Einstein's evolution equations as a system of balance laws

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The evolution system in the space plus time (3+1) decomposition of Einstein's field equations is explicitly written as a system of balance laws. This is achieved by demanding that the time coordinate be harmonic (harmonic synchronization) and the space coordinate lines be normal to the constant-time hypersurfaces. No symmetry nor special form of the metric has been assumed, so that the equations may be used as a part of a three-dimensional numerical code for general relativity. The particular case of spherical symmetry is also considered and a numerical test of this case is provided by using a nonstandard form of the Schwarzschild line element.

I. INTRODUCTION

The present work is a step in the design of a three-dimensional (3D) general-relativistic numerical code. It is clear that such a code cannot be a straightforward extension of the existing 1D or 2D ones. This is because they are strongly adapted to spherically symmetric or axially symmetric fields, respectively.[1] The jump to a code dealing with generic unsymmetrical systems is at least as radical as the one needed to go from the 1D code of May and White[2] to the 2D codes for axisymmetric collapse.[3] We must remember that in that case the equations were completely changed and the comoving coordinate system had to be abandoned.[4]

The interest of accurately studying nonsymmetrical systems in general relativity resides mainly in the fact that they are good candidates to act as strong sources of gravitational waves. One can imagine astrophysical processes in which this may actually occur, such as supernova collapse, coalescence of a binary system, or accretion of matter into a black hole. In all these cases, hydrodynamic discontinuities (shock waves) may appear.[5] The shocks are a mere consequence of the balance-law structure of the hydrodynamic equations and a solution containing them (a "weak" solution[6]) must be adequately dealt with in a realistic code.[7]

The Einstein field equations also admit weak solutions. They may arise when hydrodynamic shocks are present, but they can also appear as genuine gravitational shock waves.[8] The analogy with the situation in hydrodynamics suggests that Einstein field equations can be expressed as a system of balance laws. This would allow one to apply the same kind of numerical methods to both the field and hydrodynamic equations. A supplementary advantage of using balance-law methods is that the matching between numerical and analytic solutions could be performed in a natural way. The interest of such a combined approach has been recently stressed[9] and matched analytic-numerical results for simple flux conservative relativistic equations have been published.[10]

We shall first show that our goal is attainable. Let us start by considering the four-dimensional Einstein's field equations:

$$R_{\mu\nu}=T_{\mu\nu}-(T/2)g_{\mu\nu} \quad (\mu,\nu=0,1,2,3)\ , \tag{1}$$

where $T_{\mu\nu}$ is the stress-energy tensor and we have noted $T\equiv g^{\mu\nu}T_{\mu\nu}$.

The Ricci tensor $R_{\mu\nu}$ of the space-time metric $g_{\mu\nu}$ is defined in terms of the connection coefficients $\Gamma^{\rho}_{\mu\nu}$ (which depend on the metric and its first derivatives only):

$$R_{\mu\nu}\equiv\partial_\rho\Gamma^{\rho}_{\mu\nu}-\partial_\mu\Gamma^{\sigma}_{\sigma\nu}+\Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\sigma\rho}-\Gamma^{\rho}_{\sigma\mu}\Gamma^{\sigma}_{\rho\nu}\ , \tag{2}$$

where we must remember that

$$\Gamma^{\rho}_{\rho\mu}=\partial_\mu(\ln\sqrt{g}\,) \tag{3}$$

and g stands for the absolute value of the determinant of the metric $g_{\mu\nu}$.

The Ricci tensor $R_{\mu\nu}$ can also be written

$$R_{\mu\nu}=(1/\sqrt{g}\,)\partial_\rho[\sqrt{g}\,(\Gamma^{\rho}_{\mu\nu}-\delta^{\rho}_{\mu}\Gamma^{\sigma}_{\sigma\nu})] -\Gamma^{\rho}_{\sigma\mu}\Gamma^{\sigma}_{\rho\nu}+\Gamma^{\sigma}_{\sigma\mu}\Gamma^{\rho}_{\rho\nu} \tag{4}$$

so that the principal part (the part containing derivatives of the connection coefficients) is expressed in flux conservative form and Einstein's equations (1) can be interpreted as a system of balance laws. The right-hand side in Eqs. (1) contains the source terms of nongravitational origin. The last two terms in (4) can be interpreted as gravitational source terms reflecting the nonlinear structure of the equations.

