Interacting Lagrangian for massive spin-two field

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As is well known, Kaluza-Klein compactification of five-dimensional gravity gives a four-dimensional theory of a tower of massive spin-2 fields interacting with background gravitational, electromagnetic, and scalar fields. Working to lowest order in the massive fields, one can obtain a Lagrangian for a single massive spin-2 field interacting with gravitational and other background fields which is consistent up to terms of higher order in the massive spin-2 field. This consistency depends on massive spin-2 gauge invariances which are valid up to terms of higher order in the massive field. We explicitly exhibit the four-dimensional Lagrangian and its gauge transformations. This analysis parallels a similar analysis in string theory carried out elsewhere.

I. INTRODUCTION

The problem of finding consistent interactive Lagrangians for spin-2 massive particles was first formulated in Ref. 1. As extensively discussed in the literature,[2–5] attempts to introduce interactions in the free theory by minimal coupling result in the loss of some of the constraint equations necessary to ensure the existence of the appropriate number of degrees of freedom. The inclusion of extra terms in the Lagrangian which might allow one to circumvent this difficulty leads usually to loss of causality.[3,6]

The root of this problem has to do with gauge invariance.[7] Consistency of massless and massive fields of spin greater than one-half is always achieved by means of gauge invariance (Yang-Mills invariance, local supersymmetry, and general covariance for massless particles of spin 1, $\frac{3}{2}$, or 2; spontaneously broken Yang-Mills invariance or local supersymmetry for massive particles of spin 1 or $\frac{3}{2}$). Likewise, consistent Lagrangians for free massless fields of spin ≥ 2 achieve their consistency via gauge invariances that permit the unphysical modes to be gauged away or set to zero by constraint equations; and consistent free Lagrangians for massive fields of spin ≥ 2 are best understood as Lagrangians in which an underlying gauge invariance is spontaneously broken, with the mass terms arising via a Higgs-type mechanism. The problem with coupling massive fields of high spin to electromagnetism or gravity is that the coupling ruins the gauge invariances that, in the free theory, tame the constraint equations and decouple the unphysical modes.

A solution to the problem of gravitational couplings of massive spin-2 fields is offered in principle by Kaluza-Klein theory.[8] For instance, compactification of five-dimensional gravity to the product of four-dimensional space-time and a circle gives, as is well known, a theory that contains at the massless level a graviton, electromagnetic field, and dilaton, and in addition an infinite tower of spin-2 massive fields interacting between themselves and with the massless modes.

The consistency of five-dimensional Kaluza-Klein theory depends on the presence of the whole infinite tower of massive spin-2 modes. If one is not primarily interested in this particular theory, but one wants to extract general features associated with the attempt at consistent coupling of a massive spin-2 field, then it is natural to try to extract one massive spin-2 field from the infinite tower and to try to focus on how gauge invariance is achieved for that field. This cannot be done exactly (since the five-dimensional symmetries mix up the various massive modes), but if one works to all orders in the four-dimensional massless background and only to first order in the massive fields, then the different massive modes can be decoupled. Of course, working only to first order in the massive fields means that we will not achieve the massive spin-2 gauge invariance exactly, but only to lowest nontrivial order. At this order we explicitly work out the four-dimensional interacting Lagrangian and its gauge transformations. One of the purposes of this detailed analysis is to compare it with a somewhat parallel analysis in string theory.[9]

II. REVIEW OF KALUZA-KLEIN THEORY

We start from pure gravity in five dimensions

$$S=\int d^5x\sqrt{-{}^{(5)}g}\;{}^{(5)}R \tag{1}$$

and compactify it in $M^4\times S^1$, the four-dimensional space-time times a circle S^1. We assume the ground state to be given by the metric

$$g_{MN}=\begin{bmatrix} g_{\mu\nu}+\phi^2A_\mu A_\nu & \phi^2A_\mu \\ \phi^2A_\nu & \phi^2 \end{bmatrix}, \tag{2}$$

or, in a more compact form, by

$$g_{MN}dx^Mdx^N=g_{\mu\nu}dx^\mu dx^\nu+\phi^2(d\theta+A_\mu dx^\mu)^2 .$$

In the above, μ,ν are indices in four dimensions and M,N are indices in five dimensions, $M=(\mu,\theta)$ where θ is the coordinate on S^1.

