g}^3 \frac{1-p^{-1}}{1-p^{2-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3-\partial_3\cdot\partial_1}} \left[3\frac{(1-p^{-1})p^{1-\partial_2\cdot(\partial_3+\partial_1)}}{1-p^{1-\partial_2\cdot(\partial_3+\partial_1)}} + 1 - 2p^{-1} \right] \Phi^{(1)3} \Bigg] . \tag{3.33}$$

For the second term in (3.33), using the $O(\Phi)$ field equation (2.12), we rewrite

$$(1-p^{-1})\frac{p^{-\Box/2-1}-1}{1-p^{1-\partial_1\cdot\partial_2}}\Phi_1^{(1)}\Phi_2^{(2)} = (1-p^{-1})\frac{p^{-\partial_1\cdot\partial_2-\Box_2/2}-1}{1-p^{1-\partial_1\cdot\partial_2}}\Phi_1^{(1)}\Phi_2^{(2)}$$
$$= (1-p^{-1})\left[\frac{p^{1-\partial_1\cdot\partial_2}(p^{-\Box_2/2-1}-1)}{1-p^{1-\partial_1\cdot\partial_2}} - 1 \right] \Phi_1^{(1)}\Phi_2^{(2)}$$
$$= -(1-p^{-1})\Phi_1^{(1)}\Phi_2^{(2)} + (1-p^{-1})\frac{p^{1-\partial_1\cdot\partial_2}}{1-p^{1-\partial_1\cdot\partial_2}}\Phi_1^{(1)}(p^{-\Box_2/2-1}-1)\Phi_2^{(2)} . \tag{3.34}$$

Similarly, for the third term in (3.33), we have

$$\tfrac{1}{6}(1-p^{-1})\frac{p^{-\Box/2-1}-1}{1-p^{2-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3-\partial_3\cdot\partial_1}} \left[3\frac{(1-p^{-1})p^{1-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3}}{1-p^{1-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3}} + 1 - 2p^{-1} \right] \Phi_B^{(1)3}$$
$$= -\tfrac{1}{6}(1-p^{-1})(1-2p^{-1})\Phi^{(1)3} - \tfrac{1}{2}(1-p^{-1})^2 \frac{p^{1-\partial_1\cdot\partial_2}}{1-p^{1-\partial_1\cdot\partial_2}}\Phi_1^{(1)}\Phi_2^{(1)2} . \tag{3.35}$$

Therefore our condition (3.33) is

$$0 = \tilde{g}^3[(p^{-\Box/2-1}-1)\Phi^{(3)} - (1-p^{-1})\Phi^{(1)}\Phi^{(2)} - \tfrac{1}{6}(1-p^{-1})(1-2p^{-1})\Phi^{(1)3}]$$
$$+\tilde{g}^3\frac{(1-p^{-1})p^{1-\partial_1\cdot\partial_2}}{1-p^{1-\partial_1\cdot\partial_2}}\Phi_1^{(1)}[(p^{-\Box_2/2-1}-1)\Phi_2^{(2)} - \tfrac{1}{2}(1-p^{-1})\Phi_2^{(1)2}] . \tag{3.36}$$

At this stage, if we also use the $O(\Phi^2)$ field Eq. (3.13), all the pole terms disappear. Thus our condition (3.33) implies

$$(p^{-\Box/2-1}-1)\Phi^{(3)} = (1-p^{-1})\Phi^{(1)}\Phi^{(2)} + \tfrac{1}{6}(1-p^{-1})(1-2p^{-1})\Phi^{(1)3} , \tag{3.37}$$

which is nothing else than our expected field Eq. (2.28) with the identification $\phi^{(i)} = \Phi^{(i)}$ $(i=1,2,3)$.

