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Canonical Quantization of Cylindrical Gravitational Waves*

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The Einstein-Rosen cylindrical gravitational waves are quantized by the canonical methods due to Dirac (the constraint formalism) and to Arnowitt, Deser, and Misner (the ADM deparametrized formalism). The general reduction of geometrodynamical phase space to a mini-phase-space by intransitive groups of motion with spacelike Killing vectors is presented. The ADM classical canonical formalism restricted to the infinite-dimensional mini-phase-space generated by the cylindrical group of motions is built up. Six invariantly defined functions of one coordinate label are introduced as new canonical variables by a canonical transformation in mini-phase-space. Two canonical coordinates are identified with the Einstein-Rosen time and cylindrical radius. Canonically conjugate to them are C-energy density and C-energy flux. The third pair of canonical variables carries the ∞^1 dynamical degrees of freedom of the cylindrical wave. The canonical transformation mixes superspace with momentum space, the Einstein-Rosen time being constructed from the extrinsic curvature of the spacelike hypersurface rather than from its intrinsic geometry. The ADM classical canonical formalism for cylindrical gravitational waves is proved to be identical with the parametrized formalism for the cylindrical massless scalar waves propagating in Minkowskian spacetime. The identity of the quantum formalisms follows. The extrinsic time representation, with the Einstein-Rosen time and cylindrical radius as two of the three basic variables, is used instead of the metric representation. The Dirac contraints are imposed on the state functional. If the hypersurfaces are labeled by the Einstein-Rosen cylindrical radius, the constraints are reduced to a functional differential equation of the Schrödinger type. This equation is further reduced to a single partial differential equation by integrability conditions which ensure that the evolution of the state functional between two hypersurfaces is path-independent. The inner product of two state functionals conserved by the deformation of the hypersurface is defined. Under the coordinate conditions restricting the allowable hypersurfaces to those of a constant Einstein-Rosen time, the Dirac formalism is deparametrized into the ADM quantum formalism.

I. INTRODUCTION

The gravitational field has been quantized by many different methods. All of them have at least one feature in common: It is difficult to apply them to a concrete problem. In a sense, the aim of this paper is complementary to that of inventing a new method of quantization. It presents a simple problem on which different methods of quantization can be tried out and compared with each other. We argue that the cylindrical gravitational waves are ideal for this purpose. In this paper, two versions of canonical quantization are applied to them. Through the analysis, we gain insight into superspace, the nature of geometrodynamical time, and the ways in which the many-fingered time formalism works. It is believed that the other methods of quantization of the gravitational field (such as the Feynman integration over all paths,[1] or the Mandelstam method[2]) can also be better understood when illustrated on this simple model.

The canonical quantization of the gravitational field has been studied and improved by a number of persons. The reader interested in the historical details may find them in the paper by DeWitt.[3] The classical canonical formalism, which was originally quite cumbersome, attained its present elegant status through the work[4] of Arnowitt, Deser, and Misner. Starting from the ADM action principle, the canonical theory can be developed by two different lines of reasoning, the first one due to ADM themselves, and the second one initiated by Dirac.[5] Let us explain their basic differences.

The canonical formalism necessarily destroys the spacetime covariance of the theory by cutting spacetime into slices and investigating their geometrical properties. In the Dirac approach, all spacelike slices are equally admissible, as well as all systems of coordinates on them. The intrinsic geometry and the extrinsic curvature of the slices enter the formalism as canonical coordinates and momenta. The canonical variables cannot be freely specified, but are subject to $(1+3)\infty^3$ constraints: the super-Hamiltonian constraint and the supermomentum constraints. In quantum theory, the state functional is defined on an arbitrary slice and in an arbitrary system of coordinates. The constraints are imposed on the state functional and assume the form of functional differential equations. They govern the changes of the state functional under a deformation of the slice and under a displacement of the spatial system of coordinates.

In the ADM approach, a definite slicing of spacetime and a definite coordinatization of the slices are picked out by coordinate conditions. Four canonical coordinates and four canonical momenta are eliminated by the coordinate conditions and by a subsequent "deparametrization" of the canonical formalism. The true nonvanishing Hamiltonian depending on two unconstrained canonical pairs of variables emerges in the process of deparametrization. The state functional is defined only on a one-parameter family of slices and in a fixed system of coordinates on them. The quantization then proceeds as in standard field theory.

