Spacetime of supermassive U(1)-gauge cosmic strings

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We investigate numerically the spacetime geometry in the presence of an infinitely long, straight, static, U(1)-gauge cosmic string formed during phase transitions at energy scales larger than the grand-unified-theory scale. As the energy scale of symmetry breaking increases, we find that at radial infinity the geometry around a string changes from Minkowskian minus a wedge to an analog of a Kasner spacetime. The geometry transition occurs at $\Delta\phi=2\pi$, where the deficit angle $\Delta\phi$ is defined in the sense of comparison with flat spacetime in the absence of the string. Phase transitions producing such supermassive strings should occur before inflation to avoid contradictions with current observations.

The main importance of cosmic strings inheres in their ability to produce the desired density perturbations in a galaxy formation scenario.[1] It is understandable then that most of the work in this field has been focused primarily on the astrophysical consequences of cosmic strings. However, ever since Vilenkin[2] pointed out that the geometry around a cosmic string of linear energy density μ is that of Minkowski spacetime minus a wedge of angular size $\Delta\phi=8\pi\mu$, relativists have not been able to resist the temptation of analyzing in more detail the nature of spacetimes in the presence of cosmic strings.[2-14]

Under the cosmic-string scenario of galaxy formation, strings of astrophysical relevance were formed during phase transitions at the grand-unified-theory (GUT) scale (10^{15} GeV). They have linear energy $\mu\sim\eta^2$ of the order 10^{-6}, where η is the energy scale of the symmetry breaking that produces the strings. (We use units where $\hbar=c=G=1$.) One should then recognize that the linear energy density of these strings justifies the weak-field approximation used by Vilenkin[2] for the purpose of analyzing gravitational effects of the strings. Nevertheless, the Universe may have undergone phase transitions at energy scales higher than the 10^{15}-GeV GUT scale which may produce more massive strings, $\eta\gg10^{-3}$. If these strings were formed, they would most likely be stretched away during the inflationary era, and therefore they would not contradict current observations. On the other hand, because of their large masses, $\mu\gg10^{-6}$, this type of string can no longer be treated by means of the weak-field approximation.

The geometry of astrophysically relevant cosmic strings has been extensively investigated for global,[9,10,13] local,[3-6,8] and bosonic superconducting[11,12] strings. We emphasize "astrophysically relevant" because most of the work has been done only for values of $\mu\le10^{-6}$, in agreement with current observational constraints.[15] Recently,[7] studies of strings with $\mu\sim10^{-2}$ have shown that although the structure of spacetime inside the string is not flat, the geometry at radial infinity remains Minkowskian minus a wedge. It is for these strings that the corrections to the weak-field approximation formula for the deficit angle, $\Delta\phi=8\pi\mu$, due to gravitational self-interaction play a significant role. On the other hand, for the case of bosonic superconducting strings,[11,12] the spacetime approaches instead a version of the Kasner metric due to the electromagnetic current of the string.

The purpose of this paper is then to extend the analysis of the spacetime in the presence of a static cylindrically symmetric string formed during phase transitions above the GUT scale. In particular we want to find if there exists a maximum value for the energy scale of the symmetry breaking, such that above this maximum value the geometry at radial infinity is no longer Minkowskian minus a wedge. Although it is likely that quantum effects (including those of quantum gravity) become important for the most massive of these strings, our treatment will be completely classical.

The analysis consists of numerically solving the static coupled Einstein-scalar-gauge field equations for an infinitely long, straight, U(1)-gauge cosmic string. We use the model[16] of a cosmic string that consists of a U(1)-gauge vector field A_a coupled to a complex scalar field $\Phi\equiv Re^{i\psi}$. The Lagrangian for these fields reads

$$\mathcal{L}=-\tfrac{1}{2}\nabla_a R\,\nabla^a R-\tfrac{1}{2}R^2(\nabla_a\psi+eA_a)(\nabla^a\psi+eA^a)-\frac{\lambda}{8}(R^2-\eta^2)^2-\tfrac{1}{4}F_{ab}F^{ab}\,, \quad (1)$$

where $F_{ab}\equiv\nabla_a A_b-\nabla_b A_a$ and e, λ, and η are positive constants which constitute the parameter space. The spacetime metric is assumed to be static and cylindrically symmetric, that is,

$$ds^2=-e^{A}dt^2+e^{B}dz^2+e^{C}d\phi^2+d\rho^2\,, \quad (2)$$

where A, B, and C are functions of only the radial coordinate ρ. The coordinates t and z are chosen so that the metric functions satisfy the boundary conditions