II. THE EVOLUTION SYSTEM

The system of ten equations (1) is not suitable for the numerical construction of the space-time. This is better understood if we decompose the metric $g_{\mu\nu}$ into its space and time components:

$$g_{\mu\nu}dx^{\mu}dx^{\nu}=-\alpha^2dt^2+\gamma_{ij}(dx^{i}+\beta^{i}dt)(dx^{j}+\beta^{j}dt)\ , \tag{5}$$

where latin indices correspond to spatial components only (3+1 decomposition[11]). The subset of equations obtained when taking $\mu=i$, $\nu=j$ in (1) provides six independent second-order evolution equations for the induced metric γ_{ij}: it is the so-called evolution system of Einstein's field equations.

The remaining independent equations in (1) provide only constraints to be satisfied by the initial data, but no evolution equations for the lapse function α nor the shift vector β^i can be extracted from (1). In fact, one can choose α and β^i at will, and these four gauge degrees of freedom correspond to the arbitrary choice of coordinate systems in the space-time.

In what follows we shall express the space components of Einstein's equations (1) in terms of the quantities appearing in (5). We shall adopt for simplicity an Eulerian coordinate system ($\beta^i=0$), but the relevant results for a nonzero shift vector are given in the Appendix. Unless otherwise stated, all geometric operations (such as raising and lowering indices) and geometric objects (such as connection coefficients) are to be understood as the ones associated with the three-dimensional metric γ_{ij}.

Let us begin by listing the values of the relevant connection coefficients of $g_{\mu\nu}$:

$$\begin{aligned} {}^{(4)}\Gamma^0_{0i} &= \partial_i(\ln\alpha), \quad {}^{(4)}\Gamma^0_{ij} = -(1/\alpha)K_{ij} , \\ {}^{(4)}\Gamma^k_{0i} &= -\alpha K_i^k, \quad {}^{(4)}\Gamma^k_{ij} = \Gamma^k_{ij} , \end{aligned} \tag{6}$$

where K_{ij} is the extrinsic curvature of the $t=$const slices and it can be obtained from

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} . \tag{7}$$

Allowing for (4), the first-order form of the evolution system of Einstein's field equations is then given (when $\beta^i=0$) by the system of balance equations (7) and

$$\begin{aligned} -\partial_t(\sqrt{\gamma}K_{ij}) + \partial_k(\alpha\sqrt{\gamma}\Gamma^k_{ij}) - \partial^2_{ij}(\alpha\sqrt{\gamma}) = \alpha\sqrt{\gamma}[\, & T_{ij} - (T/2)\gamma_{ij} + \Gamma^r_{ki}\Gamma^k_{rj} + 2K_i^{\ k}K_{kj} \\ & + \partial_i(\ln\alpha)\partial_j(\ln\alpha) - \partial_i \ln(\alpha\sqrt{\gamma})\partial_j \ln(\alpha\sqrt{\gamma})] . \end{aligned} \tag{8}$$

Let us note that no evolution equation has been given for the flux terms appearing in (8). This is because of the well-known relationship between the metric and the connection coefficients:

$$\Gamma_{kij} = (\partial_i\gamma_{jk} + \partial_j\gamma_{ik} - \partial_k\gamma_{ij})/2 \tag{9}$$

which acts as a constraint. For many purposes, however, one can consider the evolution equation obtained when taking the time derivative of (9),

$$\partial_t\Gamma_{kij} + \partial_r[\alpha(\delta^r_i K_{jk} + \delta^r_j K_{ik} - \delta^r_k K_{ij})] = 0 , \tag{10}$$

so that (9) is a first integral of (10) to be imposed on the initial data only.

III. COORDINATE CONDITIONS

As stated in the preceding section, the evolution system must be supplemented with four conditions to compute α and β^i. We have imposed three of these conditions by fixing $\beta^i=0$ when writing Eqs. (7) and (8). This is the so-called Eulerian gauge and it amounts to taking the time coordinate lines to be normal to the slices $t=$const. In that way, we ensure that for a regular time slicing of the space-time the congruence of time coordinate lines will not become singular. We must remember, however, that β^i is related only with the choice of spatial coordinates at every time slice. We think that one must not be dogmatic about that point.

