Quantum states of a field partitioned by an accelerated frame

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(Received 24 November 1987; revised manuscript received 10 August 1988)

The discrete Minkowski Bessel mode technology is developed. This development includes (i) the eigenvalue spectrum, (ii) the phase-space integral formulation of the quantized Klein-Gordon (KG) field in accelerated frames with a finite bottom ($b>0$), and (iii) the procedure for making the transition to bottomless ($b=0$) frames. This technology is used to show that the Rindler vacuum state of an acceleration-partitioned real KG field is perpendicular not only to the Minkowski vacuum state, but also to every quantum state associated with a "detector" accelerating through what initially was the Minkowski vacuum. In fact, every element of the Hilbert space containing the Rindler vacuum is perpendicular to the Hilbert space containing the Minkowski vacuum.

I. INTRODUCTION AND MOTIVATION

Unruh,[1] as well as others,[2,3] considered elementary quantum-mechanical processes involving an accelerated "detector" interacting with a wave field. The "elementary" nature of their processes consisted of the fact that only a countable number of quanta (in their case one or two) were involved in each interaction process.

Although the Minkowski vacuum is a thermal state when viewed relative to a uniformly accelerated frame, they found that there is no inconsistency[2] in the quantum-mechanical description relative to an inertial frame as compared to the description relative to the accelerated frame. Their result, when formulated in terms of transition probabilities, states that the transition probability for their system (a uniformly accelerated detector plus wave field) is independent of the frame of reference.

In other words, inertial and accelerated observers make consistent, i.e., equivalent, predictions. Consider a system consisting of the "detector" interacting with a wave field, and assume that this system makes a transition from state $|\psi_A\rangle$ to state $|\phi_A\rangle$ as described by the accelerated observer A. Similarly assume that the same system makes a transition from state $|\psi_I\rangle$ to state $|\phi_I\rangle$ as viewed by the inertial observer I. Stated mathematically, the equivalence of the outcome of the gedanken experiments asserts that the transition probabilities relative to A and I satisfy

$$|\langle \phi_A|\psi_A\rangle|^2=|\langle \phi_I|\psi_I\rangle|^2 \ . \tag{1.1}$$

It is a theorem due to Wigner[4] that the correspondence $\psi_A \rightarrow \psi_I$ due to the change of observers $A\rightarrow I$ can be expressed by means of a unitary (or antiunitary) transformation U:

$$|\psi_A\rangle = U|\psi_I\rangle \ .$$

This transformation is well known.[5] Its existence implies that the A and I descriptions of an elementary quantum process can be achieved in Hilbert spaces which are essentially the same. In fact, one may visualize the two descriptions of the elementary process in one and the same Hilbert space. Then the change in coordinate systems $A\rightarrow I$ is represented by the unitary change of basis U in that Hilbert space, which in the above-mentioned gedanken experiments contains the Minkowski vacuum state.

The purpose of this paper is to point out that the result obtained by the above-mentioned workers depends in an essential way on the "elementary" nature of the processes considered by them.

Let $|0_R\rangle$ be an element corresponding to the wave field in its ground state relative to the accelerated coordinate frame. This state is variously called the "Rindler" vacuum, or "condensed" vacuum state, of the acceleration-partitioned wave field. We shall show that this state has the property that

$$\langle 0_R|\Psi_A\rangle = 0 \ . \tag{1.2}$$

In other words, the vacuum state, and hence all excited states obtained from it by letting polynomials of creation operators act on it, lie outside the set of quantum states considered by Unruh and others.

Put differently, the Rindler ("condensed") vacuum of an accelerated frame determines a Hilbert space of quantum states which is distinct from Hilbert space determined by the Minkowski vacuum. There is no unitary transformation which connects elements in these two spaces.

The orthogonality of two vacuum states can only occur in the framework of the quantum mechanics of an infinite system,[6] such as a Klein-Gordon system in an infinite volume. By contrast, a system consisting of a finite number of degrees of freedom, or even a system consisting of an infinite number confined to a finite box (but having, say, a finite particle density) does not have orthogonal vacuum states. In fact, it is the infinite-volume limit, also called the 'thermodynamic" limit, which makes possible the orthogonality of two vacuum states and their corresponding Hilbert spaces. Such an orthogonality is well known in condensed-matter physics[6,7] such as superfluidity, superconductivity, and so on. We shall show that the set of quantum states of an acceleration partitioned Klein-Gordon (KG) field is no exception.

Our objective is to exhibit the condensed ("Rindler") vacuum state for a KG field in an accelerated frame with a finite ($b>0$) bottom. This is done in Sec. IV. A comparison and contrast of (a) this condensed state and its excited states with (b) the Minkowski vacuum and its excited states is given in Sec. V. The necessary discrete-mode technology is developed in Sec. III.

This technology is new. It corresponds to the procedure by which discrete sums of plane-wave modes in a finite inertial cavity are changed into phase-space integrals. Here such integrals are developed for a pair of symmetrically placed cavity walls accelerating into opposite directions.

Section VI consists of three remarks that place the results of this paper into the realm of condensed-matter physics.

II. OBSERVERS AND COORDINATE FRAMES

The set of timelike world lines, with or without acceleration, is partitioned quite naturally into mutually exclusive and jointly exhaustive equivalence classes. Two timelike world lines lie in the same equivalence class if and only if the intersection of the causal past and causal future of one world line equals that of the other.

An equivalence class determines a unique space-time neighborhood whose boundaries are null surfaces. For example, the equivalence class determined by a world line with uniform acceleration is a space-time region which is bounded by the future and past event horizons of this world line.

There are many world lines that yield the same space-time neighborhood but the observers which trace out these world lines are all equivalent: any one lies in the causal past or causal future of the other. Thus, one world line can communicate acquired data to any other. Consequently the equivalence of the world lines extends to the physical viewpoint of acquiring measured data about the space-time neighborhood.

One therefore can, and we shall, designate arbitrarily one of the world lines as that of the "observer," while the others as those of his "assistants."

A coordinatization of the space-time neighborhood is a "coordinate frame." In mathematics it is called a "coordinate chart." We, however, would like to use the physics label "coordinate frame" because it conjures up a lattice of meter rods and clocks. They are necessary for the measurement of the properties of the field in the space-time neighborhood. A "coordinate frame" is not to be confused with a "tangent frame," which is defined only at an event.

As an example, recall an "inertial coordinate frame." Physically it consists of a global lattice of meter rods and clocks. An "inertial observer" would be one who is located at any one of these lattice sites. The other lattice sites serve as locations for his assistants. They are his eyes and ears by which he measures physical quantities at the space-time events of the lattice. From the viewpoint of physical measurements it is sufficient that only a single observer be assigned to some lattice of the given (inertial) coordinate frame. Thus one can say that there is a one-to-one relationship between inertial "coordinate frames" and inertial "observers."