$A(0)=B(0)=0$, and regularity of the metric at the axis requires that $\lim_{\rho\to 0}e^{C}/\rho^2=1$. It can be shown[5] that for the given boundary conditions the string is Lorentz invariant in the z direction, i.e., $B=A$ everywhere. Furthermore, consistent coupled solutions for the scalar and gauge field are found to have the form $R=R(\rho)$, $\psi=n\phi$, and $A_a=(n/e)[P(\rho)-1]\nabla_a\phi$, where n represents the winding or circulation number. The winding number n gives a measure of the wrapping of the scalar field phase around the string. Since the qualitative features of the fields do not depend strongly on the winding number n, we will restrict ourselves to the case $n=1$.

Following Ref. 17, the dimension of the parameter space is reduced by introducing the gauge-to-scalar mass ratio $\alpha\equiv m_A/m_\Phi$, where the masses of the vector and scalar fields are $m_A=e\eta$ and $m_\Phi=\sqrt{\lambda}\eta$, respectively. Since the radii of the fields are of order $1/m$, the parameter α measures not only the relative strength of the gauge and scalar fields but also the comparison between the radii of the core false vacuum and the magnetic field tube. Strings with $\alpha>1$ $(\alpha<1)$ behave like type-I, attractive (type-II, repulsive) flux tubes in superconductors. The energy scale of symmetry breaking η and the mass ratio α then constitute the free parameters of the string model.

As is usually done,[17] we introduce the following rescaled functions: $r\equiv\sqrt{\lambda}\eta\rho$, $X\equiv R/\eta$, and $K\equiv\sqrt{\lambda}\eta e^{A+C/2}$. Thus the functions that determine the metric, scalar, and gauge fields are A, K, X, and P. The coupled Einstein-scalar-gauge equations reduce to the following ordinary nonlinear differential equations for these functions:[5]

$$(KA')'-4\pi\eta^2[-\tfrac{1}{2}K(X^2-1)^2+2\alpha^{-2}K^{-1}e^{2A}(P')^2]=0\ , \tag{3}$$

$$K''-4\pi\eta^2[-2K^{-1}e^{2A}P^2X^2-\tfrac{3}{4}K(X^2-1)^2+\alpha^{-2}K^{-1}e^{2A}(P')^2]=0\ , \tag{4}$$

$$K(KX')'-X[\tfrac{1}{2}K^2(X^2-1)+e^{2A}P^2]=0\ , \tag{5}$$

$$e^{-2A}K(e^{2A}K^{-1}P')'-\alpha^2X^2P=0\ . \tag{6}$$

Here a prime denotes d/dr. The boundary conditions that a solution of these equations must satisfy to represent a regular isolated string are $X(0)=A(0)=K(0)=P'(0)=0$; $P(0)=K'(0)=1$; $\lim_{r\to\infty}X=1$ and $\lim_{r\to\infty}P=0$.

To solve Eqs. (3)–(6), we used a numerical technique that iterates between two sectors of the code: a string sector and a gravitational sector. In the string sector the metric functions A and K are fixed and Eqs. (5) and (6) are solved for the scalar and gauge functions X and P as a two-point boundary problem using a relaxation method. In the gravitational sector the values of A and K are updated by integrating Eqs. (3) and (4) using a fourth-order Runge-Kutta method. In the code the variable r takes on values in the interval $[0,r_\infty]$ where r_∞ is chosen to be twenty times the radius of the scalar field X. As a consistency check, the components of the string energy-momentum tensor at r_∞ were found to be $\leq 10^{-6}$ relative to their maximum values which occur at the string axis, showing that indeed the integrations were carried out to points far outside the string.