The specification of the lapse function α is much more important because it is related with the time slicing of the space-time. It is well known that this choice must avoid the so-called crushing singularities[12] and this forbids the trivial choice $\alpha=1$. The standard alternative is the maximal slicing[13]

$$\gamma^{ij}K_{ij} = 0 \tag{11}$$

which is known to avoid these singularities. The trouble with this condition is that its compatibility with Eq. (8) leads to a second-order elliptic differential equation in the spatial derivatives of the lapse function[13] which must be solved anew at each time step.

A simpler alternative is to demand the time coordinate to be harmonic (harmonic synchronization). In the Eulerian gauge it simply reads

$$\partial_t(\sqrt{\gamma}/\alpha) = 0 \tag{12}$$

which is trivially integrated allowing an algebraic specification of the lapse function

$$\alpha(t,x^i) = C(x^i)\sqrt{\gamma} . \tag{13}$$

Condition (13) was introduced as an "algebraic gauge" to show the hyperbolic character of the (appropriately written) evolution system of Einstein's equations.[14] The ability of this condition to avoid crushing singularities has been shown recently.[15]

If we substitute (13) into our system (7) and (8), we get

$$\partial_t\gamma_{ij} = -2C(\sqrt{\gamma}K_{ij}) , \tag{14a}$$

$$-\partial_t(\sqrt{\gamma}K_{ij}) + \partial_k[C\gamma(\Gamma^k_{ij} - \delta^k_i\Gamma^r_{rj} - \delta^k_j\Gamma^s_{si})] = \gamma C_{ij} + C\gamma[\Gamma^r_{si}\Gamma^s_{rj} - 3\Gamma^r_{ri}\Gamma^s_{sj} + 2K_i^{\ k}K_{kj} + T_{ij} - (T/2)\gamma_{ij}] , \tag{14b}$$

where C_{ij} stands for $\partial^2_{ij}C$. The system (14) has the balance-law structure. This has the further advantage of treating both space and time derivatives on an equal footing and, therefore, is closer to the spirit of general relativity. Let us only note that if we supplement (14) with the "constraint" (9), the system is not of first order. In order to get a first-order system, we must supplement (14) with (10) [with α defined by (13)] and (9) is to be imposed on the initial data only.

IV. SPHERICALLY SYMMETRIC SPACE-TIMES

Let us restrict ourselves to the case of space-times with spherical symmetry:

$$ds^2 = -\alpha^2(t,r)dt^2 + X^2(t,r)dr^2 + Y^2(t,r)(d\theta^2 + \sin^2\theta\, d\phi^2) , \qquad (15)$$

where one must choose the lapse function $\alpha(t,r)$ according to (13). A natural choice for the parameter $C(x^i)$ is

$$C = a(r)/\sin\theta \qquad (16)$$

so that the lapse function is given by

$$\alpha(t,r) = a(r)XY^2 \qquad (17)$$

and it is not affected by the coordinate singularity on the polar axis.

The evolution equations (14a) for the relevant components of the metric (15) are

$$\partial_t X^2 = -2a(r)(XY^2K_{rr}), \quad \partial_t Y^2 = -2a(r)(XY^2K_{\theta\theta}) \qquad (18)$$

and we need only two evolution equations for the components of K:

$$\begin{aligned}
-\partial_t(XY^2K_{rr}) - \partial_r[aX^2Y^4(\Gamma^r_{rr} + 4\Gamma^\theta_{\theta r})] &= X^2Y^4a_{rr} + aX^2Y^4\{-2(\Gamma^r_{rr})^2 - 10(\Gamma^\theta_{\theta r})^2 - 12(\Gamma^r_{rr}\Gamma^\theta_{\theta r}) + 2(K_{rr}/X)^2 \\
&\quad + [T_{rr} - (T/2)X^2]\} , \\
-\partial_t(XY^2K_{\theta\theta}) + \partial_r(aX^2Y^4\Gamma^r_{\theta\theta}) &= aX^2Y^4\{-1 + 2(\Gamma^r_{\theta\theta}\Gamma^\theta_{\theta r}) + 2(K_{\theta\theta}/Y)^2 + [T_{\theta\theta} - (T/2)Y^2]\} ,
\end{aligned} \qquad (19)$$

where we have used $\Gamma^\theta_{\theta r} = \Gamma^\phi_{\phi r}$.