As another example consider an accelerated coordinate frame and its accelerated observer. The definition is analogous. In fact, we shall consider a pair of frames and the corresponding pair of observers. We shall label each of these two frames by I and II. Each consists of a lattice work of accelerated robust clocks whose world lines are given by

$$\begin{aligned} t &= \pm\xi \sinh\tau, \quad 0<\xi , \\ x &= \pm\xi \cosh\tau , \\ y &= y, \quad z = z . \end{aligned} \tag{2.1}$$

Here + and − refer to the two frames, I and II, respectively. The ξ, y, and z held constant, characterize a lattice point. The lattice points trace out a congruence of hyperbolic world lines. Proper time and space intervals in each frame are measured by referring to the metric whose form relative to this coordinate frame is

$$ds^2 = -\xi^2 d\tau^2 + d\xi^2 + dy^2 + dz^2 . \tag{2.2}$$

The "accelerated observer" (for coordinate frame I, say) is assigned to one of the lattice sites, say $0<\xi=g^{-1}$, $y=y_0$, $z=z_0$, so that the proper acceleration at this site is g. The remaining lattice sites of frame I are occupied by his robust assistants. Their world lines lie in the intersection of the causal past and future of the observer. Their purpose is to make physical measurements whose results are transmitted along time or null lines to the observer. The intersection of the observer's past and future is the space-time of Rindler coordinate chart I ($|t|<x$). Rindler chart II ($|t|<-x$) is associated with the other hyperbolic world lines in Eq. (2.1), but the metric has the same form Eq. (2.2). The two charts I and II are causally disjoint: observer I cannot communicate with observer II.

An inertial observer is, however, different from either accelerated observers. The inertial observer traces out a straight line (e.g., x,y,z held constant) in space-time. The intersection of his causal past and future contain those of I and II. Thus, unlike an accelerated observer, he can make measurements, i.e., collect measurement data, both in I and in II.

III. NORMAL MODE EXPANSION FOR THE PARTITIONED SYSTEM

The unusual feature of a uniformly accelerated frame is that it induces a relativistic wave field to be partitioned into two causally disjoint subsystems. The natural harmonics for the Klein-Gordon (KG) field partitioned in this way are the two types of Minkowski Bessel (MB) modes:[8]

$$B_\omega^\pm(kU,kV) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[\mp ik(Ue^{\theta}+Ve^{-\theta})/2]e^{-i\omega\theta}d\theta . \tag{3.1}$$

Here

$$U=t-x,\quad V=t+x\ ,$$
$$k=(k_y^2+k_z^2+m^2)^{1/2}>0\ ,$$

and the + and − superscripts refer to positive and negative Minkowski frequency functions. In terms of these modes the real KG field has the form

$$\psi(x)=\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d^2k\left[a_{\omega k}B_\omega^+\frac{e^{i(k_y y+k_z z)}}{2\pi}+a_{\omega k}^\dagger B_{-\omega}^-\frac{e^{-i(k_y y+k_z z)}}{2\pi}\right].\tag{3.2}$$

This is an expansion corresponding to a continuous mode spectrum. We also would like to have a discrete mode sum expansion, very much like a plane-wave mode expansion in a finite volume. This will be done below after this subsection, and the impatient reader can proceed there directly. The purpose of the following subsection is to show how a certain peculiarity of the MB modes leads directly to the "condensed" ("Rindler") vacuum state.

A. Degenerate expansions

The peculiar property of the MB modes is this— *the Rindler sector* I (*or* II) *coordinate representatives of* B_ω^+ *and* B_ω^- *form a linearly dependent pair.* In other words, when the domain of definitions B_ω^+ and B_ω^- is restricted to accelerated frame I (or II) then the pair of positive and negative Minkowski frequency functions $B_\omega^\pm$ become linearly dependent. This is so despite the fact that $B_\omega^\pm$ are independent on I∪II. Compare Figs. 1(a) with 1(b). Actually such a behavior is not considered unusual in mathematics. For example, the two functions $f(x)=|x|$ and $g(x)=x$ are linearly independent on $-\infty<x<\infty$, but they are linearly dependent relative to the restricted domain $0\le x<\infty$.

The unusual feature of this linear dependence is that it prevails throughout the space-time of frame I (or II). Consequently there is an infinite degeneracy in the manner that the KG field can be expanded in terms of the MB modes. This expansion degeneracy is inherited by the whole quantum state structure of the KG system. In particular there is a corresponding infinite degeneracy in the "ground" states associated with each of the respective MB mode expansions. The resulting task is therefore this: what criteria does one use to pick out "the" ground state? Unruh gave one, and it resulted in the "Rindler" vacuum. We shall give another, and it also yields the "Rindler" vacuum.

The linear dependence of

$$B_\omega^\pm=\frac{1}{\pi}K_{i\omega}(k\xi)e^{-i\omega\tau}\times\begin{cases}e^{\pm\pi\omega/2} & \text{in I ,}\\ e^{\mp\pi\omega/2} & \text{in II ,}\end{cases}\tag{3.3}$$

when restricted to I or II implies a degeneracy in the expansion of the KG field in terms of these modes. From the standard expansion, Eq. (3.2), i.e.,

$$\psi(x)=\frac{1}{\sqrt{2}}\int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\omega\,d^2k\left[(a_{\omega k}B_\omega^+ +a_{-\omega-k}^\dagger B_\omega^-)\frac{e^{i(k_y y+k_z z)}}{2\pi}+(a_{\omega k}^\dagger B_{-\omega}^- +a_{-\omega-k}B_{-\omega}^+)\frac{e^{-i(k_y y+k_z z)}}{2\pi}\right],\tag{3.4}$$

one obtains by replacing B_ω^+ and B_ω^- with

$$u_\omega B_\omega^+ - v_\omega B_\omega^- \equiv R_{\omega 1},\quad -v_\omega B_\omega^+ + u_\omega B_\omega^- \equiv R_{\omega 2}^*\tag{3.5}$$

the new expansion

$$\psi(x)=\frac{1}{\sqrt{2}}\int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\omega\,d^2k\left[(A_{\omega k}R_{\omega 1}+A_{-\omega-k}^\dagger R_{\omega 2}^*)\frac{e^{i(k_y y+k_z z)}}{2\pi}+(A_{\omega k}^\dagger R_{\omega 1}^*+A_{-\omega-k}R_{\omega 2})\frac{e^{-i(k_y y+k_z z)}}{2\pi}\right].\tag{3.6}$$

Here the new expansion coefficients are related to the old ones by

$$A_{\omega k}=u_\omega a_{\omega k}+v_\omega a_{-\omega-k}^\dagger,\quad A_{-\omega-k}^\dagger=v_\omega a_{\omega k}+u_\omega a_{-\omega-k}^\dagger\ .\tag{3.7}$$