D. Effective action at $O(\Phi^4)$ and vanishing β function

The effective action at $N=4$ is also obtained, but the calculation is rather tedious, so that we quote here only the result of the vanishing β-function condition at $O(\Phi^4)$ (some details of the derivation are given in the Appendix):

$$\begin{aligned}0=\tilde{g}^4[&(p^{-\Box/2-1}-1)\Phi^{(4)}-\tfrac{1}{2}(1-p^{-1})\Phi^{(2)2}-\tfrac{1}{2}(1-p^{-1})(1-2p^{-1})\Phi^{(1)2}\Phi^{(2)}\\&-\tfrac{1}{24}(1-p^{-1})(1-2p^{-1})(1-3p^{-1})\Phi^{(1)4}-(1-p^{-1})\Phi^{(1)}\Phi^{(3)}]\\&+[\text{terms vanishing after the use of Eqs. (3.12), (3.13), and (3.37)}]\ ,\end{aligned} \tag{3.38}$$

which is nothing else than our expected Eq. (2.29) with the identification

$$\phi^{(i)}=\Phi^{(i)}\quad (i=1,2,3,4)\ . \tag{3.39}$$

To conclude this section, we have checked that our vanishing β-function condition (2.22) implies the tachyonic field Eq. (2.24) obtained by amplitude calculations after the identification $\tilde{g}\phi=\tilde{g}\Phi'_B=\tilde{g}\Phi^{(1)}+\cdots+\tilde{g}^4\Phi^{(4)}+\cdots$. Even though this statement is explicitly confirmed up to $O(\Phi^5)$, i.e., up to $O(\tilde{g}^5)$ in the $\tilde{g}$-perturbative expansion, we expect that such agreement between the vanishing β function and the tachyonic field Eq. (2.24) must persist to all orders; some further support for this conjecture will be given in Sec. IV.

IV. σ MODEL IN SOLITON BACKGROUND

In this section we study our σ model (2.1) in the presence of a soliton background which is an exact solution of the tachyonic field Eq. (2.24) to all orders.

In Secs. II and III we have examined the equivalence of the discrete scale invariance, i.e., the vanishing β-function condition to the background-field Eq. (2.24), up to the $O(\Phi^5)$ terms by means of perturbative expansions, and we are convinced that this equivalence is valid to all orders. On the other hand, we know that the all-order tachyonic field equation is satisfied by a soliton solution, which was obtained by Brekke, Freund, Olson, and Witten in Ref. 9:

$$\tilde{g}p^{-1}\phi+1=p^{D/[2(p-1)]}\exp\left[-\frac{c}{2}X_\mu X^\mu\right]\quad\left[c\equiv\frac{1-p^{-1}}{\ln p}\right]\ . \tag{4.1}$$

Here we are using D-dimensional Euclidean space coordinates X^μ ($\mu=1,\ldots,D$). However, in a Minkowskian space-time, we have a corresponding solution with D replaced by $D-1$. Combining this fact with the above result, it seems interesting to examine the discrete scale invariance of our σ model in the presence of the soliton background solution, instead of the perturbative approach we have taken in Secs. II and III.

In order to see how the above solution (4.1) follows from the requirement of the σ-model scale invariance, we put our σ model (2.1) in a background

$$\tilde{g}\Phi=-\tfrac{1}{2}aX_\mu X^\mu+b\ , \tag{4.2}$$

where a and b are some constants. We specify the background Φ in this form, from the observation that an exact solution (4.1) has the Gaussian dependence on X^μ. Recall also the relationship between Φ and Φ':

$$1+\tilde{g}p^{-1}\Phi'=\exp(\tilde{g}p^{-1}\Phi)\ , \tag{4.3}$$

as in (A12) in the Appendix. Accordingly, our action (2.1) is

$$\int dx\,(-\tfrac{1}{2}X^\mu\Delta X_\mu+\tfrac{1}{2}ap^K X_\mu X^\mu-bp^K)\ , \tag{4.4}$$

where $g=\tilde{g}p^K$ as before.

Our world-sheet effective potential V is defined by

$$\exp\left[-\int dx\ V\right]=\frac{\int[\mathcal{D}X^\mu]\exp\left[\int dx\,(\frac{1}{2}X^\mu\Delta X_\mu-\frac{1}{2}ap^K X_\mu^2+bp^K)\right]}{\int[\mathcal{D}X^\nu]\exp\left[\int dx\ \frac{1}{2}X^\nu\Delta X_\nu\right]}\ . \tag{4.5}$$

We can perform this path integration explicitly, by using the p-adic "momentum" representation Δ_u (Ref. 11) of the Laplacian Δ defined in (2.1):

$$\begin{aligned}\exp\left[-\int dx\ V\right]&=\exp\left[\int dx\left[bp^K+\tfrac{1}{2}D\int du\ \ln(\Delta_u-ap^K)-\tfrac{1}{2}D\int du\,(\ln\Delta_u)\right]\right]\\&=\exp\left[\int dx\left[bp^K+\tfrac{1}{2}D\int du\ \ln[1+ap^K G_K(u)]\right]\right]\ .\end{aligned} \tag{4.6}$$