The connection coefficients appearing in (19) are given by

$$\Gamma_{rrr} = \partial_r(X^2)/2, \quad \Gamma_{\theta\theta r} = -\Gamma_{r\theta\theta} = \partial_r(Y^2)/2 \qquad (20)$$

and their evolution equations (10) can be written as follows:

$$\begin{aligned}
&\partial_t\Gamma_{rrr} + \partial_r(aXY^2K_{rr}) = 0 , \\
&\partial_t\Gamma_{\theta\theta r} + \partial_r(aXY^2K_{\theta\theta}) = 0 .
\end{aligned} \qquad (21)$$

We are not claiming that (18) and (19) are the simplest way of writing the evolution system for the spherically symmetric line element (15). The interest of Eqs. (18) and (19) arises from the fact that they are obtained directly from the form (14) of the full three-dimensional evolution system.

V. A NUMERICAL TEST

At this point, it may be convenient to verify the system (18) and (19) by means of the numerical construction of a previously known analytic solution of Einstein's field equations. We will choose a vacuum metric: this means that our spherically symmetric solution will be in fact the Schwarzschild line element in some coordinate system. To get a real test, one must avoid choosing the standard system in which the metric coefficients X and Y in (15) are independent of the time coordinate. We have chosen instead

$$X^2 = (2m/Y)^4(1 - 2m/Y)[1 - (2m/Y)^4]^{-1} , \qquad (22)$$

where $Y(t,r)$ is given by the implicit equation

$$r = t + 2m[(Y/2m)^3/3 + (Y/2m)^2/2 + Y/2m + \ln(Y/2m)] \qquad (23)$$

and the coefficient $a(r)$ in (17) is

$$a(r) = (2m)^{-2} = \text{const} . \qquad (24)$$

The standard form of the Schwarzschild line element is recovered from (22)–(24) in two steps. One can use first (23) to go from the (t,r) to the (t,Y) coordinate system and then perform the transformation of the time coordinate,

$$t = \tau + 2m \ln|1 - 2m/Y| , \qquad (25)$$

to get the standard (τ, Y) Schwarzschild form. Note that the form (22)–(24) of the metric is regular for $Y > 0$; the well-known singularity of the (τ, Y) coordinate system at the horizon ($Y = 2m$) arises from the fact that the transformation (25) becomes singular there.[15]

The slicing given by the $t =$const hypersurfaces can be easily represented in a Kruskal diagram (see Fig. 1). For a given value of our time coordinate t, we can substitute the Schwarzschild time τ from (25) into the well-known definition of the Kruskal coordinates (u,v) to obtain the equation for every hypersurface in terms of the parameter Y:

$$\begin{aligned}
u &= \tfrac{1}{2}(\sqrt{Y}/2m)e^{Y/4m}[e^{t/4m} + (1 - 2m/Y)e^{-t/4m}] , \\
v &= \tfrac{1}{2}(\sqrt{Y}/2m)e^{Y/4m}[e^{t/4m} - (1 - 2m/Y)e^{-t/4m}]
\end{aligned} \qquad (26)$$

which is valid in the whole $Y > 0$ space-time domain.

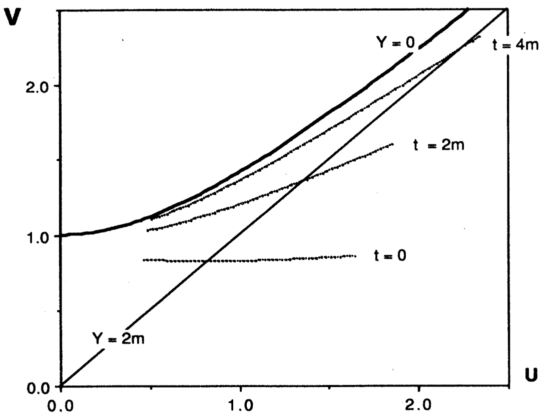

FIG. 1. Kruskal diagram showing the evolution of the spacelike slices $t=$const. It provides a clear demonstration of the singularity-avoiding properties of the harmonic synchronization. The slice $t=6m$ is not shown because it gets too close to the $Y=0$ singularity (without actually reaching it: see Fig. 2) to be represented on the same scale.

The values of the Schwarzschild coordinate Y in terms of our radial coordinate r can be computed for every fixed value of t from Eq. (23).