As already mentioned, the main purpose of this paper is to understand the canonical quantization and some of its consequences by applying it to a particularly simple problem. The basic method is to freeze all but a few of the infinitely many degrees of freedom of the gravitational field by putting a number of canonical coordinates and their conjugate momenta identically equal to zero. Such a trick is justified in the classical theory, but violates the uncertainty principle in the quantum theory. In spite of this, it is believed that at least some relevant features of the full quantum theory are preserved and that, by studying the simplified models with only a few degrees of freedom left, we can learn a lot about general quantum geometrodynamics.

The pioneer of this approach is DeWitt, who first applied the Dirac method of quantization to the Friedmann universe.[3] In appearance, this is the simplest of all possible models, because all gravitational degrees of freedom are frozen except one: the radius of the universe. In reality, the model becomes quite complicated by the very fact that it is too simple. The difficulty is that the Dirac approach needs at least one degree of freedom to describe the geometrodynamical time. The radius of the universe can do that – it is a gigantic hand of a clock telling the epoch of the universe. But what about the state functional? If it depended only on the gravitational degrees of freedom, we would run into a difficulty of interpretation. The state functional is the probability amplitude that the dynamical variables of the system have definite values at a given time – but where are the dynamical variables? The paradox is easily resolved. The classical Friedmann universe cannot exist without matter, and matter provides the additional variables for the state functional. However, the presence of matter variables is regarded as a *Schönheitsfehler* by the true believers in pure geometrodynamics, who hold that geometry needs no matter for its interpretation, though matter can be interpreted in terms of quantized geometry.

The pollution of geometrodynamics by matter is absent in the second model treated so far by the method of frozen variables – in Misner's "mixmaster" universe.[6] The mixmaster universe is an empty homogeneous universe the geometry of which is characterized by three variables: its volume V, and two anisotropy parameters $\beta_\pm$. Misner quantizes the mixmaster universe by the ADM method, the volume of the universe serving as a geometrodynamical time, and the anisotropy parameters as the dynamical degrees of freedom. However, neither the Friedmann universe nor the mixmaster universe gives us the opportunity to exhibit the full scope of the Dirac method of quantization. Because of the high symmetry of these models, a privileged slicing of the spacetime exists, such that the intrinsic geometry of the slices is homogeneous. The symmetry thus provides a unique one-parameter family of spacelike hypersurfaces on which the further formalism is based. This goes against the spirit of the Dirac method, which permits an arbitrary slicing, and, in a sense, also against the usual procedures of the ADM method, which picks out a one-parameter family of slices by coordinate conditions rather than by symmetry considerations. As a result, when we apply the Dirac method of quantization to such degenerate models, it becomes almost indistinguishable from the ADM method.

The homogeneity of the models is also responsible for the drastic reduction in the number of gravitational degrees of freedom. E.g., from the infinitely many degrees of freedom of the gravitational field, only three are left in the mixmaster universe, one of them representing the geometrodynamical time of the privileged family of slices. In a typical field theory, we expect to find several degrees of freedom at each point of space, and we can study the interaction between the degrees of freedom at neighboring points, resulting in the propagation of a wave. The requirement of homogeneity ties the corresponding degrees of freedom in different points rigidly together, forcing a typical degree of freedom in one point to imitate the behavior of the corresponding degree of freedom at any other point, so that they finally dance together like the well-disciplined Rockettes. The field aspect of gravity has thus almost completely disappeared from the model.

The same situation can also be described in a slightly different language. The mixmaster universe can be interpreted as the lowest gravitational mode fitting into a closed universe, the universe being closed by the effective energy of the mode. The lack of wave propagation is then not surprising.

It is not possible to form a wave packet from one mode of the field, and observe how this packet travels from one point to another. The quantization of the model amounts to quantizing only the lowest mode of the field, keeping all higher modes artificially frozen.

At this point, a natural question arises. Why not take a full-fledged gravitational wave with an infinity of higher modes present, and quantize it instead of quantizing the closed universe? After all, it was not a *universe* that was first quantized in quantum field theory, but the electromagnetic *wave*. There are exact wave solutions known in Einstein's theory of gravitation – namely, the cylindrical waves[7] and the plane waves.[8] So let us take, for example, the cylindrical gravitational waves, freeze all extracylindrical degrees of freedom, and quantize the cylindrical degrees of freedom by the Dirac and the ADM canonical methods. This is exactly what is done in this paper.