The inverse of the above metric is

$$g^{NR}=\begin{bmatrix} g^{\rho\nu} & -A^\rho \\ -A^\nu & \phi^{-2}+A_\mu A^\mu \end{bmatrix}. \tag{3}$$

Five-dimensional tangent space indices will be denoted as $A,B,C=1,\ldots,5$; four-dimensional tangent space indices will be denoted as $a,b,c=1,\ldots,4$. The vierbein $E^A{}_M$ is given by

$$E^a{}_\mu=e^a{}_\mu,\quad E^a{}_\theta=0,\quad E^5{}_\mu=\phi A_\mu,\quad E^5{}_\theta=\phi\ , \tag{4}$$

where $e^a{}_\mu$ is the four-dimensional vierbein associated with $g_{\mu\nu}$. The inverse vierbein is $E_A{}^M$:

$$\begin{aligned}&E_a{}^\mu=e_a{}^\mu,\quad E_a{}^\theta=-A_a=-A_\mu e_a{}^\mu\ ,\\ &E_5{}^\mu=0,\quad E_5{}^\theta=\phi^{-1}\ .\end{aligned} \tag{5}$$

The fields $g_{MN}=(g_{\mu\nu},A_\mu,\phi)$ are periodic in θ and may be Fourier expanded in the form

$$g_{MN}(x,\theta)=\sum_{n=-\infty}^{\infty} g_{MN}^{(n)}(x)e^{in\theta}\ . \tag{6}$$

We also write $g_{MN}=g_{MN}^{(0)}+h_{MN}$, where $g_{MN}^{(0)}$ is the θ-independent piece of the metric and h_{MN} is the θ-dependent metric perturbation. If one retains only the θ-independent part in the above expression, one obtains a theory of massless spin 2, $g_{\mu\nu}^{(0)}$, massless spin 1, $A_\mu^{(0)}$, and a massless spin 0, $\phi^{(0)}$. [From now on we neglect the superscript (0) in the θ-independent piece of (6), whenever this does not create confusion.] For convenience and completeness we rederive here in the vierbein formalism the well-known results for the massless modes. From the definitions

$$E^A=E^A{}_M dX^M,\quad \partial_A=E_A{}^M\partial_M$$

one derives

$$\begin{aligned}&E^a=e^a,\quad E^5=\phi(A+d\theta)\ ,\\ &\partial_a=e_a{}^\mu(\partial_\mu-A_\mu\partial_\theta),\quad \partial_5=\frac{1}{\phi}\partial_\theta\ ,\end{aligned} \tag{7}$$

and

$$\begin{aligned}&{}^{(5)}dE^a=de^a\ ,\\ &{}^{(5)}dE^5=\phi\, dA+d\phi\wedge(A+d\theta)=\frac{\phi}{2}F_{ab}e_a\wedge e_b+\frac{d\phi}{\phi}\wedge E^5\ ,\end{aligned} \tag{8}$$

where $F_{ab}=e_a{}^\mu e_b{}^\nu F_{\mu\nu}$ and $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$.

By using the definition of spin connection in four and five dimensions, respectively,

$$de^a+\omega^a{}_b\wedge e^b=0,\quad dE^A+\Omega^A{}_B\wedge E^B=0\ , \tag{9}$$

we can compute the five-dimensional connection $\Omega^A{}_B$ in terms of the four-dimensional one $\omega^a{}_b$:

$$\Omega^5{}_a=\frac{\partial_a\phi}{\phi}E^5+\frac{\phi}{2}F_{ab}E^b,\quad \Omega^a{}_b=\omega^a{}_b-\frac{\phi}{2}F^a{}_bE^5\ . \tag{10}$$

Now, by using the definition of the Riemann tensor

$$R=d\Omega+\Omega\wedge\Omega \tag{11}$$

and the definition of its components

$$R^a{}_b=\tfrac{1}{2}R^a{}_{bcd}E^c\wedge E^d+R^a{}_{bc5}E^c\wedge E^5 \tag{12}$$

we find

$$\begin{aligned}{}^{(5)}R^a{}_{bcd}&={}^{(4)}R^a{}_{bcd}-\frac{\phi^2}{4}(F^a{}_bF_{cd}+F^a{}_cF_{bd})\\ &\quad+\frac{\phi^2}{4}(F^a{}_bF_{dc}+F^a{}_dF_{bc})\ ,\\ {}^{(5)}R^a{}_{bc5}&=-(D_c\phi)F^a{}_b-\frac{\phi}{2}D_cF^a{}_b\\ &\quad-\frac{F^a{}_c}{2}D_b\phi+\frac{D^a\phi}{2}F_{bc}\ ,\\ {}^{(5)}R^5{}_{b5d}&=-\frac{\phi^2}{4}F_b{}^cF_{cd}-\frac{D_dD_b\phi}{\phi}\ .\end{aligned} \tag{13}$$