Here we have also used the general formula $\det\mathcal{M}=\exp[\mathrm{tr}(\ln\mathcal{M})]$ and $\Delta_u^{-1}=-G_K(u)$ for the p-adic "momentum" representation $G_K(u)$ of our propagator $G_K(x)$ defined in (2.7). Since the field redefinition (A12) between Φ' and Φ was the result of our special regularization used in (2.7), we have to adopt exactly the same regularization for the propagator $G_K(x)$. The explicit form of $G_K(u)$ is easily obtained by performing a p-adic Fourier transformation[14] of (2.7), as

$$G_K(u)=\begin{cases} c^{-1}\left[\frac{1}{|u|}-p^{-K-1}\right] & \text{for } |u|\le p^K , \\ 0 \quad \text{for } |u|\ge p^{K+1} . \end{cases} \tag{4.7}$$

Now the u integration in (4.6) is easily performed:

$$\begin{aligned}\int_{|u|\le p^K} du \ln\left[1+\frac{ap^K}{c|u|}(1-p^{-K-1}|u|)\right] &=(1-p^{-1})\sum_{k=-\infty}^{K} p^k \ln(1-ac^{-1}p^{-1}+ac^{-1}p^{K-k}) \\ &=p^K(p-1)\sum_{l=1}^{\infty} p^{-l}\ln(1-ac^{-1}p^{-1}+ac^{-1}p^{-1+l}) ,\end{aligned} \tag{4.8}$$

$$V=-p^K\left[b-\frac{D(p-1)}{2}\sum_{l=1}^{\infty} p^{-l}\ln(1-ac^{-1}p^{-1}+ac^{-1}p^{-1+l})\right] . \tag{4.9}$$

Our discrete scale invariance under $K\to K+1$ is recovered, when the term in large parentheses in (4.9) vanishes. This gives us a condition

$$b=\frac{D(p-1)}{2}\sum_{l=1}^{\infty} p^{-l}\ln(1-ac^{-1}p^{-1}+ac^{-1}p^{-1+l}) . \tag{4.10}$$

In addition to this condition, we have to consider the scaling property of the X^μ propagator. As in the usual case of massive propagators, we define $G_K^{(a)}(u)$ as the inverse of the Lagrangian in Eq. (4.4):

$$\begin{aligned}G_K^{(a)}(u)&\equiv\frac{1}{G_K^{(0)}(u)^{-1}+ap^K} \\ &=c^{-1}\frac{1-p^{-K-1}|u|}{(1-ac^{-1}p^{-1})|u|+ac^{-1}p^K} .\end{aligned} \tag{4.11}$$

The corresponding x-space propagator is

$$G_K^{(a)}(x)=c^{-1}p^{-K-1}\times\int_{|u|\le p^K} du\frac{1-p^{-K-1}|u|}{1-\delta(1-p^{-K-1}|u|)}e^{2\pi iux} , \tag{4.12}$$

where $a\equiv(1-\delta)cp$.

There are two special cases of δ in (4.12). The first is when $\delta=1$; this is the massless case, corresponding to $G_K(x)$ of Eq. (2.7). In this case, therefore we know that the integral of (4.12) gives $G_K^{(0)}(x)\approx-\ln|x|$ for $|x|\ge p^{-K}$ up to an irrelevant infinite constant. Another special case is $\delta=0$ (or equivalently $a=cp$); in this case the integration (4.12) is easily performed to give

$$G_K^{(cp)}(x)=c^{-1}\frac{p^{-2K}}{1+p}\frac{1}{|x|^2} , \tag{4.13}$$

for $|x|\ge p^{-K}$. In both of these two cases we see that the propagator $G_K^{(a)}(x)$ has the right scaling property under $x\to px$. In fact, when $\delta=1$, the propagator is essentially logarithmic, so that it has a constant shift under this scaling, like two-dimensional massless bosonic field theory, while in the case of $\delta=0$, there is only one single term of $|x|^{-2}$ which scales like $G_K^{(cp)}(x)\to p^2G_K^{(cp)}(x)$.