Following Ref. 5, we examine the properties of the spacetime at large r by using the expression

$$KA'(K'-\tfrac{3}{4}KA')=8\pi\eta^2K^2P_r\ , \tag{7}$$

where P_r is the radial component of the energy-momentum tensor of the string in units of $\lambda\eta^4$ and is given by

$$P_r=\tfrac{1}{2}[(X')^2-\tfrac{1}{4}(X^2-1)^2]+\tfrac{1}{2}K^{-2}e^{2A}[\alpha^{-2}(P')^2-X^2P^2]\ . \tag{8}$$

One can derive Eq. (7) as follows: use the field equations (3)–(6) and expression (8) to evaluate $8\pi\eta^2(K^2P_r)'$ and show that it is the derivative of the left-hand side of Eq. (7); the boundary conditions then yield Eq. (7). In Ref. 5 it is shown that with the assumption that $\int_0^\infty K\sigma\,dr$ converges (where σ is the energy density of the string in units of $\lambda\eta^4$) then K' and KA' reach constant asymptotic values $(K')_\infty$ and $(KA')_\infty$ as $r\to\infty$. It was also shown that under the assumption that $\lim_{r\to\infty}K^2\sigma=0$ the right-hand side of Eq. (7) also vanishes in that limit. Thus we find

$$KA'(K'-\tfrac{3}{4}KA')|_\infty=0\ . \tag{9}$$

From Eq. (9) one can then conclude that as $r\to\infty$ the string must approach either a vacuum metric which has $(KA')_\infty=0$ or one which has $(K')_\infty-\frac{3}{4}(KA')_\infty=0$. If we denote by K_∞ and A_∞ the values of the metric fields far from the string, one obtains that for $(KA')_\infty=0$ integration yields $A_\infty=a_1$ and $K_\infty=a_2r+a_3$ where a_i's are constants.

Thus the metric outside the string approaches

$$ds^2=e^{a_1}(-dt^2+dz^2)+\lambda^{-1}\eta^{-2}[dr^2+e^{-2a_1}(a_2r+a_3)^2d\phi^2]\ . \tag{10}$$

It is straightforward to check[5] that Eq. (10) represents the metric of flat spacetime minus a wedge. On the other hand, for $(K')_\infty-\frac{3}{4}(KA')_\infty=0$ one gets $K_\infty=b_1r+b_2$ and $A_\infty=\frac{4}{3}\ln(b_1r+b_2)+b_3$ with b_i's constants of integration. The spacetime metric for this case is

$$ds^2=e^{b_3}(b_1r+b_2)^{4/3}(-dt^2+dz^2)+\lambda^{-1}\eta^{-2}[dr^2+e^{-2b_3}(b_1r+b_2)^{-2/3}d\phi^2]\ . \tag{11}$$

This metric is an analog of a Kasner metric (here r and t are reversed from the usual Kasner cosmology situation) and has the property that as r increases the circumference of circles ($r=$const) asymptotically approaches zero; the spacetime becomes effectively three dimensional.

If when increasing the energy scale of symmetry breaking a continuous transition occurs from a spacetime with flat metric (10) to one given by (11), one would have to go in a continuous way from a stage where $(KA')_\infty=0$ (flat) to one with $(K')_\infty-\frac{3}{4}(KA')_\infty=0$. Hence the transition point must be when $(KA')_\infty=(K')_\infty=0$. Thus, $K_\infty=c_1$

and $A_\infty = c_2$. For this case the metric outside the string would read

$$ds^2 = e^{c_2}(-dt^2+dz^2)+\lambda^{-1}\eta^{-2}(dr^2+c_1^2 e^{-2c_2} d\phi^2) . \tag{12}$$

This metric is essentially that of flat spacetime with one compact dimension.

In order to examine the changes of the spacetime metric as one increases the energy scale of symmetry breaking, it is useful to define the quantity D_∞ by

$$1-\frac{D_\infty}{2\pi} \equiv e^{-A}(K'-\tfrac{1}{2}KA')|_{r_\infty} . \tag{13}$$

One can then show that D_∞ is given by

$$D_\infty = 8\pi\mu_\infty + \frac{\pi}{2}\int_0^{r_\infty} dr\, e^{-A}K(A')^2 , \tag{14}$$

where μ_∞ is the mass per unit length of that part of the string with $r<r_\infty$ and is given by

$$\mu_\infty = \pi\eta^2 \int_0^{r_\infty} dr\, e^{-A}K[(X')^2+K^{-2}e^{2A}X^2P^2+\tfrac{1}{4}(X^2-1)^2 + \alpha^{-2}K^{-2}e^{2A}(P')^2] . \tag{15}$$

In our case the stress energy of the string falls off sufficiently quickly that μ_∞ is essentially equal to μ the mass per unit length of the string. One can derive Eq. (14) as follows: evaluate $[e^{-A}(K'-\frac{1}{2}KA')]'$ using Eqs. (3), (4), (7), and (8) and then integrate from 0 to r_∞ to obtain Eq. (14).