The constants u_ω and v_ω are quite arbitrary, except that

$$u_\omega^2-v_\omega^2=1\ .$$

This guarantees that Eqs. (3.4) and (3.6) are equal. To guarantee that the commutation relations and hence the Klein-Gordon ("Wronskian") inner product be preserved by the transformations (3.5) and (3.7), we demand without loss of generality that u_ω and v_ω are real. Consequently one parametrizes the coefficients by the parameter α_ω,

$$u_\omega(\alpha)=\cosh\alpha_\omega,\quad 0<\omega<\infty;\quad v_\omega(\alpha)=\sinh\alpha_\omega\ ,\tag{3.8}$$

and one may speak of an α_ω-parametrized collection of degenerate expansions. A typical member is given by

$$\psi(x)=\frac{1}{\sqrt{2}}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty d\omega\, d^2k\left[A_{\omega k}[\alpha]R_{\omega 1}\frac{e^{i(k_y y+k_z z)}}{2\pi}+A_{-\omega-k}[\alpha]R_{\omega 2}\frac{e^{-i(k_y y+k_z z)}}{2\pi}+\text{c.c.}\right] . \tag{3.9a}$$

Here the operators

$$\begin{aligned}A_{\omega k}[\alpha]&=a_{\omega k}\cosh\alpha_\omega+a^\dagger_{-\omega-k}\sinh\alpha_\omega ,\\ A_{-\omega-k}[\alpha]&=a_{-\omega-k}\cosh\alpha_\omega+a^\dagger_{\omega k}\sinh\alpha_\omega\end{aligned} \tag{3.9b}$$

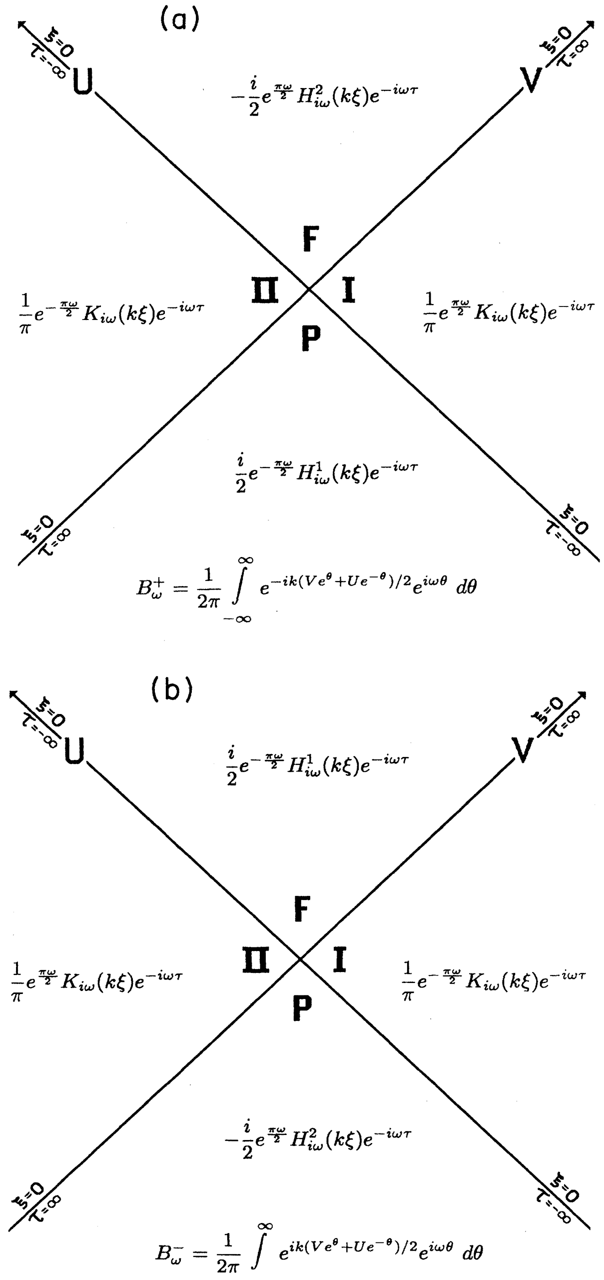

FIG. 1. Minkowski Bessel (MB) modes $B^\pm_\omega(kU,kV)$ and their coordinate representatives in the respective Rindler charts I, II, F, and P.

are actually *functionals* of α because $\alpha_\omega=\alpha(\omega)$ is actually a function of ω. Thus for every function $\alpha(\omega)$ there is one and only one expansion given by Eqs. (3.9).

The two most important expansions give rise to the Minkowski vacuum and the Rindler vacuum, respectively. The expansion $\alpha(\omega)\equiv 0$ is simply the standard expansion equation (3.4). It has operators $a_{\omega k}$ and $a^\dagger_{\omega k}$ which refer to absorption and emission processes relative to an inertial frame. The reference state $|0(\alpha\equiv 0)\rangle$ of the KG system for this $\alpha(\omega)\equiv 0$ expansion is determined by

$$a_{\omega k}|0(\alpha\equiv 0)\rangle=0,\quad a_{-\omega-k}|0(\alpha\equiv 0)\rangle=0 .$$

It is, of course, the familiar Minkowski vacuum state

$$|0_M\rangle=|0(\alpha\equiv 0)\rangle .$$

The second well-known expansion is characterized by the condition that, in I,

$$\begin{aligned}0&=R_{\omega 2|I}\\ &\equiv(-e^{\pi\omega/2}\sinh\alpha_\omega\\ &\qquad+e^{-\pi\omega/2}\cosh\alpha_\omega)\frac{1}{\pi}K_{i\omega}(k\xi)e^{i\omega\tau} ,\\ &\qquad\qquad \omega>0 .\end{aligned} \tag{3.10}$$

What is the meaning of this condition? It is an expression of the quantum absorption principle. Recall that in quantum mechanics atoms, detectors, and so on, are caused to have absorptive transitions only by fields consisting of positive-frequency harmonics ($e^{i\omega\tau}$; $\omega>0$). Negative-frequency harmonics ($e^{i\omega\tau}$) do not cause any absorptive transitions. Consequently quantum mechanics demands that the field absorption operators have *no* negative-frequency harmonics. This is precisely the condition expressed by Eq. (3.10). This quantum-mechanical absorption principle is the one which allowed Unruh[1] to arrive at the fact that the Minkowski vacuum state viewed relative to an accelerated frame with an event horizon has a physically relevant character given by the universal Davies-Unruh temperature $T=(1/k)(\hbar/c)(g/2\pi)$. Equation (3.10) yields an expansion for $\psi(x)$ where the function $\alpha_\omega=\alpha_{\text{crit}}(\omega)$:

$$\begin{aligned}e^{\pi\omega/2}/\sqrt{2\sinh\pi\omega}&=\cosh\alpha_\omega\equiv u_\omega(\alpha_{\text{crit}}) ,\\ e^{-\pi\omega/2}/\sqrt{2\sinh\pi\omega}&=\sinh\alpha_\omega\equiv v_\omega(\alpha_{\text{crit}}) .\end{aligned} \tag{3.11}$$

The absorption operators for this expansion are no longer the Minkowski absorption operators. Instead, the new operators are those obtained by inserting (3.11) into (3.7). The corresponding reference state $|0(\alpha)\rangle$ is determined by

$$\begin{aligned}A_{\omega k}[\alpha_{\text{crit}}]|0(\alpha_{\text{crit}})\rangle&=0 ,\\ A_{-\omega-k}[\alpha_{\text{crit}}]|0(\alpha_{\text{crit}})\rangle&=0 .\end{aligned} \tag{3.12}$$

It is variously called the "Killing" vacuum,[5] or the "Rindler" vacuum.[3] Its form given in Sec. VI. Actually it is a type of condensed vacuum state with nonzero and nonsharp (in fact, infinite) photon number. Macroscopically this state manifests itself, it turns out, as a photon superfluid. (*Nota bene*: The appellation "photon" is our way of implying that the $a_{\omega k}$ particles are archetypical and, except for intrinsic spin effects, have the same properties as the familiar Maxwell wave quanta in Minkowski space-time.)