From (13) we can compute the components of the Ricci tensor. Setting them to zero gives the background equations of motion $R_{AB}=0$ for the massless modes:

$$\begin{aligned}{}^{(5)}R_{bd}&\equiv{}^{(5)}R^a{}_{bad}+{}^{(5)}R^5{}_{b5d}\\ &={}^{(4)}R_{bd}-\frac{D_bD_d\phi}{\phi}+\tfrac{1}{2}\phi^2F_b{}^cF_{cd}=0\ ,\\ {}^{(5)}R_{55}&=\frac{\phi^2}{4}F_{cd}F^{cd}-\frac{D^aD_a\phi}{\phi}=0\ ,\\ {}^{(5)}R_{5c}&=-\tfrac{3}{2}(D^d\phi)F_{dc}-\frac{\phi}{2}D^dF_{dc}=0\ .\end{aligned} \tag{14}$$

Finally the curvature scalar is

$${}^{(5)}R={}^{(4)}R-\tfrac{1}{4}\phi^2F_{ab}F^{ab}-2\frac{D^aD_a\phi}{\phi} \tag{15}$$

which is the well-known action for $g_{\mu\nu}$, A_μ, and ϕ derived by Kaluza-Klein compactification of the action (1).

III. THE MASSIVE MODES

The purpose of this paper is to derive the four-dimensional action for the massive modes propagating in a massless background, and its gauge transformations as well. We will start in five dimensions. Since we will work in the vierbein formalism, we will derive the action for the massive θ-dependent modes contained in the Fourier expansion of the θ-dependent vierbein $\tilde{E}^A{}_M(x,\theta)$. We are going to treat the θ dependence as a perturbation: namely, we will write it as

$$\tilde{E}^A=E^A+\epsilon^A\ , \tag{16}$$

where E^A is the θ-independent vierbein just computed and ϵ^A is the θ-dependent part

$$\epsilon^A=\sum_{n\neq 0}\epsilon^A_{(n)}e^{in\theta}\ . \tag{17}$$

We will assume ϵ_A to be small and keep only terms linear in ϵ_A.

Similarly, for the spin connection, we write

$$\tilde{\Omega}^A{}_B=\Omega^A{}_B+\lambda^A{}_B\ , \tag{18}$$

where $\lambda^A{}_B$ is the θ-dependent part of the spin connection.

Starting from the definition of $\tilde{\Omega}^A{}_B$, namely, from

$$d\tilde{E}^{A}+\tilde{\Omega}^{A}{}_{B}\wedge\tilde{E}^{B}=0\ , \tag{19}$$

one can solve for $\lambda^{A}{}_{B}$ in the linearized approximation and derive

$$\lambda^{A}{}_{B}=(D_{B}\tilde{\epsilon}^{A}{}_{C}-D^{A}\tilde{\epsilon}_{BC})E^{C}\ . \tag{20}$$

Above $\tilde{\epsilon}^{A}{}_{C}$ is defined by $\tilde{\epsilon}^{A}=\tilde{\epsilon}^{A}{}_{C}E^{C}$, $\tilde{\epsilon}_{AB}=\eta_{AC}\tilde{\epsilon}^{C}{}_{B}$ is symmetric in its indices, and D_B is the covariant derivative in terms of the "unperturbed" spin connection $\Omega^{A}{}_{B}$ associated with the massless modes only. Precisely

$$D_{A}\tilde{\epsilon}^{B}{}_{C}=\partial_{A}\tilde{\epsilon}^{B}{}_{C}=\Omega^{B}{}_{F,A}\wedge\tilde{\epsilon}^{F}{}_{C}-\Omega^{F}{}_{C,A}\wedge\tilde{\epsilon}^{B}{}_{F}\ , \tag{21}$$

where $\Omega^{A}{}_{B}=\Omega^{A}{}_{B,C}E^{C}$. From here we can compute the correction $\hat{R}^{A}{}_{B}$ to the zeroth-order Ricci tensor $R^{A}{}_{B}$ and get the full Ricci tensor $\tilde{R}^{A}{}_{B}$:

$$\begin{aligned}&\tilde{R}^{A}{}_{B}=R^{A}{}_{B}+\hat{R}^{A}{}_{B},\quad \hat{R}^{A}{}_{B}=E^{L}D_{L}\lambda^{A}{}_{B}\ ,\\&\hat{R}^{A}{}_{BCD}=D_{C}\lambda^{A}{}_{B,D}-D_{C}\lambda^{A}{}_{B,C}\ ,\end{aligned} \tag{22}$$

and finally

$$\tilde{R}_{AB}=(\eta_{AE}+\tilde{\epsilon}_{AE})(R^{E}{}_{BCD}+\hat{R}^{E}{}_{BCD})(\eta^{BD}-\tilde{\epsilon}^{BD})\ .$$

Assuming that the background metric obeys $R_{AB}=0$, the condition that $\tilde{R}_{AB}=0$ to lowest order in $\tilde{\epsilon}$ is

$$D^{C}D_{C}\tilde{\epsilon}_{AB}-D^{C}D_{B}\tilde{\epsilon}_{CA}-D^{C}D_{A}\tilde{\epsilon}_{CB}+D_{A}D_{B}\tilde{\epsilon}=0\ , \tag{23}$$

where $\tilde{\epsilon}$ is the trace $\eta^{AB}\tilde{\epsilon}_{BA}$.