For all values of δ other than these two special cases, we see that the integral (4.12) does not scale like $G_K^{(a)}(x)\to p^wG_K^{(a)}(x)$ for a fixed w. This can be easily confirmed by various methods, e.g., by expanding the integrand in (4.12) around $\delta=0$, and noticing the appearance of a polynomial of $|x|^{-1}$ with more than a single term. Only in the special case $\delta=0$, we can absorb the scaling of the propagator by appropriate redefinition of the field X^μ, and hence *recover* the scale invariance of the total Lagrangian.

Therefore we are left with the two conditions

$$\begin{aligned}b&=\frac{D(p-1)}{2}\sum_{l=1}^{\infty} p^{-l}\ln(1-ac^{-1}p^{-1}+ac^{-1}p^{l-1}) , \\ \delta&=0 .\end{aligned} \tag{4.14}$$

These equations imply that

$$b=\frac{D\ln p}{2(1-p^{-1})}=\tfrac{1}{2}c^{-1}D, \quad a=cp . \tag{4.15}$$

This means nothing else than the exact soliton solution: i.e.,

$$\begin{aligned}\tilde g\Phi&=-\tfrac{1}{2}cpX_\mu X^\mu+\tfrac{1}{2}c^{-1}D , \\ p^{-1}\tilde g\phi&=p^{-1}\tilde g\Phi'=\exp(\tilde g p^{-1}\Phi)-1 \\ &=p^{D/[2(p-1)]}\exp(-\tfrac{1}{2}cX_\mu^2)-1 .\end{aligned} \tag{4.16}$$

Here we have identified ϕ with Φ', and used the relationship (A12) between Φ and Φ'.

V. CONCLUSIONS

In this paper we have established a closed form for the Nth-order effective action in a tachyonic background as an integral expression (2.18). Applying this form to the $N=1$, 2, 3, and 4 cases, we gave the vanishing β-function condition (2.22) up to $O(\Phi^5)$, and we have shown that this condition actually gives the tachyonic background-field Eq. (2.24) (obtained from amplitude calculations[9]) up to $O(\Phi^5)$. Therefore, as in ordinary string theory, the equivalence between the vanishing β-function condition and the field equation of the effective field theory is also valid in the p-adic case. In the case of p-adic string theories, this equivalence is more illustrative than ordinary string theories. This is because we can explicitly get the effective action at $O(\Phi^N)$ by summing up geometric series. [See, e.g., (3.16)–(3.27).] In ordinary string theory, because of the infinitely many massive states present in the theory, we have to take a special on-shell limit with respect to the physical mode under question, in order to see the above equivalence.[3,4] In p-adic string theory, on the other hand, the calculation is carried out in a more rigorous way, by the use of $\tilde{g}$ expansions both for the full field equation and the effective action, as in Eqs. (2.26)–(2.29), and (3.12), (3.13), or (3.37).

In Sec. IV we have checked the discrete scale invariance of our σ model in the presence of a soliton solution which supplies a nontrivial background. We saw that a previously derived soliton solution can be arrived at by using the two conditions, i.e., the requirement of the discrete scale invariance of the effective action, and the condition for the right scaling property of the X^μ propagators. The result of this section introduces several new concepts. First, it supports our claim of the all-order equivalence between the vanishing β-function condition and the full tachyonic field equation, because of the soliton solution satisfying the tachyonic equation *to all orders.* The second noteworthy concept is that, unlike ordinary string theories, where we use either a weak-field expansion around "flat" backgrounds or the α' expansion, we can put our σ model directly in a nontrivial background, as an exact solution of a full-order field equation. In addition, this technique can be useful in searching for new soliton solutions of the tachyonic field equation, based on path integrals in the σ-model approach to the p-adic string.

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APPENDIX

In this appendix we give some details for the derivation of the $\tilde{W}^{(4)}$ and the vanishing β-function condition at $O(\Phi^4)$.