We find numerically that the behavior of the metric outside the string depends on D_∞. When $D_\infty<2\pi$ the metric outside the string has the form given in Eq. (10), i.e., flat space minus a wedge. When $D_\infty>2\pi$ the asymptotic form of the metric is given in Eq. (11) and is the analog of Kasner spacetime. The metric takes the form in Eq. (12) when $D_\infty=2\pi$. We consider each of these cases in turn.

First consider the case where $D_\infty<2\pi$ and the metric far from the string is flat space minus a wedge. We now demonstrate that in this case D_∞ is equal to $\Delta\phi$ where $\Delta\phi$ is the deficit angle. First note that $\Delta\phi$ is given by[5]

$$2\pi-\Delta\phi = \lim_{r\to\infty}\frac{dl}{d\rho} , \tag{16}$$

where l is defined to be the length of an orbit of the angular Killing field $(\partial/\partial\phi)^a$. By the definition of l one has

$$l=\int_0^{2\pi}\sqrt{g_{\phi\phi}}\,d\phi$$

which in our case becomes

$$l=\frac{2\pi}{\sqrt{\lambda}\eta}e^{-A}K . \tag{17}$$

Substitution of (17) into (16) yields

$$1-\frac{\Delta\phi}{2\pi} = e^{-A}(K'-KA')|_\infty . \tag{18}$$

However, since the spacetime approaches flat space minus a wedge, it follows that $(KA')_\infty=0$. Then using

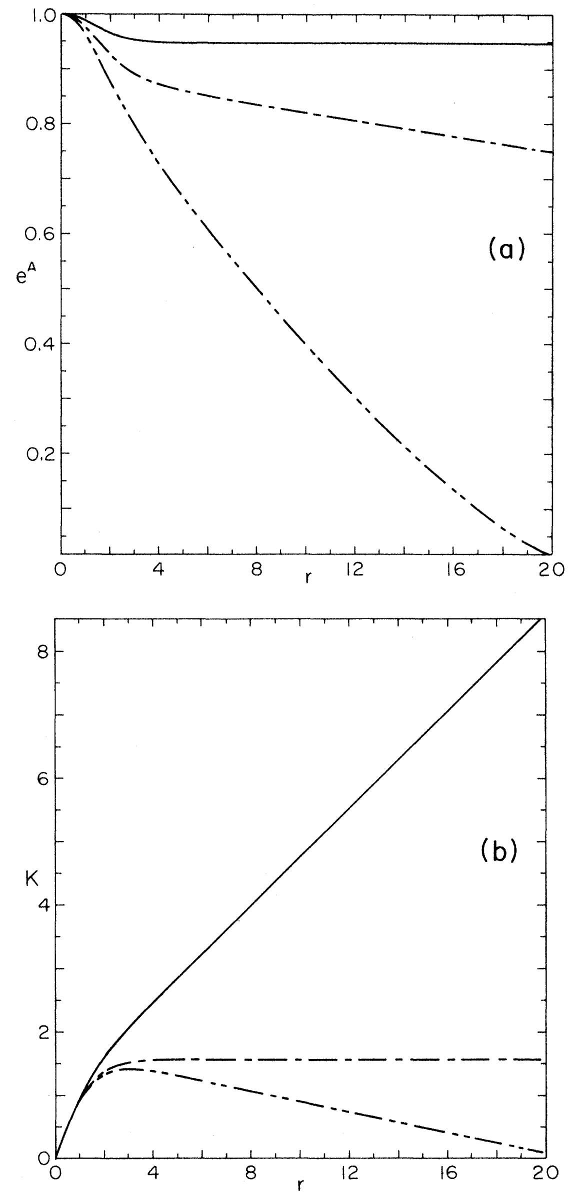

FIG. 1. Solutions for the metric fields (a) e^A and (b) K as functions of the radial coordinate r, in the case $\alpha=1/\sqrt{2}$. The dash patterns (——), (– – –), and (—·—) correspond to $D_\infty=\pi$, 2π, and 3π, respectively.