Notice that the above equation is nothing but the well-known linearized equation for the metric perturbation h_{MN} (Ref. 10). In fact it reduces to it if $\tilde{\epsilon}_{AB}=E_{A}{}^{M}E_{B}{}^{N}h_{MN}$. It is convenient to make the change of variables $\tilde{\epsilon}_{AB}\to\epsilon_{AB}=\tilde{\epsilon}_{AB}-\frac{1}{2}\tilde{\epsilon}\eta_{AB}$ since then the field equations (23) become symmetric in A and B,

$$D^{C}D_{C}\epsilon_{AB}-D^{C}D_{A}\epsilon_{CB}-D^{C}D_{B}\epsilon_{CA}-\tfrac{1}{3}\eta_{AB}D^{C}D_{C}\epsilon=0\ , \tag{24}$$

and obviously following from the Lagrangian density

$$\begin{aligned}\mathcal{L}=\epsilon^{*AB}(&D^{C}D_{C}\epsilon_{AB}-D^{C}D_{A}\epsilon_{CB}\\&-D^{C}D_{B}\epsilon_{CA}-\tfrac{1}{3}\eta_{AB}D^{C}D_{C}\epsilon)\ ,\end{aligned} \tag{25}$$

or equivalently from

$$\begin{aligned}\mathcal{L}=&(D^{C}\epsilon^{*AB})(D_{C}\epsilon_{AB})-2(D^{C}\epsilon^{*AB})(D_{A}\epsilon_{CB})\\&-\tfrac{1}{3}(D^{C}\epsilon^{*})(D_{C}\epsilon)\ .\end{aligned} \tag{26}$$

Notice that because of the θ integration, in the linearized approximation, the modes in the Fourier expansion of ϵ_{AB},

$$\epsilon_{AB}=\sum_{n}e^{in\theta}\epsilon^{(n)}_{AB}\ ,$$

are decoupled. As is well known,[8,11,12] these modes are massive, with mass proportional to $|n|$, and charged, with charge n. Electromagnetic gauge invariance is automatically built in the covariant derivative D_A. Electromagnetic gauge invariance follows from the five-dimensional symmetry

$$\theta\to\theta+f,\quad x\to x'=x \tag{27}$$

which implies that

$$A_{\mu}\to A_{\mu}-\partial_{\mu}f\ . \tag{28}$$

Under this transformation, $\epsilon^{(n)}_{AB}$ transforms as

$$\epsilon^{(n)}_{AB}\to\epsilon^{(n)}_{AB}e^{-inf}\ .$$

The derivative $\partial_{a}\epsilon^{(n)}_{AB}=e_{a}{}^{\mu}(\partial_{\mu}-inA_{\mu})\epsilon^{(n)}_{AB}$ equals the usual four-dimensional gauge-covariant derivative for the charged field $\epsilon^{(n)}$. The field $\epsilon^{(n)}$ also possesses an additional gauge invariance, whose spontaneous breaking can be regarded as the origin of the mass of this field;[13] our purpose here is to elucidate this.

IV. THE MASSIVE SPIN-2 FIELD GAUGE INVARIANCE

It can be shown[6] that, if the background equation of motion (14) are satisfied, that is, if the massless background fields are on shell, then (23) is invariant under the gauge transformation

$$\tilde{\epsilon}'_{AB}=\tilde{\epsilon}_{AB}-D_{A}\xi_{B}-D_{B}\xi_{A}\ , \tag{29}$$

where ξ_B is an infinitesimal parameter. Equivalently (24) is invariant under the gauge transformation

$$\epsilon'_{AB}=\epsilon_{AB}-D_{A}\xi_{B}-D_{B}\xi_{A}+\eta_{AB}D^{C}\xi_{C} \tag{30}$$

obtained by expressing (29) in terms of the new variables $\epsilon_{AB}=\tilde{\epsilon}_{AB}-\frac{1}{2}\tilde{\epsilon}\eta_{AB}$. Notice that the invariance (29) is a "piece" of the general coordinate invariance in five dimensions:

$$g'_{MN}=g_{MN}-\mathcal{D}_{M}\xi_{N}-\mathcal{D}_{N}\xi_{M}\ . \tag{31}$$

Here $\mathcal{D}_M$ is different from $D_M=E^{A}{}_{M}D_{A}$, since D_M is the covariant derivative associated with the spin connection $\Omega^{A}{}_{B}$, i.e., with the θ-independent massless fields only, while in the equation of general covariance $\mathcal{D}_M$ is the full covariant derivative, i.e., associated with the full metric $g_{MN}=\sum_{n}g^{(n)}_{MN}(x)e^{in\theta}$ (Ref. 13).