Using our general formula (2.18) for $N=4$, we see that we have to evaluate the integral

$$I^{(4)}=\int dx\,dy\,dz\,|x|_0^\alpha\,|y|_0^\beta\,|z|_0^\gamma\,|x-y|_0^\delta\,|y-z|_0^\epsilon\,|z-x|_0^\eta\ . \tag{A1}$$

The total integration region is divided into the following subregions:

$$\begin{aligned}
&\text{(i)}\ \ |x|<1,\ \ |y|<1,\ \ |z|<1\ ;\\
&\text{(ii)}\ \ |x|\ge 1,\ \ |y|<1,\ \ |z|<1\ ;\\
&\text{(iii)}\ \ |x|\ge 1,\ \ |y|\ge 1,\ \ |z|<1\ ;\\
&\text{(iv)}\ \ |x|>|y|>|z|\ge 1\ ;\\
&\text{(v)}\ \ |x|=|y|>|z|\ge 1\ ;\\
&\qquad \text{(a)}\ \ |x-y|\ge 1,\ \ \text{(b)}\ \ |x-y|<1\ ,\\
&\text{(vi)}\ \ |x|>|y|=|z|\ge 1\ ;\\
&\qquad \text{(a)}\ \ |y-z|\ge 1,\ \ \text{(b)}\ \ |y-z|<1\ ,\\
&\text{(vii)}\ \ |x|=|y|=|z|\ge 1\ ;\\
&\qquad \text{(a)}\ \ |x-y|<1,\ \ |y-z|<1,\ \ |z-x|<1\ ,\\
&\qquad \text{(b)}\ \ |x-y|\ge 1,\ \ |y-z|\ge 1,\ \ |z-x|\ge 1\ ,\\
&\qquad \text{(c)}\ \ |x-y|<1,\ \ |y-z|\ge 1,\ \ |z-x|\ge 1\ .
\end{aligned} \tag{A2}$$

We also have other regions associated by the permutations of x, y, and z.

As a typical example, we perform the integration over the subregion (iv):

$$\begin{aligned}
I^{(4)}(\text{iv})&=\int_{|x|>|y|>|z|\ge 1} dx\,dy\,dz\,|x|_0^\alpha\,|y|_0^\beta\,|z|_0^\gamma\,|x-y|_0^\delta\,|y-z|_0^\epsilon\,|z-x|_0^\eta\\
&=\int_{|x|>|y|>|z|\ge 1} dx\,dy\,dz\,|x|_0^{\alpha+\delta+\eta}\,|y|_0^{\beta+\epsilon}\,|z|_0^\gamma\ .
\end{aligned} \tag{A3}$$

By parametrizing $|x|=p^k$, $|y|=p^l$, $|z|=p^m$, $k>l>m\ge 0$, we get

$$I^{(4)}(\text{iv})=(1-p^{-1})^3\sum_{k=2}^{\infty}\sum_{l=1}^{k-1}\sum_{m=0}^{l-1}p^{(\alpha+\delta+\eta+1)k+(\beta+\epsilon+1)l+(\gamma+1)m}\ . \tag{A4}$$

The result of this summation is considerably simplified by the use of identities, such as

$$\frac{1}{1-a}-\frac{b}{1-ab}=\frac{1-b}{(1-a)(1-ab)}\ . \tag{A5}$$

After this arrangement, we get the triple-pole expression

$$I^{(4)}(\text{iv})=\frac{1-p^{-1}}{1-p^{\alpha+\beta+\gamma+\delta+\epsilon+\eta+3}}\frac{(1-p^{-1})p^{\alpha+\beta+\delta+\epsilon+\eta+2}}{1-p^{\alpha+\beta+\delta+\epsilon+\eta+2}}\frac{(1-p^{-1})p^{\alpha+\delta+\eta+1}}{1-p^{\alpha+\delta+\eta+1}} . \tag{A6}$$

We can similarly perform the integration over other subregions. Like the cases of $\tilde{W}^{(2)}$ and $\tilde{W}^{(3)}$ in Secs. III A and III B the contribution of all the nonconnected graphs can be seen to vanish. With the use of the operators $J^{(i)}$ defined by (3.31), $W^{(4)}$ can be given by

$$\tilde{W}^{(4)}(\Phi_B)=J^{(1)}\cdot\left[\frac{1}{4!}\tilde{g}^3p^{-3}\Phi_B^4\right]+J^{(2)}\cdot(\tfrac{1}{2}\tilde{g}p^{-1}\Phi_B^2)^2+2J^{(2)}\cdot\left[\Phi_B\cdot\frac{1}{3!}\tilde{g}^2p^{-2}\Phi_B^3\right]+3J^{(3)}(\cdot\Phi_B\cdot\Phi_B\cdot\tfrac{1}{2}\tilde{g}p^{-1}\Phi_B^2)+J^{(4)}\cdot(\Phi_B^4) , \tag{A7}$$