Besides Unruh's there is an alternative criterion for arriving at the second expansion characterized by $\alpha(\omega)$ in Eq. (3.11) and hence by the corresponding reference state (3.12). This criterion is based on a Rayleigh-Ritz variational principle. Using that principle one finds the quantum state which minimizes the subsystem I (and/or II) Hamiltonian.[9,10] That quantum state is also characterized by Eq. (3.11).

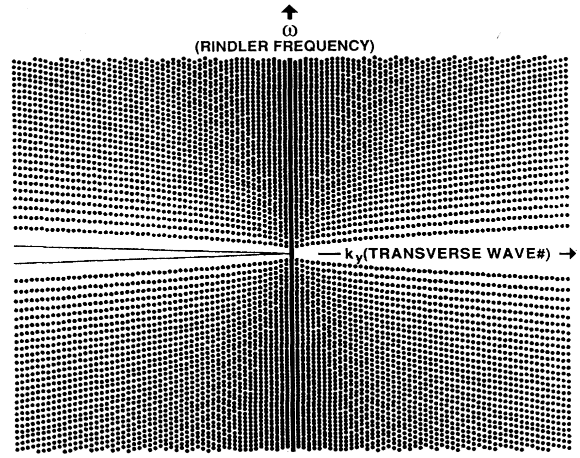

FIG. 2. Minkowski Bessel (MB) mode eigenvalue spectrum. Each solid dot represents a MB mode allowed by a symmetrically placed pair of accelerated cavity walls ($x=\pm b\sinh\tau$). The lattice of allowed MB modes is obtained from Eqs. (3.17). It is analogous to the plane-wave lattice of a symmetrically placed pair of inertial cavity walls. There the modes are arranged in a cubical lattice and the mode density is uniform. In the MB lattice, by contrast, one has a nonuniform mode density: it has, as one can see, a maximum along the ω axis ($k_y=k_z=0$); this maximum is infinite for a zero rest mass scalar field. Furthermore no modes are allowed on ($n=0$) or outside the double cone $\omega^2/b^2=k_y^2+k_z^2$, or outside the two-sheeted hyperboloid $\omega^2/b^2=k_y^2+k_z^2+m^2$, if the field has mass m. In this picture $m=0$. The uniform horizontal k_y spacing between the allowed MB modes is in units of $\pi/(\text{cavity width})=\pi/L$. The units along the vertical axis are dimensionless. Their natural length is $\pi b/L$. It is determined by the double cone boundary $\omega=\pm k_y b\equiv\pm n_y\pi b/L$ which separates the region (in $\omega-k_y$ space) where MB modes are allowed ($\omega^2>k_y^2b^2$) from the region where their existence is forbidden ($\omega^2<k_y^2b^2$). In the limit of a bottomless ($b\to 0$) accelerated cavity, the two sheets of the hyperboloid degenerate into the k_y-k_z plane (here the k_y axis), and the mode density is nonzero everywhere. This limit is the thermodynamic limit for an accelerated frame. It corresponds to the thermodynamic ("infinite volume") limit of an inertial cavity.

B. Discrete mode expansion

One of the most direct ways of discussing the quantum mechanics (QM) of infinite systems is to first discuss the QM of a system confined to a finite volume and then consider the infinite-volume ("thermodynamic") limit. This is a standard procedure for making a transition from a discrete mode spectrum to its continuum limit. This is a useful procedure for plane-wave modes, and we shall extend it to the MB modes of the acceleration-partitioned KG system. The payoff will be a proof of Eq. (1.2).

Consider a KG field partitioned by a pair of accelerated frames, one in Rindler chart I, the other in Rindler chart II. These frames are to be the interior of a pair of semi-infinite boxes. Let the transverse area be L^2, so that

$$0\le y,z\le L\ ,$$

and let the bottom of each be located symmetrically at $\xi=b>0$ in I and II, respectively. Their tops are open. Thus

$$b\le\xi<\infty\ .$$

It is the pseudogravitational field of each accelerated frame that prevents the escape of the wave field toward $\xi=\infty$. The spectrum of MB modes forms therefore a discrete set. See Fig. 2. The coordinate representatives[8] of these modes satisfy the Sturm-Liouville problem

$$\left[\xi\frac{d}{d\xi}\xi\frac{d}{d\xi}+\omega^2-k^2\xi^2\right]K_{i\omega}(k\xi)=0$$

with the same fixed and given homogeneous boundary condition at $\xi=b$,

$$a_1K_{i\omega}(kb)+a_2\frac{d}{d\xi}K_{i\omega}(kb)=0\ , \qquad (3.13)$$

in I and in II.

1. *Minkowski Bessel modes: Their WKB form*

In the WKB approximation the solution the Sturm-Liouville equation has the form (for $x<\omega$)

$$K_{i\omega}(x)=\left[\frac{\pi}{\sinh\pi\omega}\right]^{1/2}\frac{1}{(\omega^2-x^2)^{1/4}}\times\cos\left[\int_x^{\omega}\sqrt{\omega^2-x^2}\,\frac{dx}{x}-\frac{\pi}{4}\right]\ . \qquad (3.14a)$$

The integration constant in the phase of cos has been determined by the condition that this solution match at $x=\omega$ that solution which decays exponentially as $x\to\infty$.

The normalization amplitude has been adjusted by the requirement that $K_{i\omega}(x)$ agree with the small-x expansion of the standard definition of the modified Bessel function:

$$K_{i\omega}\simeq\frac{\pi}{2i\sinh\pi\omega}\left[\left[\frac{x}{2}\right]^{-i\omega}\frac{1}{\Gamma(1-i\omega)}-\left[\frac{x}{2}\right]^{i\omega}\frac{1}{\Gamma(1+i\omega)}\right]$$
$$=\left[\frac{\pi}{\omega\sinh\pi\omega}\right]^{1/2}\cos\left[\arg\Gamma(i\omega)-\omega\ln\left[\frac{x}{2}\right]\right]. \tag{3.14b}$$

The phase of $K_{i\omega}(x)$ in the WKB approximation in Eq. (3.14a) is

$$\text{phase}(x)=\omega\ln\frac{\omega+\sqrt{\omega^2-x^2}}{x}-\sqrt{\omega^2-x^2}-\frac{\pi}{4}. \tag{3.15}$$