We have used our Eqs. (18)–(20) with $a(r)$ given by (24) to compute the values of the metric coefficients X, Y in the domain given by $2m<r<8m$, $0<t<6m$ (see Fig. 1). The analytic expressions (22) and (23) have been used to provide both the initial and boundary conditions. We have chosen the well-known "staggered leapfrog" method[16] to obtain a finite-differenced version of our equations. The numerical results for the canonical coordinate Y after 200, 400, and 600 iterations are shown in Fig. 2. The agreement with the values computed from the analytic expressions (22) and (23) is excellent (errors are of the order 10^{-4}) and this confirms the correctness of the spherically symmetric balance-law system (18) and (19).

At this point, we can check whether or not our numerical results actually verify the constraint equations in (1). Analytically, the constraints are propagated by the evolution equations, but a finite-differenced version of the evolution system does not necessarily do this. We must note that one needs a finite-differenced version of the constraint equations to perform this check. This is also a

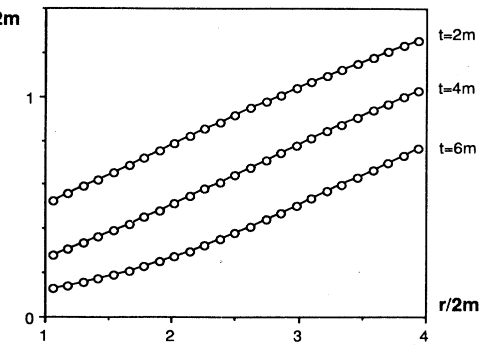

FIG. 2. The numerical values of the Schwarzschild coordinate Y are plotted after 200, 400, and 600 iterations. Only 25 of the 50 computed points are shown for clarity. The continuous line is the one obtained from the analytical solution [Eq. (23) in the text].

source of errors which can be estimated by using the analytic expressions (22) and (23) to compute the constraints. We have found in that way that the order of magnitude of the finite-differencing errors in the energy constraint is 10^{-3} and it is 10^{-5} in the momentum constraint. As far as the typical errors in our numerical results are of the order 10^{-4}, we see that only the momentum constraint will provide a meaningful check. We have then evaluated the momentum constraint by using our numerical results and we have found that the errors remain of the same order (10^{-4}) as in the evolution system.

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APPENDIX

In the generic case (nonzero shift vector β^i) it is very hard to proceed directly from the four-dimensional form (4). A more convenient starting point is provided by the standard form of the evolution system in the 3+1 decomposition.[11] The balance-law form of this system [the analog of Eqs. (7) and (8) in the text] is

$$\partial_t \gamma_{ij} - \partial_k(\delta_i^k \beta_j + \delta_j^k \beta_i) = -2\alpha K_{ij} - 2\Gamma_{ij}^k \beta_k \ , \tag{A1}$$

$$\begin{aligned}\partial_t(\sqrt{\gamma}K_{ij}) - \partial_k(\sqrt{\gamma}\beta^k K_{ij} + \alpha\sqrt{\gamma}\Gamma_{ij}^k) - \partial_{ij}^2(\alpha\sqrt{\gamma}) \\ = \sqrt{\gamma}(K_{ik}\partial_j\beta^k + K_{jk}\partial_i\beta^k) - \alpha\sqrt{\gamma}[T_{ij} - (T/2)\gamma_{ij} + \Gamma_{ki}^r\Gamma_{rj}^k + 2K_i^k K_{kj} + \partial_i(\ln\alpha)\partial_j(\ln\alpha) - \partial_i(\ln\alpha\sqrt{\gamma})\partial_j(\ln\alpha\sqrt{\gamma})]\end{aligned} \tag{A2}$$

and the supplementary equation (10) reads now

$$\partial_t \Gamma_{kij} + \partial_r[\delta_i^r(\alpha K_{jk} + \Gamma_{jk}^s\beta_s) + \delta_j^r(\alpha K_{ik} + \Gamma_{ik}^s\beta_s) - \delta_k^r(\alpha K_{ij} + \Gamma_{ij}^s\beta_s) - \delta_i^r(\partial_j\beta_k)] = 0 \ . \tag{A3}$$

The coordinate condition (12) (harmonic synchronization) becomes

$$\partial_t(\sqrt{\gamma}/\alpha)-\partial_k(\beta^k\sqrt{\gamma}/\alpha)=0 \tag{A4}$$

and one must provide three supplementary conditions to determine β^i.

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