We will take advantage of this gauge invariance to prove that (23) is the equation for a massive spin-2 particle interacting with gravitational and electromagnetic background. Reduction to four dimensions will provide consistent interacting equations and Lagrangians for spin-2 massive particles. To this purpose we will make use of the gauge transformation (30) to reduce the 15 components of ϵ_{AB} down to five propagating components. We will see that we can impose the usual condition of vanishing divergence $D^{A}\epsilon_{AB}=0$. These are constraints that a massive spin-2 field needs to obey in any dimension. In five dimensions the above constraints reduce the number of components from 15 down to 10. Moreover a residual gauge symmetry which follows from (30) will allow us to impose five more constraints $\epsilon=0$ and $\epsilon_{a5}=0$. These constraints will also imply $\epsilon_{55}=0$. Therefore, we are reduced to five propagating components only. The ϵ_{A5} fields have been absorbed into the surviving spin-2 fields to provide them masses via the

Higgs mechanism.

In more detail, let us first choose a gauge such that $D^A\epsilon'_{AB}=0$; i.e., from (30) let us choose a ξ_A such that

$$D^A\epsilon'_{AB}=D^A\epsilon_{AB}-D^AD_A\xi_B-D^AD_B\xi_A+D_BD^C\xi_C=0\ . \tag{32}$$

Notice that the last two terms cancel by permuting the order of covariant derivatives of one of the terms and using the equations of motion for the massless background, so that (32) is simply

$$0=D^A\epsilon'_{AB}=D^A\epsilon_{AB}-D^AD_A\xi_B\ . \tag{33}$$

Obviously (33) can be satisfied with a residual gauge invariance left over such that $D^AD_A\xi_B=0$. If the massless background satisfies its equation of motion, this residual gauge invariance suffices to impose the tracelessness condition $\epsilon'=0$:

$$\epsilon'=\epsilon+3D^C\xi_C=0\ . \tag{34}$$

This determines one of the gauge parameters, say, ξ_5, completely.

Notice that the equation of motion (24) and the constraint (33) imply (if the background equation of motion $R_{AB}=0$ is satisfied, in which case we can commute covariant derivatives), that $D^AD_A\epsilon=0$. This is compatible with (34) only if the gauge parameter ξ_C obeys the residual gauge condition $D^AD_A\xi_C=0$. Moreover, we are now still free to make additional gauge transformations provided that $D^AD_A\xi_B=D^A\xi_A=0$. Indeed, we choose our last gauge condition to be

$$\epsilon'_{AB}C^B=\epsilon_{AB}C^B-(D_A\xi_B+D_B\xi_A)C^B=0\ , \tag{35}$$

where $C^A=C_A=(0,0,0,0,\phi)$ is a Killing vector in the background space

$$D_AC_B+D_BC_A=0\ . \tag{36}$$

If we take the divergence of this equation, use the previous gauge choice $D^A\epsilon_{AB}=0$, the fact that C^A is a Killing vector, and the equation of motion for the massless background, we see that the residual gauge conditions $D^AD_A\xi_B=D^B\xi_B=0$ are indeed required by compatibility. Actually we can use (35) only to solve for the vanishing of four of the quantities, say, $\epsilon_{aB}C^B$, because we have already determined ξ_5 completely. This implies $\epsilon_{a5}=0$ [since it follows from (2) that ϕ cannot be zero]. However, it is easy to see that if $\epsilon_{a5}=0$ and $D^A\epsilon_{AB}=0$, then $\epsilon_{55}=0$ as well. Indeed

$$\begin{aligned}0=D^A\epsilon_{A5}&=\partial^A\epsilon_{A5}-\Omega^C{}_{A,}{}^A\epsilon_{C5}-\Omega^C{}_{5,}{}^A\epsilon_{AC}\\&=\partial^5\epsilon_{55}=\frac{in}{\phi}\epsilon_{55}=0\ .\end{aligned}$$

The main ingredient in the above derivation is the fact that follows from (10) that $\Omega^5{}_{a,b}$ is antisymmetric in its indices a and b. The complete gauge choice is therefore

$$D^A\epsilon_{AB}=0,\quad \epsilon=0,\quad \epsilon_{A5}=0\ . \tag{37}$$

Again, the 15 degrees of freedom have been reduced to five propagating degrees of freedom, as is appropriate for a spin-2 field in four dimensions. We will now write some of these relations in four-dimensional terms. To this purpose we express the five-dimensional covariant derivative in terms of four-dimensional covariant derivatives. For instance,

$$\begin{aligned}{}^{(5)}D_a\xi_b&=\partial_a\xi_b-\Omega^c{}_{ba}\xi_c-\Omega^5{}_{ba}\xi_5\\&={}^{(4)}D_a\xi_b+\frac{\phi}{2}F_{ab}\xi_5\ ,\\{}^{(5)}D^A\xi_A&=\partial^a\xi_a+\partial^5\xi_5-\Omega^a{}_{b,}{}^b\xi_a-\Omega^a{}_{5,}{}^5\xi_a-\Omega^5{}_{a,}{}^a\xi_5\\&=D^a\xi_a+\frac{in}{\phi}\xi_5+\frac{D^b\phi}{\phi}\xi_b\ .\end{aligned} \tag{38}$$