Not surprisingly this is entirely consistent with the phase in the $x\ll\omega$ approximation, Eq. (3.14b), because there, by Stirling's formula,

$$\arg\Gamma(i\omega)\simeq\omega\ln\omega-\omega-\frac{\pi}{4}.$$

A typical normal mode

$$\Psi_{\omega k_y k_z}(\tau,\xi,y,z)=e^{-i\omega\tau}K_{i\omega}(k\xi)e^{ik_y y}e^{ik_z z}$$

is the product the longitudinal (i.e., τ- and ξ-dependent) eigenfunction $e^{-i\omega\tau}K_{i\omega}(k\xi)$ and the transverse (i.e., y- and z-dependent) eigenfunctions. The boundary condition, Eq. (3.13), at $\xi=b$ implies that the functions $e^{-i\omega\tau}K_{i\omega}(k\xi)$ have discrete frequencies

$$\omega=\omega_n,\quad n=1,2,\ldots.$$

For the case of the Dirichlet ($a_2=0$) boundary condition at the bottom ($\xi=b$) of the accelerated cavity, these frequencies are determined by the "Bohr quantization" conditions

$$\text{phase}(kb)\equiv\int_{kb}^{\omega_n}\sqrt{\omega_n^2-x^2}\frac{dx}{x}-\frac{\pi}{4}=(n+\tfrac{1}{2})\pi,\quad n=1,2,\ldots. \tag{3.16a}$$

For the case of the Neumann ($a_1=0$) boundary condition the Bohr-quantization condition is

$$\int_{kb}^{\omega_n}\sqrt{\omega_n^2-x^2}\frac{dx}{x}-\frac{\pi}{4}=n\pi,\quad n=1,2,\ldots. \tag{3.16b}$$

For the general boundary condition, Eq. (3.13), the Bohr-quantization condition is intermediate between these two extreme cases.

The transverse eigenfunctions are also discrete, and their eigenvalues are

$$k_y=\frac{2\pi}{L}n_y,$$
$$k_z=\frac{2\pi}{L}n_z,\quad n_y,n_z=0,\pm1,\pm2,\ldots.$$

2. *Discrete mode spectrum*

Consider the eigenvalue spectrum of the normal modes of a wave field in an accelerated coordinate frame with a finite bottom ($\xi=b>0$). This spectrum forms a lattice inside the cone

$$(\omega/b)^2=k_y^2+k_z^2$$

in (ω,k_y,k_z) space. For a field with a finite rest mass m the cone becomes a two-sheeted hyperboloid,

$$(\omega/b)^2=k_y^2+k_z^2+m^2.$$

Given a triplet of integers (n,n_y,n_z), the corresponding lattice point is determined by the Bohr quantum condition and the transverse eigenvalue conditions: namely,

$$n+\frac{1}{4}=\frac{1}{\pi}\left[\omega\ln\frac{\omega+\sqrt{\omega^2-k^2b^2}}{kb}-\sqrt{\omega^2-k^2b^2}\right], \tag{3.17a}$$

$$n_y=\frac{1}{2\pi}Lk_y, \tag{3.17b}$$

$$n_z=\frac{1}{2\pi}Lk_z, \tag{3.17c}$$

with

$$k=\sqrt{k_y^2+k_z^2+m^2}.$$

The normal-mode eigenvalue lattice, exhibited in Fig. 2, is determined by these triplets of integers. It is analogous to the familiar plane-wave eigenvalue spectrum for an inertial frame. There

$$n_x=\frac{1}{2\pi}Lk_x,\quad n_y=\frac{1}{2\pi}Lk_y,\quad n_z=\frac{1}{2\pi}Lk_z.$$

The main difference between these two lattices is this: in the plane-wave lattice the mode density is uniform over the whole (k_x,k_y,k_z) space, while in the Minkowski Bessel wave lattice the mode density [see Eq. (3.18) below] is highly nonuniform in (ω,k_y,k_x) space. This can readily be seen from the computer-generated spectral lattice, Fig. 2.

Furthermore, unlike a finite inertial cavity, a pair of finite accelerated cavities with finite bottom ($b>0$) has a spectral lattice that is confined entirely inside each of the two ($\omega=\pm|\omega|$) cones

$$(\omega/b)^2>k_x^2+k_y^2,$$

or inside each of the two hyperboloids,

$$(\omega/b)^2>k_x^2+k_y^2+m^2,$$

if the wave field has rest mass m. This follows directly from the Bohr-quantization condition, Eq. (3.16).

In the limit of a bottomless ($b\to0$) accelerated frame, the two hyperboloidal boundaries are spread out flat and touch each other in the k_y-k_z plane $\omega=0$. Furthermore, as $b\to0$ there are normal modes everywhere in (ω,k_y,k_z) space.

3. Mode sums: Their phase volume integrals

The longitudinal and transverse eigenfunction spectra form a lattice inside the "upper" and "lower" cone, $(\omega/b)^2=k_y^2+k_z^2\equiv k^2$, in (ω,k_y,k_z) space. The density of modes is not uniform. It follows from the eigenvalue formulas, Eqs. (3.17), that the mode density ρ is

$$\rho=\frac{\partial(n,n_y,n_z)}{\partial(\omega,k_y,k_z)}=\left[\frac{L}{2\pi}\right]^2\frac{dn}{d\omega}=\left[\frac{L}{2\pi}\right]^2\frac{1}{\pi}\ln\frac{\omega+\sqrt{\omega^2-k^2b^2}}{kb}\ , \tag{3.18}$$

where

$$kb<\omega=\omega_n \quad (\text{"interior of the upper cone"})\ .$$

A mode sum can thus be approximated by a phase-space integral inside the semi-infinite upper cone in (ω,k_y,k_z) space:

$$\sum_n\sum_{n_y}\sum_{n_z}(\cdots)\approx\int_0^\infty d\omega\int_0^{\omega/b}k\,dk\int_0^{2\pi}d\theta_k\left[\frac{L}{2\pi}\right]^2\frac{dn}{d\omega}(\cdots)\ . \tag{3.19}$$

As a useful example consider the total number of modes inside the finite upper cone whose height is $\omega=\varpi$. Let us call this number No. (ϖ,L^2,b). It is given by

$$\text{No.}(\varpi,L^2,b)\equiv\int_0^{\varpi}d\omega\int_0^{\omega/b}k\,dk\int_0^{2\pi}d\theta_k\rho\left[\frac{L}{2\pi}\right]^2\frac{dn}{d\omega}=\left[\frac{1}{2\pi}\right]^2\left[\frac{L}{b}\right]^2\frac{\varpi^3}{3}\ . \tag{3.20}$$

This result was obtained with the help of Eq. (3.18), and doing the integration over the interior of the finite cone:

$$kb=\omega,\quad 0<\omega<\varpi\ .$$

The number equals the total number of all those normal modes that (a) are permitted to exist in an accelerated cavity whose transverse area is L^2 and whose bottom is at $\xi=b$ and (b) have "Rindler frequency" ω less than the maximum value ϖ.