where $J^{(4)}$ is defined by

$$J^{(4)}\cdot(\Phi_B^4)\equiv-\tfrac{1}{24}\tilde{g}^4\frac{1}{1-p^A}\left[3\frac{(1-p^{-1})^3(1-p^{A+D})p^D}{(1-p^B)(1-p^C)(1-p^D)}+12\frac{(1-p^{-1})^2p^E}{1-p^E}\frac{(1-p^{-1})p^B}{1-p^B}\right.$$
$$+6\frac{(1-p^{-1})^2(1-2p^{-1})p^C}{1-p^C}+4\frac{(1-p^{-1})^2(1-2p^{-1})p^E}{1-p^E}$$
$$\left.+(1-p^{-1})(1-2p^{-1})(1-3p^{-1})\right]\Phi_B^4 , \tag{A8}$$

where

$$A\equiv-\partial_4\cdot\partial_1-\partial_4\cdot\partial_2-\partial_4\cdot\partial_3-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3-\partial_3\cdot\partial_1+3 ,$$
$$B\equiv-\partial_4\cdot\partial_1-\partial_4\cdot\partial_2-\partial_4\cdot\partial_3-\partial_2\cdot\partial_3-\partial_3\cdot\partial_1+2 ,$$
$$C\equiv-\partial_4\cdot\partial_1-\partial_4\cdot\partial_2-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3-\partial_3\cdot\partial_1+2 , \tag{A9}$$
$$D\equiv-\partial_4\cdot\partial_1-\partial_4\cdot\partial_2-\partial_2\cdot\partial_3-\partial_3\cdot\partial_1+1 ,$$
$$E\equiv-\partial_4\cdot\partial_1-\partial_4\cdot\partial_2-\partial_4\cdot\partial_3+1 .$$

Therefore, if we neglect higher-order terms in Φ, we have

$$\tilde{W}^{(1)}(\Phi_B)+\tilde{W}^{(2)}(\Phi_B)+\tilde{W}^{(3)}(\Phi_B)+\tilde{W}^{(4)}(\Phi_B)$$
$$=J^{(1)}\cdot(\Phi'_B)+J^{(2)}\cdot(\Phi'^2_B)+J^{(3)}\cdot(\Phi'^3_B)+J^{(4)}\cdot(\Phi'^4_B) , \tag{A10}$$

with the field redefinition

$$\tilde{g}\Phi'_B=\tilde{g}\Phi_B+\tfrac{1}{2}\tilde{g}^2p^{-1}\Phi_B^2+\frac{1}{3!}\tilde{g}^3p^{-2}\Phi_B^3+\frac{1}{4!}\tilde{g}^4p^{-3}\Phi_B^4 . \tag{A11}$$

At $O(\Phi^2)$ and $O(\Phi^3)$, respectively, we saw in Eqs. (3.8) and (3.31) that the terms $-\frac{1}{2}\tilde{g}^2p^{-1}\Phi_B^2$ and $-\frac{1}{6}\tilde{g}^3p^{-2}\Phi_B^3$, respectively, in $\tilde{W}^{(2)}(\Phi_B)$ and $\tilde{W}^{(3)}(\Phi_B)$ are absorbed into the field redefinition $\tilde{g}\Phi'_B=\tilde{g}\Phi_B+\frac{1}{2}\tilde{g}^2p^{-1}\Phi_B^2+(1/3!)\tilde{g}^3p^{-2}\Phi_B^3$. At $O(\Phi^4)$, we observe a similar phenomenon that the term $-\frac{1}{24}\tilde{g}^4p^{-3}\Phi_B^4$ arising in $W^{(4)}$ from the subregion (i) in (A2) is absorbed into the $O(\Phi^4)$ term of the field redefinition (A11). In general for $W^{(N)}$, the term $-(1/N!)\tilde{g}^Np^{-N+1}\Phi_B^N$ arises from the integral subregion, where the p-adic norms of all the variables are smaller than one, i.e., $|z_i|<1$ $(i=1,\dots,N-1)$ in the integral (2.18), which is absorbed into the $O(\Phi^N)$ term in the general expression of the field redefinition:

$$\tilde{g}\Phi'_B=\tilde{g}\Phi_B+\tfrac{1}{2}\tilde{g}^2p^{-1}\Phi_B^2+\frac{1}{3!}\tilde{g}^3p^{-2}\Phi_B^3+\frac{1}{4!}\tilde{g}^4p^{-3}\Phi_B^4+\cdots$$
$$=p[\exp(\tilde{g}p^{-1}\Phi_B)-1] . \tag{A12}$$