The cylindrical gravitational wave (by which we mean the wave symmetric with respect to reflections in the planes containing the axis of symmetry and perpendicular to the axis of symmetry, as well as with respect to translations along the axis and rotations around it) has only one polarization, the other one being eliminated by the reflection symmetry. On the other hand, there are ∞^1 degrees of freedom contained in this polarization, one degree of freedom for each cylindrical surface drawn around the axis of symmetry. The degrees of freedom on one cylindrical surface still dance in unison, but those on another surface may lag behind them, giving rise to the radial propagation of a wave. An infinite number of higher modes are present in the wave. Furthermore, the slicing of the spacetime is not completely fixed by the symmetry. All slices with rotational symmetry around the axis of symmetry are allowed. Such slices are too numerous to be fitted into a one-parameter family. In this way, both objections against the cosmological models are met by the cylindrical waves. The field aspect of the gravitational theory is truly represented and the Dirac formalism has a chance to show its specific features, distinguishing it from the ADM method.

The choice of the free canonical variables and the freezing process are done almost intuitively in the models we have discussed. However, observing how the formalism works in these cases, we can abstract an underlying algorithm. The reduction of the canonical formalism is accomplished by a group of motions which imposes supplementary conditions on the intrinsic geometry and the extrinsic curvature of the slices which are invariant varieties of the group. These conditions reduce the superspace (the space of all three-dimensional geometries) to a mini-superspace, and the corresponding geometrodynamical phase space to a mini-phase-space. The procedure is quite general, and we hope to study the systematics of models arising from the various group structures in a subsequent paper.

Perhaps the most interesting aspect of the cylindrical gravitational waves is the light they shed on the nature of geometrodynamical time. In the Dirac formalism, one degree of freedom at each point of space represents a Tomonaga-Schwinger time variable. This was already noted in a more specialized situation in which the privileged slicing implying homogeneity reduced the Tomonaga-Schwinger time function to one real parameter labeling the slices. The problem is how to disentangle the time variable from the truly dynamical degrees of freedom. Let us place this problem in a broader setting.

The general theory of relativity is a culmination of long efforts to understand the nature of space and time. "What then is time?" asks Saint Augustine in Book XI of his *Confessions*, and replies: "If no one asks me, I know; if I wish to explain it to one that asketh, I know not: yet I say boldly, that I know, that if nothing passed away, time past were not; and if nothing were coming, a time to come were not; and if nothing were, time present were not."[9] "For if eternity and time are rightly distinguished, in that time exists not without a varying changeableness, whereas in eternity is no change, who seeth not that times could have been, had no creature come into existence, which should vary something by some change?"[10] "The course of time began with the motions of creation, wherefore it is idle to ask about time before creation, which were to ask for time before time.... Time, therefore, rather hath its commencement from the creation, than creation from time, but both from God."[11] This is perhaps the first exposition of the idea that time is a measure of change of the existing things, and that in the absence of changing things there would be no time.

There is another stream of ideas flowing across the territory of the general theory of relativity. We can summarize its basic tenet by saying that all existing things are made out of geometry. Plato's vision that the world consists of four perfect solids,[12] Clifford's picture of particles as wave packets of geometry,[13] Einstein's whole life quest for a unified field theory,[14] and Wheeler's conception of elementary particles as geometrodynamical excitons,[15] follow this stream down from antiquity to the present time. Now, if we bring together Saint Augustine's philosophy of time with the Plato-Clifford-Einstein-Wheeler tradition, we conclude almost by a syllogism that *time is a measure of change of geometry*.

This is well known from the classical general theory of relativity, though in a less pretentious wording. The proper time τ along a timelike curve $x^\iota = x^\iota(t)$ is defined by the integral

$$\tau = \int_{t_1}^{t_2} \left(-{}^4g_{\iota\kappa} \frac{dx^\iota}{dt} \frac{dx^\kappa}{dt}\right)^{1/2} dt \ ,$$

and therefore constructed out of geometry. Incidentally, this formula is usually interpreted the other way around: The proper time is measured independently by a standard clock moving along the worldline $x^\iota = x^\iota(t)$, and its connection with geometry is used to define operationally what geometry means. Geometry is thus explained in terms of "material objects," like standard clocks, rather than material objects being explained in terms of geometry.