The expression for the number of modes, Eq. (3.20), is correct only for a massless KG field. If the field is massive, then the corresponding number of modes is the one found inside the finite upper hyperboloid of revolution

$$(k_y^2+k_z^2+m^2)b^2=\omega^2,\quad mb<\omega<\varpi\ .$$

One must make the replacements $k^2=k_x^2+k_y^2\to k^2=k_x^2+k_y^2+m^2$ in the expression for $dn/d\omega$, Eq. (3.18), so that the phase-space volume integral, Eq. (3.20), gets replaced by

$$\text{No.}(\varpi,L^2,b)\equiv\int_{mb}^{\varpi}d\omega\int_m^{\omega/b}k\,dk\int_0^{2\pi}d\theta_k\rho\left[\frac{L}{2\pi}\right]^2\frac{dn}{d\omega}=\left[\frac{1}{2\pi}\right]^2\left[\frac{L}{b}\right]^2\left[\frac{\bar{k}^3}{3}+m^2b^2\left[\bar{k}-\varpi\ln\frac{\varpi+\bar{k}}{mb}\right]\right]\ . \tag{3.21}$$

Here $\bar{k}$ stands for

$$\bar{k}=\sqrt{\varpi^2-m^2b^2},\quad \varpi>mb\ .$$

4. Bottomless frames versus frame with bottom: Quantum theory

We shall now show how to take what in an inertial frame corresponds to taking the infinite-volume ("thermodynamic") limit of the real KG field quantized in a finite box. In the case of our acceleration-partitioned field, this transition to the limit corresponds to going from the field defined in a pair of accelerated frames each with a finite bottom $(0<b)$ to a pair of bottomless $(b=0)$ frames.

Consider the KG field partitioned by a pair of bottomless accelerated frames I and II. The field has a continuous spectrum mode expansion given by Eq. (3.4). If, however, the pair of frames have finite bottoms $(b>0)$ and finite transverse areas L^2, then the mode expansion is necessarily discrete. It is given by

$$\psi(x)=\sum_i[a_if_i(x)+a_{-i}f_{-i}(x)+\text{c.c.}]\ . \tag{3.22}$$

The mode labels i will be specified shortly.

Suppose the absorption and emission operators satisfy the usual relations

$$[a_i,a_j^\dagger]=\delta_{ij},\quad [a_i,a_{-i}^\dagger]=0 \tag{3.23}$$

and the KG ("Wronskian") inner product of two functions $f(x)$ and $g(x)$ is defined by

$$(g,f)=i\int_{-L/2}^{L/2}\int_{-L/2}^{L/2}dy\,dz\left[\int_\infty^b\left[g^*\frac{\partial f}{\partial\tau}-f\frac{\partial g^*}{\partial\tau}\right]_{\text{II}}\frac{d\xi}{\xi}+\int_b^\infty\left[g^*\frac{\partial f}{\partial\tau}-f\frac{\partial g^*}{\partial\tau}\right]_{\text{I}}\frac{d\xi}{\xi}\right]\ . \tag{3.24}$$

Then the discrete modes f_i in Eq. (3.22) satisfy

$$(f_i,f_j)=\delta_{ij},\quad (f_i,f_{-i})=0\ . \tag{3.25}$$

Our task is to furnish an asymptotic integral approximation of the discrete sum, Eq. (3.22), which in the limit as $b\to 0$ ("bottomless" accelerated frames) approaches (3.2). This approximation is given by

$$\psi(x)=\int_0^\infty d\omega \int_0^{\omega/b} k\,dk \int_0^{2\pi} d\theta_k \rho \left[\frac{a_{\omega k}}{\rho^{1/2}} \frac{1}{\rho^{1/2}} \frac{B_\omega^+}{\sqrt{2}} \frac{e^{i(k_y y+k_z z)}}{2\pi} + \frac{a_{-\omega -k}}{\rho^{1/2}} \frac{1}{\rho^{1/2}} \frac{B_{-\omega}^+}{\sqrt{2}} \frac{e^{-i(k_y y+k_z z)}}{2\pi} + \text{c.c.} \right] . \quad (3.26)$$

That this integral approaches (3.2), or equivalently (3.4), as $b \rightarrow 0$ is obvious by inspection. That this integral is an asymptotically accurate expression for the discrete sum, Eq. (3.22), can readily be verified by first using[11] Eq. (3.19),

$$\sum_i(\cdots)\equiv \sum_n \sum_{n_y} \sum_{n_z}(\cdots) = \int_0^\infty d\omega \int_0^{\omega/b} k\,dk \int_0^{2\pi} d\theta_k \rho(\cdots) ; \quad (3.27)$$

second, letting

$$a_i \equiv a_{n n_y n_z} = \frac{a_{\omega k}}{\rho^{1/2}}, \quad a_{-i}=\frac{a_{-\omega -k}}{\rho^{1/2}}, \quad \omega>0 ; \quad (3.28)$$

and finally letting

$$f_i \equiv f_{n n_y n_z} = \frac{1}{\rho^{1/2}} \frac{B_\omega^+}{\sqrt{2}} \frac{e^{i(k_y y+k_z z)}}{2\pi} , \qquad f_{-i}=\frac{1}{\rho^{1/2}} \frac{B_{-\omega}^+}{\sqrt{2}} \frac{e^{-i(k_y y+k_z z)}}{2\pi} . \quad (3.29)$$

In all three equations (3.27)–(3.29) the continuous variables ω, k_y, and k_z are given by their discrete approximations ω_n, k_y, k_z as determined by Eqs. (3.17).

Do the operators $a_{\pm i}$, Eq. (3.28), and their discrete modes, Eq. (3.29), satisfy the correct commutation relations

$$[a_{n n_y n_z}, a^\dagger_{n' n_y' n_z'}]=\delta_{nn'}\delta_{n_y n_y'}\delta_{n_z n_z'} \quad (3.30)$$

and the correct KG inner products

$$(f_{n n_y n_z}, f_{n' n_y' n_z'})=\delta_{nn'}\delta_{n_y n_y'}\delta_{n_z n_z'} \ ? \quad (3.31)$$

That the answer is "yes" is readily verified. To show Eq. (3.31), insert Eq. (3.29) into (3.24). Use (3.3), integrate, and use the identity[12]

$$\int_{kb}^\infty K_{i\omega}(x) K_{i\omega'}(x) \frac{dx}{x} = \frac{\pi}{2\omega \sinh \pi\omega} \left[\frac{2\pi}{L} \right]^2 \pi \rho \delta_{nn'} .$$

The result is Eq. (3.31). Here $\omega=\omega_n$, $\omega'=\omega_{n'}$, and ρ is given by Eq. (3.18). Note that $(f_i, f_{-i})=0$.