Eqs. (13) and (18) we find that

$$\Delta\phi = D_\infty . \tag{19}$$

Now we consider the case where $D_\infty>2\pi$ and the spacetime outside of the string is the analog of Kasner spacetime. We demonstrate that the spacetime becomes singular at some finite value of r which we denote $r_{\max}$. First note that smoothness of the metric requires that $K>0$. Hence the metric becomes singular at $r_{\max}$ if $K(r_{\max})=0$. Since $(K'-\frac{3}{4}KA')_\infty=0$ it follows from Eq. (13) that $K'_\infty=3e^{A_\infty}(1-D_\infty/2\pi)<0$. Therefore, in the expression $K_\infty=b_1r+b_2$ for the metric function (11), the constants b_1 and b_2 satisfy $b_1<0$, $b_2>0$. Clearly the singular point $K(r_{\max})=0$ occurs at $r_{\max}\equiv -b_2/b_1$. A calculation of the Riemann tensor for the metric in Eq. (11) shows that the curvature scalar $R^{abcd}R_{abcd}$ diverges like $(r-r_{\max})^{-4}$ as $r\to r_{\max}$. Thus the string spacetime contains a curvature singularity at $r=r_{\max}$.

In the case where $D_\infty=2\pi$ it follows from Eq. (13) that $(K'-\frac{1}{2}KA')_\infty=0$. Since $KA'(K'-\frac{3}{4}KA')|_\infty=0$, it then follows that $K'_\infty=KA'_\infty=0$ and the metric far from the string has the form given in Eq. (12).

We now present some results of the numerical analysis. Throughout the simulations the scalar and gauge fields of the string, which are solved for as a two-point boundary-value problem, did not exhibit qualitative changes other than those previously[7] analyzed for $\mu\le 10^{-2}$. Therefore we will fix our attention on the metric functions e^A and K.

Figures 1(a) and 1(b) show the gravitational fields for a U(1)-gauge cosmic string in the case $\alpha=1/\sqrt{2}$ when D_∞ takes on the values π, 2π, and 3π. This corresponds to values for η of 0.2, 0.255, and 0.26, respectively. At $D_\infty=\pi$ one notifies that the asymptotic values $(e^A)_\infty$ and K_∞ closely resemble the flat metric functions of Eq. (9). That is, $(e^A)_\infty$ becomes constant, and K_∞ grows linearly with the radial coordinate r. The quantity $(KA')_\infty$ was found to satisfy the flat metric condition $(KA')_\infty=0$ to a relative accuracy of $\le 10^{-4}$. At $D_\infty=2\pi$ both $(e^A)_\infty$ and K_∞ are constant. Thus the metric asymptotically has the