However, this is not exactly the point we would like to stress. Our primary concern is the modification in the concept of time necessitated by the quantum principle. Unfortunately, the proper time has no proper place in quantum geometrodynamics. Quantum geometrodynamics speaks about the intrinsic and extrinsic geometries of spatial hypersurfaces, rather than about the spacetime geometry along timelike curves. The intrinsic and extrinsic geometries stand to each other as the canonical coordinate to its conjugate momentum, and we cannot know them simultaneously in quantum geometrodynamics. The proper time becomes an operator which has in general no sharp value. If time is to be constructed out of geometry, one question still remains to be answered: Out of what geometry, the intrinsic geometry or the extrinsic geometry?

Wheeler and DeWitt gave their preference to the intrinsic geometry. In classical geometrodynamics, Wheeler formulated the sandwich conjecture, according to which the intrinsic geometry freely specified on two closed spacelike hypersurfaces uniquely determines the spacetime geometry between them and therefore carries information about the proper time.[16] In quantum geometrodynamics, DeWitt opted for the metric representation, in which the components of the intrinsic metric tensor are chosen as diagonal operators. He showed that the pure dilation of the intrinsic metric tensor has the character of a timelike displacement, if a natural metric is imposed on superspace.[3] Misner also identified a combination of the components of the intrinsic metric tensor (the logarithm of the volume of the universe) with the geometrodynamical time, when quantizing the mixmaster universe.[6]

The rival candidate for time is the extrinsic geometry. Now, in spite of the ingenious arguments by Wheeler and DeWitt, we are firmly convinced that time should be identified rather with an extrinsic curvature variable than with an intrinsic geometry variable. Such an identification was first attempted by ADM,[4] used by Peres,[17] and discussed by the present author.[18] However, the formalism looks natural only for the linearized theory, and becomes implicit and involved for the full nonlinear theory. At this point, the cylindrical waves have something to say. As far as we know, they are the first model in which it is possible to define an extrinsic time explicitly and elegantly even for strong gravitational fields, and to show the great advantages of the extrinsic time representation.

One of these advantages looks so important that it should guide the choice of the time variable. Namely, the canonical variable conjugate to a good extrinsic time enters the super-Hamiltonian *linearly*. It follows that the main Dirac constraint has the form of a Schrödinger equation. This is in sharp contrast to the metric representation, in which the Dirac constraint is an equation of the Klein-Gordon type. It is usual to interpret this Klein-Gordon-type constraint as a time evolution equation for the state functional. It seems to us that this interpretation is misleading and that a much better analogy to the Klein-Gordon-type constraint is given by the time-independent Schrödinger equation for stationary states. It would be interesting to study the extrinsic time variables in other simple models and see if they can bring the Dirac constraint into the Schrödinger form.

After all these remarks concerning motivation and the general significance of the quantization of the cylindrical gravitational waves, let us indicate the main results. The most important conclusion reached is that the Dirac quantization of the cylindrical gravitational waves is completely isomorphic to the quantization of a single cylindrical massless scalar field on a Minkowskian spacetime background, if this field is quantized in curvilinear coordinates by the constraint method. It is shown that only two Dirac constraints remain in the cylindrical case, and these can be reduced further to one main constraint of the Schrödinger type. The formal solution of this remaining constraint is written down. The ADM quantization of the cylindrical gravitational waves is proved to be equivalent to the standard quantization of the cylindrical massless scalar field on a Minkowskian spacetime background.