The covariant derivatives on the right are now four-dimensional covariant derivatives. Proceeding this way, calling $\xi_5=\xi$ we write the gauge transformations (30) in four-dimensional terms

$$\begin{aligned}\epsilon'_{ab}&=\epsilon_{ab}-D_a\xi_b-D_b\xi_a+\eta_{ab}\left[D^c\xi_c+\frac{in\xi}{\phi}+\frac{D^c\phi}{\phi}\xi_c\right]\ ,\\\epsilon'_{a5}&=\epsilon_{a5}-D_a\xi-\frac{\phi}{2}F^c{}_a\xi_c-\frac{in}{\phi}\xi_a+\frac{D_a\phi}{\phi}\xi\ ,\\\epsilon'_{55}&=\epsilon_{55}-\frac{in}{\phi}\xi-\frac{D^c\phi}{\phi}\xi_c+D^c\xi_c\ .\end{aligned} \tag{39}$$

We can also return to space-time indices by using the four-dimensional vierbein $e^a{}_\mu$ by introducing the four-dimensional tensor $\epsilon_{\mu\nu}=e^a{}_\mu e^b{}_\nu\epsilon_{ab}$, the four-dimensional vector $B_\mu=e^a{}_\mu\epsilon_{a5}$, and the scalar $\psi=\epsilon_{55}$. In terms of these fields the four-dimensional gauge invariance reads

$$\begin{aligned}\epsilon'_{\mu\nu}&=\epsilon_{\mu\nu}-D_\mu\xi_\nu-D_\nu\xi_\mu+g_{\mu\nu}\left[D^\rho\xi_\rho+\frac{in}{\phi}\xi+\frac{D^\rho\phi}{\phi}\xi_\rho\right]\ ,\\B'_\mu&=B_\mu-D_\mu\xi-\frac{\phi}{2}F^\nu{}_\mu\xi_\nu-\frac{in}{\phi}\xi_\mu-\frac{D_\mu\phi}{\phi}\xi\ ,\\\psi'&=\psi-\frac{in\xi}{\phi}-\frac{D^\mu\phi}{\phi}\xi_\mu+D^\mu\xi_\mu\ ,\end{aligned} \tag{40}$$

where $D_\mu=\nabla_\mu-inA_\mu$ and ∇_μ is the ordinary four-dimensional covariant derivatives for a space with metric $g_{\mu\nu}$. As said, these equations are manifestly invariant under the electromagnetic gauge transformation $A_\mu\to A_\mu-\partial_\mu f,\xi\to\xi e^{-inf}$. The transformations (40) are very similar to the gauge transformations of the spin-2 massive multiplet in the bosonic string.[14,9] The gauge condition $\epsilon_{A5}=\epsilon=0$ yields the four-dimensional gauge conditions $B_\mu=0$, $\psi=0$, and $\epsilon=\epsilon^\mu{}_\mu=0$. With these choices the remaining gauge condition becomes $D^\mu(\phi\epsilon_{\mu\nu})=0$. The tensor $\phi\epsilon_{\mu\nu}$ has zero trace and vanishing covariant divergence and, hence, corresponds to a propagating field having five degrees of freedom as required for a massive spin-2 field.

V. THE LAGRANGIAN

The gauge-invariant Lagrangian density (25) can be translated in four-dimensional language by expressing the five-dimensional derivatives in terms of four-dimensional ones as in (38). A very lengthy calculation gives the following result (to be integrated over $\sqrt{-{}^{(4)}g}\,\phi\,d^4x$):