As we can see from (A7), if we substitute the field redefinition (A11) into (A10), we also have contributions to $\tilde{W}^{(4)}(\Phi_B)$ from $J^{(2)}(\Phi'^2_B)$ and $J^{(3)}(\Phi'^3_B)$. It is easily shown that those contributions are obtained exactly from the integral subregions of (A2) in which at least one of $|z_i|$ or $|z_i-z_j|$ is less than one, just as in the $\tilde{W}^{(3)}(\Phi_B)$ case discussed in Sec. III. From this observation, we expect that the field redefinition (A12) holds to all orders in $\tilde{g}$.

After this field redefinition (A11), our vanishing β-function condition up to $O(\Phi^5)$ is

$$0=(p^{-\Box/2-1}-1)[J^{(1)}\cdot(\Phi'_B)+J^{(2)}\cdot(\Phi'^2_B)+J^{(3)}\cdot(\Phi'^3_B)+J^{(4)}\cdot(\Phi'^4_B)]$$
$$=(p^{-\Box/2-1}-1)\left[-\tilde{g}\Phi'_B-\tfrac{1}{2}\tilde{g}^2\frac{1-p^{-1}}{1-p^{1-\partial_1\cdot\partial_2}}\Phi'^2_B-\tfrac{1}{6}\tilde{g}^3\frac{1-p^{-1}}{1-p^{2-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3-\partial_3\cdot\partial_1}}\left[3\frac{(1-p^{-1})p^{1-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3}}{1-p^{1-\partial_1\cdot\partial_2-\partial_2\cdot\partial_3}}+1-2p^{-1}\right]\Phi'^3_B\right.$$
$$-\tfrac{1}{24}\tilde{g}^4\frac{1}{1-p^A}\left[3\frac{(1-p^{-1})^3(1-p^{A+D})p^D}{(1-p^B)(1-p^C)(1-p^D)}+12\frac{(1-p^{-1})^2p^E}{1-p^B}\frac{(1-p^{-1})p^B}{1-p^B}\right.$$
$$+6\frac{(1-p^{-1})^2(1-2p^{-1})p^C}{1-p^C}+4\frac{(1-p^{-1})^2(1-2p^{-1})p^E}{1-p^E}$$
$$\left.\left.+(1-p^{-1})(1-2p^{-1})(1-3p^{-1})\right]\Phi'^4_B\right] . \tag{A13}$$

As we already did in Secs. III B and III C we can use the lower-order field equations (3.12), (3.13), and (3.37). After arranging terms, and eliminating terms vanishing after the use of those lower-order field equations, we are left with the condition

$$0=\tilde{g}^4[(p^{-\Box/2-1}-1)\Phi^{(4)}-\tfrac{1}{2}(1-p^{-1})\Phi^{(2)2}-\tfrac{1}{2}(1-p^{-1})(1-2p^{-1})\Phi^{(1)2}\Phi^{(2)}$$
$$-\tfrac{1}{24}(1-p^{-1})(1-2p^{-1})(1-3p^{-1})\Phi^{(1)4}-(1-p^{-1})\Phi^{(1)}\Phi^{(3)}]$$
$$+\tfrac{1}{8}\tilde{g}^4(1-p^{-1})^3\left[2\frac{p^C}{1-p^C}\frac{1}{1-p^D}+4\frac{p^B}{1-p^B}+\frac{p^D}{1-p^D}-6\frac{p^C}{1-p^C}-\frac{(1-p^{A+D})p^D}{(1-p^B)(1-p^C)(1-p^D)}\right]\Phi^{(1)4}\,. \tag{A14}$$

We see here that the terms in the large parentheses cancel each other completely, and only the square brackets remain. This implies nothing else than the $O(\Phi^4)$ tachyonic field Eq. (2.29) with the identification $\phi^{(4)}=\Phi^{(4)}$.

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*Present address: Department of Physics, Tohoku University, Sendai 980, Japan.

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