We start the presentation by summarizing the traditional treatment of the Einstein-Rosen waves. In Sec. II, the general form of the cylindrically symmetric line element is written down, and an invariant meaning of the standard coordinates and the standard metric coefficients is emphasized. This line element is a starting point of the ADM formalism, and the notation is adapted to it. In Sec. III, the Einstein vacuum equations are used to cast the line element to the Einstein-Rosen form. The physical meaning of the remaining Einstein equations is discussed. The transformation from the Einstein-Rosen coordinates to the general cylindrical coordinates is given, and the coefficients of the general cylindrically symmetric line element are connected with the coefficients of the Einstein-Rosen line element and the Einstein-Rosen coordinates. The boundary conditions at spatial infinity, at the axis of symmetry, and in the remote past and future are formulated. In Sec. IV, the ADM canonical formalism is reviewed and the Dirac and the ADM methods of quantization are recapitulated. Section V deals with the general reduction of the ADM canonical formalism by the spatial groups of motions. The concepts of mini-superspace and mini-phase-space are introduced. It is shown that the geometrodynamical trajectory stays in the mini-phase-space, if its initial point lies there. The general formalism of Sec. V is applied to the Einstein-Rosen waves in Sec. VI. The super-Hamiltonian and the supermomentum of the Einstein-Rosen waves are written down. In Sec. VII, it is shown that the Einstein-Rosen time can be constructed from the canonical momenta, and has therefore the character of an extrinsic time. By a canonical transformation, the Einstein-Rosen time is introduced as a canonical coordinate. Canonically conjugate to it is the energy density of the wave. The second pair of canonical variables is the Einstein-Rosen cylindrical radius and the energy flux. It is only the third pair of canonical variables which carries the ∞^1 dynamical degrees of freedom of the cylindrical wave. The super-Hamiltonian and the supermomentum are significantly simplified, if the Einstein-Rosen time is used as a new canonical coordinate. Even greater simplification is achieved by the introduction of the Einstein-Rosen advanced and retarded times as two of the canonical coordinates in Sec. VIII. To prove that the formalism built for the gravitational waves is isomorphic to a formalism for the cylindrical massless scalar field in a Minkowskian spacetime, we interrupt the investigation of the Einstein-Rosen waves and insert Sec. IX dealing with the cylindrical scalar waves. Cylindrical-type curvilinear coordinates in Minkowskian spacetime are introduced as supplementary canonical variables and the resulting "parametrized formalism" is shown to be identical with the formalism for the Einstein-Rosen waves. Fixing the system of coordinates on the slices, but leaving the slices themselves arbitrary, we reduce the parametrized formalism of Sec. IX to the half-parametrized formalism of Sec. X. In Sec. XI, cylindrical waves are quantized by the Dirac method. The Dirac constraints are imposed on the state functional. The supermomentum constraint implies the invariance of the state functional under the relabeling of the hypersurface. In Sec. XII, the hypersurface is labeled by the Einstein-Rosen cylindrical radius. In this half-parametrized formalism of Sec. X, the super-Hamiltonian constraint turns out to be a functional differential Schrödinger equation. If we know its solution, we construct the solution of the super-Hamiltonian and supermomentum constraints of the fully parametrized theory. However, the functional differential Schrödinger equation still represents an infinite set of equations, one equation for each value of the radial coordinate. In Sec. XIII, we show that these equations are mutually dependent, satisfying an infinite number of integrability conditions. The integrability conditions ensure that the evolution of the state functional is path-independent and they reduce the functional differential Schrödinger equation to a partial differential Schrödinger equation. The formal solution of this single equation is written down, expressing the state functional on an arbitrary slice by means of its initial value on an initial slice. In Sec. XIV, the state functional is interpreted as the probability amplitude. The Schrödinger equation implies that the probability satisfies an equation of continuity. The inner product of two state functionals that is left unchanged by the deformation of the hypersurface is defined. In Sec. XV, the realization of the extrinsic time representation is discussed from the point of view of the quantum theory of measurement. In Sec. XVI, the coordinate conditions are imposed. They permit only the slices of a constant Einstein-Rosen time, labeled by the Einstein-Rosen cylindrical radius. The Dirac formalism is then replaced by the much simpler formalism of ADM. In Sec. XVII, Table I shows the main steps followed in quantizing cylindrical gravitational waves.

Let us explain our notation. Greek indices run through the values 0, 1, 2, 3, Latin indices through the values 1, 2, 3. The spacetime metric has the signature −, +, +, +. The spacetime quantities bear the superscript 4, which distinguishes them from the corresponding spatial quantities. E.g., ${}^4g^{ik}$ denotes the spatial part of the contravariant spacetime metric tensor, whereas g^{ik} denotes the contravariant spatial metric tensor. The determinant of g_{ik} is denoted by g, the Levi-Civita pseudotensor by ϵ_{ikl}. Partial differentiation is denoted by a comma, covariant differentiation with respect to the spatial metric by a stroke. The Riemann tensor, the Ricci tensor, and the scalar curvature are constructed from the affine connection $\Gamma^i{}_{kl}$ according to the conventions

$$R^i{}_{klm}=\Gamma^i{}_{km,l}-\Gamma^i{}_{kl,m}+\Gamma^n{}_{km}\Gamma^i{}_{nl}-\Gamma^n{}_{kl}\Gamma^i{}_{nm},$$
$$R_{ik}=R^l{}_{ilk},\quad R=R^i_i.$$