$$
\begin{aligned}
\mathcal{L}=\;& D^\rho\epsilon^{*\mu\nu}D_\rho\epsilon_{\mu\nu}+2\,D^\mu B^{*\nu}D_\mu B_\nu+\tfrac{2}{3}D^\mu\psi^*D_\mu\psi-\tfrac{1}{3}D^\mu\epsilon^*D_\mu\epsilon-2\,D^\mu\epsilon^{*\nu\rho}D_\nu\epsilon_{\mu\rho}-2\,D^\mu B^{*\nu}D_\nu B_\mu\\
&+4\frac{D_\mu\phi}{\phi}\mathrm{Re}(B_\nu D^\mu B^{*\nu})+4\frac{D_\mu\phi}{\phi}\mathrm{Re}(B_\nu D^\nu B^{*\mu})-\tfrac{2}{3}\mathrm{Re}(D^\mu\epsilon^*D_\mu\psi)-4\phi F^{\mu\nu}\mathrm{Re}(\epsilon_{\rho\mu}D^\rho B^*_\nu)\\
&-4\phi F^{\mu\nu}\mathrm{Re}(B^{*\rho}D_\nu\epsilon_{\mu\rho})-4\frac{D^\mu\phi}{\phi}\mathrm{Re}(D^\nu\psi^*\epsilon_{\mu\nu})-\frac{4n}{\phi}\mathrm{Im}(\epsilon_{\mu\nu}D^\mu B^{*\nu})+\frac{4n}{\phi}\mathrm{Im}(B_\mu D^\mu\psi^*)\\
&+4\frac{D_\mu\phi}{\phi}\mathrm{Re}[(D^\mu\psi^*)\psi]+\frac{n^2}{\phi^2}\epsilon^{*\mu\nu}\epsilon_{\mu\nu}-\frac{1}{3}\frac{n^2}{\phi^2}\epsilon^*\epsilon-\left[\frac{4}{3}\frac{n^2}{\phi^2}-\phi^2F^{\alpha\beta}F_{\alpha\beta}\right]\psi^*\psi\\
&+\left[2\left[\frac{D\phi}{\phi}\right]^2+\phi^2F_{\alpha\beta}F^{\alpha\beta}\right]B^{*\mu}B_\mu-2\frac{D^\mu\phi}{\phi}\frac{D^\nu\phi}{\phi}B^{*\mu}B^\nu+\phi^2F_{\mu\nu}F_{\mu'\nu'}\epsilon^{*\nu\nu'}\epsilon^{\mu\mu'}\\
&-\frac{2}{3}\frac{n^2}{\phi^2}\mathrm{Re}(\epsilon^*\psi)+\frac{4n}{\phi}F_{\mu\nu}\mathrm{Im}(B^{*\mu}B^\nu)+4\,D_\mu\phi F_\nu{}^\rho\mathrm{Re}(\epsilon^{*\mu\nu}B_\rho)\\
&+4n\frac{D^\mu\phi}{\phi^2}\mathrm{Im}(\epsilon_{\mu\nu}B^{*\nu})+4n\frac{D^\mu\phi}{\phi^2}\mathrm{Im}(\psi B^*{}_\mu)+4\,D_\nu\phi F^{\mu\nu}\mathrm{Re}(\psi B^*{}_\mu)+2\phi^2F_{\mu\alpha}F^\alpha{}_\nu\mathrm{Re}(\psi^*\epsilon^{\mu\nu})\;. \qquad (41)
\end{aligned}
$$

This rather formidable looking Lagrangian can be reduced to manageable proportions in the case where the massless background obeys $A_\mu=0$, $\phi=1$ (so that we are considering the propagation of a massive spin-2 field in a gravitational background only). In this case, we get

$$
\begin{aligned}
\mathcal{L}=\;&(\nabla^\rho\epsilon^{*\mu\nu})(\nabla_\rho\epsilon_{\mu\nu})+n^2\epsilon^{*\mu\nu}\epsilon_{\mu\nu}-2(\nabla^\rho\epsilon^{*\mu\nu})(\nabla_\mu\epsilon_{\rho\nu})-\tfrac{1}{3}(\nabla^\mu\epsilon^*)(\nabla_\mu\epsilon)-\tfrac{1}{3}n^2\epsilon^*\epsilon+2\nabla^\mu B^{*\nu}\nabla_\mu B_\nu-2\nabla^\mu B^{*\nu}\nabla_\nu B_\mu\\
&+\tfrac{2}{3}\nabla^\mu\psi^*\nabla_\mu\psi-\tfrac{2}{3}\mathrm{Re}(\nabla^\mu\epsilon^*\nabla_\mu\psi)-4n\;\mathrm{Im}(\epsilon_{\mu\nu}\nabla^\mu B^{*\nu})+4n\;\mathrm{Im}(B_\mu\nabla^\mu\psi^*)-\tfrac{4}{3}n^2\psi^*\psi-\tfrac{2}{3}n^2\mathrm{Re}(\epsilon^*\psi)\;. \qquad (42)
\end{aligned}
$$

By varying this Lagrangian and then imposing the gauge conditions $B_\mu=\psi=\epsilon=0$, one gets the equation

$$\nabla^\rho\nabla_\rho\epsilon_{\mu\nu}-n^2\epsilon_{\mu\nu}+2R_{\rho\mu\sigma\nu}\epsilon^{\rho\sigma}=0 \qquad (43)$$

which can be also obtained from a four-dimensional transcription of our earlier equation (24). In the absence of the gravitational field this equation obviously describes a free propagating massive spin-2 field. Notice that if there was no mass term, (43) would be the four-dimensional equation of linearized gravity in Lorentz gauge and it would have causal propagation.[10] The mass term, of the right sign, indeed does not alter this property. More generally, since our theory is obtained by compactifying a five-dimensional causal theory, the compactified theory should be causal as well.