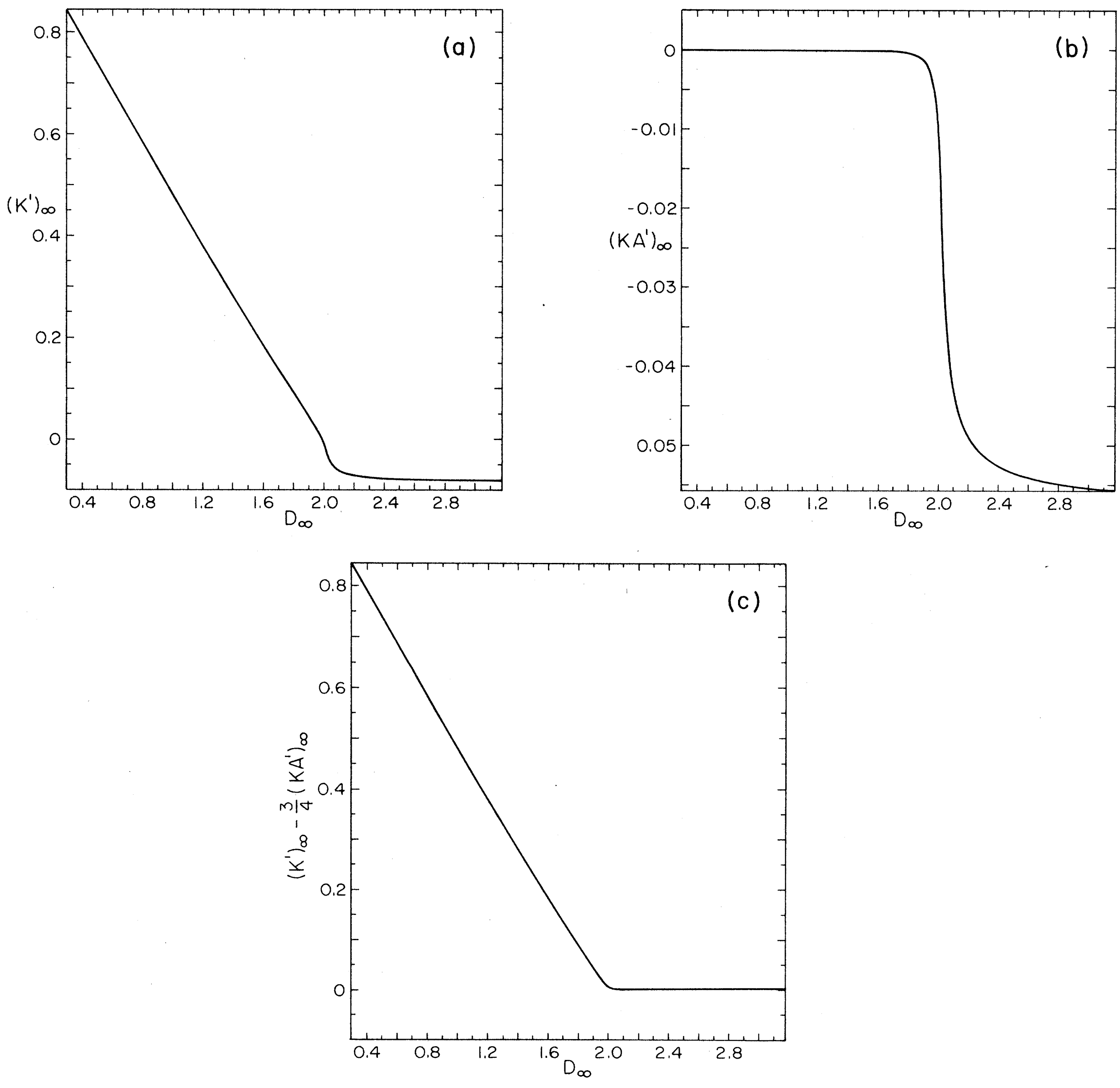

FIG. 2. Plots of the asymptotic values (a) $(K')_\infty$, (b) $(KA')_\infty$, and (c) $(K')_\infty-\frac{3}{4}(KA')_\infty$ as functions of D_∞ in units of π.

form given in Eq. (11). For $D_\infty = 3\pi$ the quantity K_∞ again depends linearly on the radial coordinate r but now with a negative slope. In order to verify that the metric is that of the analog of Kasner spacetime, we computed the quantity $(K')_\infty - \frac{3}{4}(KA')_\infty$, and found that it vanishes to a relative accuracy of $\leq 10^{-4}$.

To have a complete picture of the Minkowski-Kasner transition, we plotted in Figs. 2(a)–2(c) the behavior of the asymptotic quantities $(K')_\infty$, $(KA')_\infty$, and $(K')_\infty - \frac{3}{4}(KA')_\infty$, respectively, as a function of D_∞. For values of D_∞ smaller than 2π, one notices that $(KA')_\infty = 0$, while $(K')_\infty$ and $(K')_\infty - \frac{3}{4}(KA')_\infty$ decrease approximately linearly with D_∞. The spacetime is Minkowskian. As mentioned before, at $D_\infty = 2\pi$ both $(KA')_\infty = 0$ and $(K')_\infty = 0$, thus also $(K')_\infty - \frac{3}{4}(KA')_\infty = 0$. The spacetime has the metric given in Eq. (11). Finally, for values of D_∞ greater than 2π, $(KA')_\infty$ and $(K')_\infty$ have nonvanishing negative values that satisfy the Kasner spacetime condition $(K')_\infty - \frac{3}{4}(KA')_\infty = 0$.