The symbol $\pounds$ is used for the Lie derivative. Square brackets emphasize the dependence of functionals on function variables. E.g., $\Psi[T(R),\psi(R)]$ is a functional of two function arguments $T(R)$ and $\psi(R)$, but $\mathcal{K}(T(R))$ is a function of $T(R)$ (a composite function of R). We put $h/2\pi=c=16\pi G=1$ (h is the Planck constant, c is the velocity of light, and G is the Newton gravitational constant).

II. LINE ELEMENTS WITH CYLINDRICAL SYMMETRY

Cylindrical gravitational fields are characterized by the existence of a two-parameter Abelian group of motions ${}^4G_2$ with two mutually orthogonal, hypersurface-orthogonal, spacelike Killing vectors ${}^4\xi^\iota_{(\varphi)}$ and ${}^4\xi^\iota_{(z)}$. It is assumed that the group of finite motions generated by ${}^4\xi^\iota_{(\varphi)}$ ("the translations along the axis of symmetry") acts freely on space-time, whereas the group of finite motions generated by ${}^4\xi^\iota_{(z)}$ ("the rotations around the axis of symmetry") does not act freely.[19]

There are still three classes of such fields possible, according to whether $R_{,\iota}$, where

$$R=({}^4\xi^\iota_{(\varphi)}\,{}^4\xi_{(\varphi)\iota}\,{}^4\xi^\kappa_{(z)}\,{}^4\xi_{(z)\kappa})^{1/2}, \tag{1}$$

is a spacelike, lightlike, or timelike vector.[20] We shall limit our discussion to the first class of cylindrical waves, for which $R_{,\iota}$ is a spacelike vector everywhere. It is this class of cylindrical waves that was originally investigated by Einstein and Rosen.

It is well known that spacetime is cylindrically symmetric if and only if there exists a coordinate system t,r,φ,z, $t\in(-\infty,+\infty)$, $r\in[0,\infty)$, $\varphi\in[0,2\pi)$, $z\in(-\infty,+\infty)$, in which the line element assumes the form

$$\begin{aligned}ds^2=&-(N^2-e^{\psi-\gamma}N_1{}^2)dt^2+2N_1dtdr\\&+e^{\gamma-\psi}dr^2+R^2e^{-\psi}d\varphi^2+e^{\psi}dz^2,\end{aligned} \tag{2}$$

where γ, $R\geq 0$, ψ, and N_1, N are functions of t and r. The particular dependence of the five nonvanishing coefficients g_{11}, g_{22}, g_{33}, and g_{01}, g_{00} on these five functions was tailored to suit the ADM canonical formalism. Specifically, N is the lapse function, and N_1 is the radial shift function. We shall rederive the line element (2) in Sec. VI, when reducing the canonical formalism by the cylindrical group of motions.

In the t,r,φ,z system of coordinates, the Killing vectors have the components

$${}^4\xi^\iota_{(\varphi)}=(0,0,1,0)\,,\quad {}^4\xi^\iota_{(z)}=(0,0,0,1)\,.$$

The group ${}^4G_2$ is therefore intransitive, its minimal invariant varieties being two-dimensional cylindrical surfaces $t=$const, $r=$const. Any hypersurface containing these two-dimensional cylindrical surfaces is also an invariant variety. The reduction of the canonical formalism in Sec. VI is based on using only such invariant hypersurfaces $t=t(r)$ as the allowed slices.