The commutator relation (3.30) is also easily verified. Recall that the operators $a_{\omega k}$ satisfy

$$[a_{\omega k}, a^\dagger_{\omega' k'}]=\delta(\omega-\omega')\delta(k_y-k_y')\delta(k_z-k_z') , \qquad -\infty<\omega,k_y,k_z<\infty .$$

Use the fact that the Dirac delta functions are asymptotically related to the Kronecker deltas by the mode density ρ, Eq. (3.18),

$$\delta(\omega_n-\omega_{n'})\delta\left[\frac{2\pi}{L}n_y-\frac{2\pi}{L}n_y'\right]\delta\left[\frac{2\pi}{L}n_z-\frac{2\pi}{L}n_z'\right] = \rho \delta_{nn'}\delta_{n_y n_y'}\delta_{n_z n_z'} .$$

Then with the help of (3.28) obtain the desired relations (3.30).

To summarize, we have shown how to replace summations over discrete mode with phase-space integrals. This we have done, not for the familiar case of plane waves in an inertial box, but for Minkowski Bessel modes in an accelerated frame with a finite bottom ($b>0$). The key results are Eqs. (3.27)–(3.29).

IV. RINDLER GROUND STATE FOR AN ACCELERATED FRAME WITH A FINITE BOTTOM

A field partitioned by a pair of accelerating frames with finite bottoms ($b>0$) can be decomposed into a discrete sum of modes:

$$\psi(x)=\sum_i [a_i f_i(x)+a^\dagger_{-i} f^*_{-i}(x)+a_i^\dagger f_i(x)+a_{-i} f_{-i(x)}] . \quad (4.1)$$

In the thermodynamic limit ($b=0$, "no bottom") this sum becomes with the help of Eqs. (3.27)–(3.29) the mode integral, Eq. (3.4).

The key idea leading to a ground state different from the familiar Minkowski vacuum is the expansion degeneracy (see Sec. III A) of the KG field $\psi(x)$. This field can be given many different mode integral expansions, Eq. (3.9). This expansion degeneracy holds not only for bottomless ($b=0$) accelerated frames but also for accelerated frames with a finite bottom ($b>0$). There the expansions are discrete and the standard Minkowski-Bessel mode expansion is given by Eq. (4.1).

As Eq. (3.9), the expansion coefficients for each pair of modes i and $-i$ are arbitrarily interrelated by the transformation

$$A_i=a_i u_i+a^\dagger_{-i} v_i, \quad A^\dagger_{-i}=a^\dagger_{-i} u_i+a_i v_i , \quad (4.2)$$

with

$$u_i^2-v_i^2=1 .$$

The degeneracy is expressed by the arbitrariness of each parameter u_i. This arbitrariness is removed by resorting either to the quantum absorption principle [see comments following Eq. (3.10)] or to a Rayleigh-Ritz-type minimum principle[9,10] for the moment of mass energy of the KG field in I and II. Either of these principles yields

$$u_i=e^{\pi\omega/2}/\sqrt{2\sinh \pi\omega} , \qquad v_i=e^{-\pi\omega/2}/\sqrt{2\sinh \pi\omega} . \quad (4.3)$$

Here ω has the discrete value corresponding to the mode $i\equiv(n,n_y,n_z)$. The Rindler ground state is the ground state for the new set of quantum operators, Eq. (4.2). This ground state is determined by

$$0=A_i|0_R\rangle=(a_i u_i+a^\dagger_{-i} v_i)|0_R\rangle , \qquad 0=A_{-i}|0_R\rangle=(a_{-i} u_i+a_i^\dagger v_i)|0_R\rangle . \quad (4.4)$$

It can be expressed as a pure photon state

$$|0_R\rangle = f(a_i^\dagger, a_{-i}^\dagger)|0_M\rangle \ .$$

Using $a_{\pm i} = \partial/\partial a_{\pm i}^\dagger$ one obtains the solution to Eq. (4.4):

$$|0_R\rangle = Z^{-1/2}\prod_i \exp\left[\frac{-v_i}{u_i} a_i^\dagger a_{-i}^\dagger\right]|0_M\rangle$$
$$= Z^{-1/2}\prod_i \sum_{n_i=0}^{\infty}\left[\frac{-v_i}{u_i}\right]^{n_i} a\,|n_i\rangle \otimes |n_{-i}\rangle \ . \quad (4.5)$$

Here $n_i = n_{-i}$ and

$$Z^{-1/2} = \prod_i \frac{1}{u_i} = \exp\left[-\sum_i \ln u_i\right]$$

is the normalization constant for the normalized ground state. The value of this normalization constant can be obtained with the help of Eq. (3.27). Because $u_i = u_\omega$ is independent of the "polar coordinates" k and θ_k,

$$\ln Z = \sum_i \ln u_i^2$$
$$= \int_0^\infty \ln u_\omega^2 \left[\int_0^{\omega/b} k\,dk \int_0^{2\pi} d\theta_k \left[\frac{L}{2\pi}\right]^2 \frac{dn}{d\omega}\right] d\omega$$
$$= \left[\frac{1}{2\pi}\right]^2 \left[\frac{L}{b}\right]^2 \int_0^\infty \omega^2 \ln u_\omega^2 d\omega \ . \quad (4.6)$$

The second line is obtained from Eqs (3.27) and (3.18). The content of the square brackets is evaluated by taking the derivative of Eq. (3.20). This yields the third line.

The Rindler ground state $|0_R\rangle$, Eq. (4.5), is therefore

$$|0_R\rangle = \exp\left[-\left[\frac{L}{2\pi b}\right]^2 \int_0^\infty \omega^2 \ln u_\omega d\omega\right] \prod_i \sum_{n_i=0}^{\infty}\left[-\frac{v_i}{u_i}\right]^{n_i} |n_i\rangle \otimes |n_{-i}\rangle \ . \quad (4.7)$$

Using the expression, Eq. (4.3), one finds that the value of the integral in exp[$\cdots$] is $\pi/180$. Consequently the Rindler ground state is

$$|0_R\rangle = \exp\left[-\left[\frac{L}{b}\right]^2 \frac{1}{720\pi}\right] \prod_i \sum_{n_i=0}^{\infty}\left[-\frac{v_i}{u_i}\right]^{n_i} |n_i\rangle \otimes |n_{-i}\rangle \ . \quad (4.8)$$

V. RINDLER VACUUM VERSUS THE MINKOWSKI VACUUM

(1) Consider the Minkowski ground state

$$|0_M\rangle = \prod_i |0_i\rangle \otimes |0_{-i}\rangle \ .$$

It is determined by

$$a_i|0_M\rangle = 0, \quad a_{-i}|0_M\rangle = 0 \ .$$

The inner product of $|0_M\rangle$ with the Rindler ground state $|0_R\rangle$, Eq. (4.8), is

$$\langle 0_M|0_R\rangle = \exp\left[-\left[\frac{L}{b}\right]^2 \frac{1}{720\pi}\right] \ . \quad (5.1)$$

(2) A related quantity is

$$\langle 0_R|N^{\text{photon}}|0_R\rangle = \langle 0_R|\sum_i (a_i^\dagger a_i + a_{-i}^\dagger a_{-i})|0_R\rangle \ ,$$

the expectation value of the total number of photon when the field is in the Rindler ground state. This value is

$$\langle 0_R|N^{\text{photon}}|0_R\rangle = 2\sum_i v_i^2$$
$$= 2\left[\frac{L}{2\pi b}\right]^2 \int_0^\infty \omega^2 v_\omega^2 d\omega$$
$$= \left[\frac{L}{b}\right]^2 \frac{1}{648.2\pi} \ . \quad (5.2)$$

The first line is obtained by inverting Eq. (4.2) and using (4.4), the second line is the same reasoning leading to Eq. (4.6), and the last line is obtained by inserting Eq. (4.3) into the integral and then evaluating it.