Figures 3(a) and 3(b) show how the Minkowski-Kasner ($D_\infty = 2\pi$) transition values of the linear energy density μ and the energy scale of symmetry breaking η, respectively, depend on the gauge-to-scalar mass ratio α. In particular one observes from Fig. 3(a) that the transition occurs near $\mu = \frac{1}{4}$ for most values of α. This means that the second term on the right-hand side of Eq. (14) is not as important as the first term. Finally from Fig. 3(b) one sees that the transition point $D_\infty = 2\pi$ occurs at larger values of η when the strength of the scalar field dominates than in the opposite case when the gauge field strength is dominant. To understand this effect, let us for the moment neglect the effect of gravity on the internal structure of the string and recall[7,18,19] that for nongravitating strings μ is well approximated by $\mu \approx \pi\eta^2\alpha^{-0.4}$. Since the transition takes place near $\mu = \frac{1}{4}$, we then find that the transition value of η is well approximated by $\eta \approx (1/2\sqrt{\pi})\alpha^{0.2}$.

In summary, we have found that, for supermassive cosmic strings $\mu \gg 10^{-6}$ formed during phase transitions above the GUT scale, the spacetime geometry far from the string remains Minkowskian minus a wedge as long as the relation $D_\infty \leq 2\pi$ is satisfied. Above the limiting value of D_∞ the geometry becomes an analog of a Kasner spacetime and has a curvature singularity at a finite distance from the string. Strings with $D_\infty > 2\pi$ are thus somewhat pathological. Since surviving cosmic strings with $\mu \gg 10^{-6}$ are inconsistent with present observations, such supermassive strings (if they exist) would have to be formed before an inflationary era and subsequently inflated away.

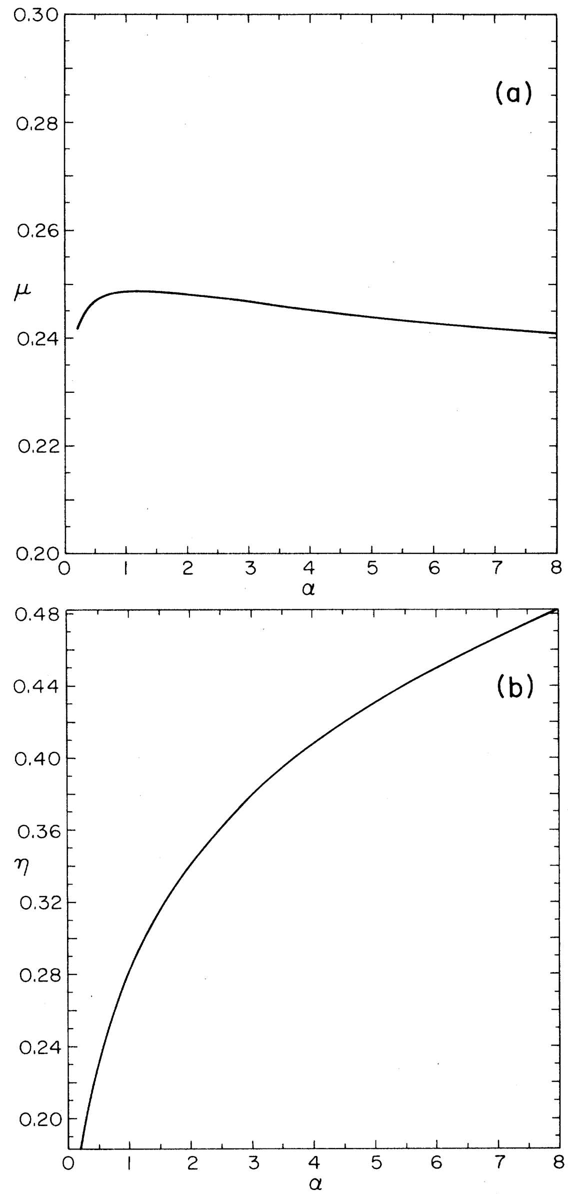

FIG. 3. Plots of (a) the linear energy density μ and (b) the symmetry-breaking energy scale η as functions of the gauge-to scalar mass ratio α for the Minkowski-Kasner transition value $D_\infty = 2\pi$.

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