The coordinates φ and z in the line element (2) are adapted to the cylindrical symmetry. We can define them invariantly by means of the Killing vectors. Indeed, in the t,r,φ,z system of coordinates it is easily checked that

$$\begin{aligned}\varphi_{,\iota}&={}^4\xi_{(\varphi)\iota}/{}^4\xi^\kappa_{(\varphi)}\,{}^4\xi_{(\varphi)\kappa},\\ z_{,\iota}&={}^4\xi_{(z)\iota}/{}^4\xi^\kappa_{(z)}\,{}^4\xi_{(z)\kappa}.\end{aligned} \tag{3}$$

Equations (3) provide the desired invariant characterization of the two scalars φ and z. Two coefficients of the line element (2), namely R and ψ, also have an invariant meaning, R being given by Eq. (1), and ψ by the formula

$$\psi=\ln({}^4\xi^\iota_{(z)}\,{}^4\xi_{(z)\iota})\,. \tag{4}$$

Equation (1) has a simple intuitive interpretation. If we draw a two-dimensional cylindrical surface $t=$const, $r=$const around the axis of symmetry, and take its part between the "planes" $z=z_0$, $z=z_0+1$, the proper surface area of this part is $2\pi R$, i.e., the same as the proper surface area of a cylindrical surface of radius R in a Euclidean space.

The coordinates φ and z are fixed up to the trivial transformation $\varphi\to\pm\varphi+\varphi_0$, $z\to\alpha z+z_0$. On the other hand, the coordinates t and r can be subject to an arbitrary transformation

$$t\to\bar t=\bar t(t,r)\,,\quad r\to\bar r=\bar r(t,r) \tag{5}$$

without changing the general form of the line element (2). This is precisely the minimum flexibility we desire for a model to which the Dirac quantization is to be applied in a nontrivial way.

The transformations (5) are frequently used to simplify further the general form (2) of the line element. Because the metric coefficients depend only on t and r, we can cast the t,r part of the line element into a conformally flat form,

$$ds^2=e^{\bar\gamma-\psi}(-d\bar t^2+d\bar r^2)+R^2e^{-\psi}d\varphi^2+e^{\psi}dz^2\,. \tag{6}$$

We have written here R and ψ instead of $\bar R$ and $\bar\psi$, because we already know from Eqs. (1) and (4) that R and ψ behave as scalars. Let us also note that the spacetime metric is simplified, the shift function N_1 being transformed away and the lapse function N being correlated with the spatial metric. On the other hand, the general form of the *spatial* metric is left unaffected.

The new coordinates $\bar t,\bar r$ have an invariant meaning which is best described in geometrical terms. Because $g_{11}=-g_{00}$ for the line element (6), the light signal emitted perpendicularly to the axis of symmetry travels with the unit velocity. Let us therefore write an eikonal equation for the light front Φ in an arbitrary system of coordinates

$$ {}^4g^{\iota\kappa}\Phi_{,\iota}\Phi_{,\kappa}=0\,, \tag{7}$$

and assume that the light front has the cylindrical symmetry,

$$ {}^4\xi^{\iota}_{(\varphi)}\Phi_{,\iota}={}^4\xi^{\iota}_{(z)}\Phi_{,\iota}=0\,. \tag{8}$$

Equations (7) and (8) have two independent solutions, $\Phi^{(1)}$ and $\Phi^{(2)}$. If we choose these two solutions as the advanced and the retarded time variables,

$$\begin{aligned} \bar{t}+\bar{r}&=\Phi^{(1)}\,, \quad \bar{t}=\tfrac{1}{2}(\Phi^{(1)}+\Phi^{(2)})\,,\\ \bar{t}-\bar{r}&=\Phi^{(2)}\,, \quad \bar{r}=\tfrac{1}{2}(\Phi^{(1)}-\Phi^{(2)})\,, \end{aligned} \tag{9}$$

the coordinates $\bar{t}$ and $\bar{r}$ bring the line element (2) to the form (6). Equations (7)–(9) therefore give an invariant definition of $\bar{t}$ and $\bar{r}$.

This system of coordinates is, of course, still not unique, because the solutions $\Phi^{(1)}$ and $\Phi^{(2)}$ of Eqs. (7) and (8) are not unique; if Φ is a solution of Eqs. (7) and (8), an arbitrary function $G(\Phi)$ is also a solution of Eqs. (7) and (8). Let us therefore pick out two arbitrary functions $G^{(+)}(\bar{t}+\bar{r})$ and $G^{(-)}(\bar{t}-\bar{r})$, and take them as the new advanced and retarded coordinates, i.e., let us put either

