Duality is an exact symmetry of string perturbation theory

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(Received 27 February 1989)

Both β duality and R duality in the toroidal compactification of heterotic strings are shown to be symmetries of the contribution of arbitrary genus to the free energy. This symmetry relates physics at radius R and coupling κ with physics at radius α'/R and coupling $\kappa(\alpha'/R^2)^{d/2}$ where d is the number of dimensions compactified. Some comments on the possible breaking of duality by nonperturbative effects are also included.

The physical relevance of duality in string theories has recently been stressed by many authors.[1,3,4,5] There are actually two related symmetries: R duality, which corresponds essentially to the exchange of winding modes with momentum modes in toroidally compactified strings, and β duality, which is slightly more complicated although there is a definite[2,3] relationship between strings at finite temperature and the said toroidal compactification.

To be specific, the genus-1 contribution to the free energy per unit volume of the heterotic string satisfies[4,3]

$$F(\beta)=\frac{\pi^2}{\beta^2}F\left[\frac{\pi^2}{\beta}\right], \tag{1}$$

whereas for heterotic strings compactified in a Cartesian torus of radius R the free energy satisfies

$$F(R)=F\left[\frac{\alpha'}{R}\right]. \tag{2}$$

R duality has been advocated[9] as a fundamental symmetry in string theory. On the other hand, β duality is most relevant for the issue of the phase transition at the critical temperature.[5,6] It is obviously of great interest to examine whether the higher-genus contributions still enjoy symmetries of the duality type.

In the simplest framework of the bosonic string, we can write the genus-g contribution to the free energy density as

$$F_g(\beta)=-\int_{\mathcal{F}_g} d\mu(m)\theta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}(0|\Omega)\Lambda(\tau,\bar{\tau}), \tag{3}$$

where the genus-g contribution to the cosmological constant is

$$\Lambda_g=-\int_{\mathcal{F}_g} d\mu(m)\Lambda(\tau,\bar{\tau}).$$

$d\mu(m)$ is the measure in moduli space, $\mathcal{M}_{3g-3}$. $\Lambda_g(\tau,\bar{\tau})$ is a function of the period matrix of the Riemann surface, $\tau\equiv\tau_1+i\tau_2\in C^{g(g+1)/2}$ and $\mathcal{F}_g$ denotes a fundamental region for the action of $\mathrm{Sp}(2g,\mathbf{Z})$ on $\mathcal{M}_{3g-3}$ (which is only known explicitly for $g<3$). The physical content of Riemann's theta function

$$\theta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}(0|\Omega)\equiv\sum_{(\mathbf{n},\mathbf{m})\in Z^{2g}}\exp[i\pi^t(\mathbf{m},\mathbf{n})\Omega(\mathbf{m},\mathbf{n})], \tag{4}$$

$$\Omega\equiv\frac{i\beta^2}{2\pi^2}\begin{bmatrix}\tau_1\tau_2^{-1}\tau_1+\tau_2 & -\tau_1\tau_2^{-1}\\ -\tau_2^{-1}\tau_1 & \tau_2^{-1}\end{bmatrix} \tag{5}$$

represents the contribution of the "thermal solitons"[5,7] and includes all β-dependent terms. In the particular case of genus two, Moore's[10] result is recovered in another guise:

$$F_2(\beta)=-\int_{\mathcal{F}_2}\prod_{i\le j}d^2\tau_{ij}\theta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}(0|\Omega)\chi_{10}(\tau)|^{-2}\det{}^{-13}\tau_2,$$

where $\chi_{10}\equiv\prod_{\text{even},\alpha}\theta_\alpha^2(0|\tau)$, and $\theta_\alpha(\mathbf{z}|\tau)$ are Jacobi's elliptic theta functions.

If we now perform upon Ω the symplectic transformation generated by

$$\begin{bmatrix}0 & -I\\ I & 0\end{bmatrix},$$

the theta function transforms in a known way:[8]

$$\theta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}(0|-\Omega^{-1})=\det{}^{1/2}\left[\frac{\Omega}{i}\right]\theta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}(0|\Omega),$$

which implies

$$F_g(\beta)=\left[\frac{2\pi^2}{\beta^2}\right]^g F_g\left[\frac{2\pi^2}{\beta}\right], \tag{6}$$

namely, the generalization of β duality to arbitrary genus for the bosonic string. This is only a formal relation though, because of the tachyon divergence.

The whole perturbative series in the string coupling constant κ will be

$$F(\kappa,\beta)=\sum_{g=1}^{\infty}\kappa^{2(g-1)}F_g(\beta). \tag{7}$$

Duality implies now the property

$$F(\kappa,\beta)=\frac{2\pi^2}{\beta^2}F\left[\frac{\kappa\pi\sqrt{2}}{\beta},\frac{2\pi^2}{\beta}\right]; \tag{8}$$

that is, it relates, for example, high temperatures with low temperatures. We shall add some further comments on the physical meaning of duality at the end of this paper.