Our results can be reexpressed as follows. We have explicitly derived a Lagrangian for a massive spin-2 field interacting with an electromagnetic, gravitational, and scalar field. This Lagrangian $\mathcal{L}$ is invariant under the gauge transformations (40) as long as the background fields are on shell, i.e., if Eqs. (14) are obeyed. This gauge invariance is essential in showing the consistency of the theory. Of course, this is not a fully consistent formalism, since Eqs. (14) do not include the back reaction of the massive fields on the massless ones. This is the price we pay for working only to lowest nontrivial order in the massive fields. Working to lowest nontrivial order in the massive fields has permitted us to write a four-dimensional action for just one massive field which is probably more or less as simple as possible and which is consistent in lowest nontrivial order but not exactly.

The situation can be clarified by modifying the gauge transformation laws somewhat so as to permit the background to be off shell. Indeed the invariance (30), the five-dimensional version of (40), is a piece of the five-dimensional general coordinate invariance (31) which holds for the action (1). Therefore, even if the massless background is off shell, the total five-dimensional action must be invariant under (30). This total five-dimensional action includes the action $\mathcal{L}$ for the θ-dependent mode plus the action $R^{(0)}$ built with the θ-independent piece of the five-dimensional metric $g^{(0)}_{MN}$. [To avoid confusion, we have reintroduced here the superscript (0) to indicate the θ-independent modes.] Of course, we must take $g^{(0)}_{MN}$ to obey the appropriate transformation laws derivable from (31). More explicitly, if we vary the action

$$I=\int d^5x\sqrt{-g^{(0)}}(R^{(0)}+\mathcal{L}) \qquad (44)$$

under (31), we obtain

$$\delta I=\int d^5x\sqrt{-g^{(0)}}(R^{(0)}_{MN}-\tfrac{1}{2}g^{(0)}_{MN}R^{(0)})\delta g^{MN}_{(0)}+\int d^5x\sqrt{-g^{(0)}}(R^{(0)}_{MN}-\tfrac{1}{2}g^{(0)}_{MN}R^{(0)})Z^{MN}+\int d^5x\frac{\delta}{\delta g^{MN}_{(0)}}(\sqrt{-g^{(0)}}\mathcal{L})\delta g^{MN}_{(0)} \quad (45)$$

for some Z_{MN}. The first term comes from varying $\sqrt{-g^{(0)}}R^{(0)}$ and the last two from varying $\sqrt{-g^{(0)}}\mathcal{L}$. The ability to write $\delta\mathcal{L}$ in the above form is a restatement of the fact that $\delta\mathcal{L}$ is zero under (30) if the equations of the massless background are obeyed (as they have been until this point). If we want to release the request that the massless fields are on shell, we can supplement the gauge transformation law (30) with

$$\delta g^{(0)}_{MN}=-Z_{MN}\ . \quad (46)$$

Then the first two terms in (45) vanish. It can be checked that the first two terms in (45) are of order $\xi\cdot\epsilon$ where ξ is the gauge parameter and ϵ is the strength of the massive field, while the third term is of order $\xi\cdot\epsilon^3$. The point is that by supplementing (30) with (46), one has gauge invariance *off shell* (without requiring the background field equations to be obeyed) up to terms of order $\xi\cdot\epsilon^3$. If the background is *on shell*, one achieves gauge invariance to the same order without the need for (46). This is related to the fact that the back reaction of the massive fields on the massless background does not make its appearance at order $\xi\cdot\epsilon$.

The question now arises of whether by adding additional terms to the Lagrangian and/or transformation laws, one can obtain full off-shell gauge invariance (and thus consistency) not just to order $\xi\cdot\epsilon^3$ but exactly. Kaluza-Klein theory gives one way to do this, and string theory gives another. Both of these solutions to the problem involve introducing an infinite tower of additional fields. One might wish to ask whether the problem has a solution without introducing such an infinite tower. One might guess that such a solution does not exist, but this is not entirely clear. For recent work on the subject we refer to Refs. 15 and 16 and references therein.

Note added. In a report motivated by this paper, M. J. Duff, C. N. Pope, and K. S. Stelle [Report No. CTP-TAHO-14/89 (unpublished)] restate the arguments why Kaluza-Klein compactification will necessarily break massive spin-2 gauge invariance, if the infinite tower of massive fields is truncated to a finite set. This is true, of course, but does not settle the question of what happens if one considers a deviation from straightforwardly truncated Kaluza-Klein compactification.

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