(3) One can also ask the standard question, what is the expected number of quasiparticles (fulling particles) in each of the accelerated frames I and II if the field is in the Minkowski vacuum $|0_M\rangle$? The answer is

$$\langle 0_M|N_{\text{I}}^{\text{quasip}} + N_{\text{II}}^{\text{quasipp}}|0_M\rangle$$
$$= \langle 0_M|\sum_i (A_i^\dagger A_i + A_{-i}^\dagger A_{-i})|0_M\rangle$$
$$= 2\sum_i v_i^2$$
$$= \left[\frac{L}{b}\right]^2 \frac{1}{648.2\pi}$$

which is obtained in the same way as Eq. (5.2).

(4) In the thermodynamic limit corresponding to each of the pair of accelerated frames I and II becoming bottomless ($b \to 0$), the amplitudes (5.1) and (5.2) become

$$\lim_{b\to 0}\langle 0_M|0_R\rangle = \lim_{b\to 0}\exp\left[-\left[\frac{L}{b}\right]^2 \frac{1}{720\pi}\right] = 0 \ , \quad (5.3)$$

$$\lim_{b\to 0}\langle 0_R|N^{\text{photon}}|0_R\rangle = \lim_{b\to 0}\left[\frac{L}{b}\right]^2 \frac{1}{648.2\pi} = \infty \ , \quad (5.4)$$

$$\lim_{b\to 0}\langle 0_M|N_1^{\text{quasi}p}|0_M\rangle=\lim_{b\to 0}\left[\frac{L}{b}\right]^2\frac{1}{648.2\pi}$$
$$=\infty\ . \qquad (5.5)$$

(5) These related results can be strengthened considerably. Let $p(a_i,a_j^\dagger)$ be a polynomial in the Minkowski quantum operators corresponding to some arbitrary element

$$p(a_i,a_j^\dagger)|0_M\rangle$$

of the Hilbert space of quantum states generated from the Minkowski vacuum. One sees that

$$\lim_{b\to 0}\langle 0_R|p(a_i a_j^\dagger)|0_M\rangle=0\ .$$

In other words, the Rindler vacuum is perpendicular to *every* quantum state involving detectors in Refs. 1–3.

Now consider a polynomial in the Rindler quantum operators. Such a polynomial can be changed with the help of Eq. (4.2) into a polynomial of Minkowski quantum operators. It follows that, in the thermodynamic limit *any* state in the Hilbert space generated from $|0_R\rangle$ is perpendicular to every state in the Hilbert space generated from $|0_M\rangle$. Let $\mathcal{H}_R$ and $\mathcal{H}_M$ denote these two respective Hilbert spaces. Our result can be summarized by the statement that the two Hilbert spaces are orthogonal to each other:

$$\mathcal{H}_R\perp\mathcal{H}_M\ .$$

(6) Equations (5.4) and (5.5) can be strengthened in a corresponding way: (i) *Any* quantum state in $\mathcal{H}_R$ has an infinite number of expected photons. (ii) *Any* quantum state in $\mathcal{H}_M$ has an infinite number of quasiparticles ("Fulling particles") in frame I and in frame II.

VI. CONCLUDING REMARKS

(1) A Rindler vacuum state $|0_R\rangle$ is very different from the quantum states generally considered in physics. To prepare a relativistic wave field in this state, place into a cylindrical pipe of cross-sectional area L^2 a pair of refrigerators and have them accelerate into opposite directions. Surprisingly the vacuum they produce in each of their Rindler sectors will not be empty when viewed jointly by a global inertial reference frame. Instead the cylindrical pipe will be filled by a nonsharp number of photons given by

$$\langle 0_R|N_{\text{photon}}|0_R\rangle=\left[\frac{L}{b}\right]^2\frac{1}{\pi}\ .$$

These photons are in the pure quantum state

$$|0_R\rangle=\exp\left[-\left[\frac{L}{b}\right]^2\frac{1}{720\pi}\right]\times\prod_i\sum_{n_i=0}^{\infty}(-1)^{n_i}e^{-2\pi n_i\omega_i}|n_i\rangle\otimes|n_{-i}\rangle\ .$$

(2) That a ground (i.e., "vacuum") state is characterized by a nonzero number of particles is not unfamiliar in condensed-matter physics. For bosons the most famous example is superfluid ^{4}He. At $T=0°$ K this system is in its ground state: there are no quasiparticles, i.e., no "first" sound quanta. Nevertheless, the system is composed of particles, the helium atoms.

There is, however, an important difference between the liquid-He system and the KG system considered here. In the liquid-He system *particle number* is conserved and the corresponding gauge symmetry is broken in the thermodynamic (volume $\to\infty$) limit. Here it is the longitudinal *linear momentum* which is conserved and it is the corresponding translation symmetry that is broken in the thermodynamic ($b\to 0$) limit. We are led to believe that the Davies-Unruh temperature (or the Bekenstein-Hawking temperature for a black hole) is a critical temperature which signals the macroscopic condensation of linear momentum into a state of correlated MB modes.

The parallelism with liquid He is much more extensive[13,14] than one is led to believe from the brevity of these cursory remarks. In fact, the parallelism is so striking—it extends from the nature of the quantum states and the dynamics of the field to the macroscopic behavior at finite temperature—that one cannot help but feel that nature is trying to tell us something important.

(3) Many workers consider the infinite thermodynamic limit, Eq. (5.5), a "difficulty." Such an attitude, however, does not take into account that the sets of quantum states $\mathcal{H}_R$ and $\mathcal{H}_M$ are two spaces perpendicular to each other. This perpendicularity is an example of the central feature of the quantum mechanics of an infinite system:[6,7] its quantum states decompose into *unitarily distinct representation spaces.* The existence of such representations expresses the key properties (e.g., symmetry breaking, phase transition, order parameter, and so on) in condensed-matter physics.

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[11]The correct version of Eq. (5.18) as well as Eq. (5.22) in U. Gerlach, Phys. Rev. D **38**, 514 (1988):

$$\sum(\cdots) \overset{b_g \ll 1}{\longrightarrow} L^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{kb}^{\infty} \frac{d^2k}{(2\pi)^2} \frac{d\omega}{\pi} \times \ln \frac{\omega + \sqrt{\omega^2 - k^2 b^2}}{kb} (\cdots)$$

Note that the integration order in this expression is the reverse of the one in Eq. (3.19) of this paper. Both are correct.

[12]See Eq. (5.11) in Ref. 8.

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