It is not difficult to generalize the preceding to heterotic strings. The genus-g contribution to the free energy per unit volume can be written as

$$F_g(\beta)=-\int_{\mathcal{F}_g} d\mu(m)\sum_s \Lambda_s(\tau,\bar\tau)\theta\begin{bmatrix}0 & 0\\ \mathbf{s}_2 & \mathbf{s}_1\end{bmatrix}(0|\tilde\Omega)$$

provided that the cosmological constant is

$$\Lambda_g=\int_{\mathcal{F}_g} d\mu(m)\sum_s \Lambda_s(\tau,\bar\tau)\ .$$

Here

$$s\equiv\begin{bmatrix}\mathbf{s}_1\\ \mathbf{s}_2\end{bmatrix},$$

where $\mathbf{s}_1,\mathbf{s}_2\in[(Z/2)/Z]^g$ are the characteristics defining the 2^{2g} spin structures on the Riemann surface. [Actually, owing to the existence of zero modes, only the $2^{g-1}(2^g+1)$ even spin structures contribute.] As in the case of the bosonic string, all β dependence is included in a Riemann's theta function which corresponds to a matrix

$$\tilde\Omega=\Omega+\frac{1}{2}\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}.$$

This term includes the contribution of the thermal solitons as well as some signs.[5] Using some theta-function gymnastics[8] one can easily rewrite this as

$$F_g(\beta)=-\int_{\mathcal{F}_g} d\mu(m)\sum_s \Lambda_s\sum_t(-1)^{4(\mathbf{s}_2\mathbf{t}_1+\mathbf{s}_1\mathbf{t}_2+\mathbf{t}_1\mathbf{t}_2)}\times\theta\begin{bmatrix}\mathbf{t}_1 & \mathbf{t}_2\\ \mathbf{0} & \mathbf{0}\end{bmatrix}(0|4\Omega)\ . \tag{9}$$

Performing again a $Sp(4g,\mathbf{Z})$ transformation, and using the curious property

$$\Omega^{-1}=\left[\frac{2\pi^2}{i\beta^2}\right]^2\begin{bmatrix}0 & I\\ -I & 0\end{bmatrix}\Omega\begin{bmatrix}0 & I\\ -I & 0\end{bmatrix}$$

one easily obtains

$$F_g\left[\frac{\pi^2}{\beta}\right]=-\int_{\mathcal{F}_g} d\mu(m)\sum_s \Lambda_s\sum_t(-1)^{4(\mathbf{s}_2\mathbf{t}_1+\mathbf{s}_1\mathbf{t}_2+\mathbf{t}_1\mathbf{t}_2)}\left[\frac{\beta^2}{2\pi^2}\right]^g\sum_r(-1)^{4(\mathbf{r}_2\mathbf{t}_1+\mathbf{r}_1\mathbf{t}_2)}\theta\begin{bmatrix}\mathbf{r}_1 & \mathbf{r}_2\\ \mathbf{0} & \mathbf{0}\end{bmatrix}(0|4\Omega)\ ,$$

which is equal to

$$F_g\left[\frac{\pi^2}{\beta}\right]=\left[\frac{\beta^2}{\pi^2}\right]^g F_g(\beta)\ . \tag{10}$$

Owing to the easily proved identity

$$\sum_t(-1)^{4[\mathbf{t}_1\mathbf{t}_2+\mathbf{t}_1(\mathbf{r}_2+\mathbf{s}_2)+\mathbf{t}_2(\mathbf{r}_1+\mathbf{s}_1)]}=2^g(-1)^{4(\mathbf{s}_2\mathbf{r}_1+\mathbf{s}_1\mathbf{r}_2+\mathbf{r}_1\mathbf{r}_2)}\ . \tag{11}$$

This duality implies for the whole series a relationship completely analogous to (8): namely,

$$F(\kappa,\beta)=\frac{\pi^2}{\beta^2}F\left[\frac{\kappa\pi}{\beta},\frac{\pi^2}{\beta}\right]. \tag{12}$$

It is most remarkable that in the proof of the duality equation (12), we have not dwelled at any point upon the accepted conjecture that $\sum_s\Lambda_s=0$ (that is, the vanishing of the integrand of the cosmological constant). Our proof relies only upon the fact that only even spin structures appear in the sum and in modular invariance.

In the same vein, using the formula for the toroidal compactification of the heterotic string, one easily gets, for the free energy,

$$F_g(\beta,R)=\left[\frac{\alpha'}{R^2}\right]^{d(g-1)}F_g\left[\beta,\frac{\alpha'}{R}\right], \tag{13}$$

which implies, for the whole series,

$$F(\kappa,R)=F\left[\frac{\kappa(\alpha')^{d/2}}{R^d},\frac{\alpha'}{R}\right], \tag{14}$$

which correlates small distances with long distances both at weak and strong couplings.

This concludes our examination of duality in string perturbation theory. This is potentially one of the most important symmetries to study in this context, because of its "stringy" character. One can even think of using it to make specific predictions allowing us to tell strings apart from ordinary quantum fields.

There are, however, strong physical arguments for β duality to be broken.[5,6] What we have shown is that this will be possible only through nonperturbative effects if at all.

We know, indeed, that the whole perturbative series is divergent,[11] and not even Borel summable. This probably means that the results obtained in string perturbation theory are not a good indication of the physical predictions of the theory. We are now performing a computation in infinite-genus surfaces, in the hope of getting some information on the nonperturbative regime.

We acknowledge many discussions with Luis Alvarez-Gaumé. This work has been partially supported by Comisión Interministerial de Ciencia y Tecnologia (Spain).

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