$$\bar{\bar{t}}\pm\bar{\bar{r}}=G^{(\pm)}(\bar{t}\pm\bar{r}) \tag{10}$$

or

$$\bar{\bar{t}}\mp\bar{\bar{r}}=G^{(\pm)}(\bar{t}\pm\bar{r})\,. \tag{11}$$

The system of coordinates $\bar{t},\bar{r}$, in which the line element assumes the form (6), is arbitrary exactly up to the transformations (10) and (11). We can easily check that by virtue of Eq. (10),

$$\bar{\bar{t}}_{,\bar{t}}=\bar{\bar{r}}_{,\bar{r}}\,, \quad \bar{\bar{t}}_{,\bar{r}}=\bar{\bar{r}}_{,\bar{t}}\,, \tag{12}$$

and by virtue of Eq. (11),

$$\bar{\bar{t}}_{,\bar{t}}=-\bar{\bar{r}}_{,\bar{r}}\,, \quad \bar{\bar{t}}_{,\bar{r}}=-\bar{\bar{r}}_{,\bar{t}}\,, \tag{13}$$

so that in both cases $\bar{\bar{t}}$ and $\bar{\bar{r}}$ are harmonic functions of $\bar{t}$ and $\bar{r}$,

$$\bar{\bar{r}}_{,\bar{t}\bar{t}}-\bar{\bar{r}}_{,\bar{r}\bar{r}}=0\,, \tag{14}$$

$$\bar{\bar{t}}_{,\bar{t}\bar{t}}-\bar{\bar{t}}_{,\bar{r}\bar{r}}=0\,. \tag{15}$$

Having a function $\bar{\bar{r}}$ that satisfies Eq. (14), we can always choose it for a new radial coordinate, provided $\bar{\bar{r}}_{,\iota}$ is a spacelike vector. By solving Eqs. (12) or (13), we find the time coordinate $\bar{\bar{t}}$ corresponding to $\bar{\bar{r}}$. Equation (14) ensures that this is always possible because it is the integrability condition of the system of Eqs. (12) or (13).

III. THE EINSTEIN-ROSEN WAVES

In this section, we want to recall the familiar properties of the Einstein-Rosen waves, before rederiving them from the canonical formalism. Until now, no use was made of the Einstein vacuum field equations. If they are written for the line element (6), one of their immediate consequences is that R must be a harmonic function, $R_{,\bar{t}\bar{t}}-R_{,\bar{r}\bar{r}}=0$. Assuming $R_{,\iota}$ is a spacelike vector, we can choose R as the new radial coordinate and, integrating Eqs. (12) or (13), find the time coordinate T corresponding to R. By picking out the Einstein-Rosen coordinates T and R, we have removed the last ambiguity remaining in the choice of the "isothermal" coordinate system $\bar{t},\bar{r}$. The Einstein-Rosen coordinates can be uniquely defined by invariant prescriptions. Moreover, the remaining Einstein vacuum equations assume a very convenient form in the Einstein-Rosen coordinate system. Writing the line element (6) in these coordinates as

$$ds^2=e^{\Gamma-\psi}(-dT^2+dR^2)+R^2e^{-\psi}d\varphi^2+e^{\psi}dz^2\,, \tag{16}$$

the Einstein vacuum equations reduce to a set of three equations

$$\psi_{,TT}-\psi_{,RR}-R^{-1}\psi_{,R}=0\,, \tag{17}$$

$$\Gamma_{,R}=\tfrac{1}{2}R(\psi_{,T}{}^2+\psi_{,R}{}^2)\,, \tag{18}$$

$$\Gamma_{,T}=R\psi_{,T}\psi_{,R} \tag{19}$$

for two functions $\psi(T,R)$ and $\Gamma(T,R)$. This set of equations has a very remarkable structure. Equation (17) looks exactly like the ordinary wave equation for the cylindrically symmetric massless scalar field ψ propagating on a Minkowskian spacetime background. Moreover, if we determine the energy density of this field in the cylindrical coordinates, we get the expression on the right-hand side of Eq. (18), and if we determine the radial energy current density (or the radial momentum density), we get the expression on the right-hand side of Eq. (19). These densities are generated by differentiating a single function Γ with respect to the space and time coordinates, respectively. The law of conservation of energy is the integrability condition of the system of equations (18), (19), this integrability condition being satisfied by virtue of Eq. (17). The function Γ is therefore an energy superpotential. Because of this analogy with the scalar field in the Minkowskian spacetime, we shall call expression (18) the energy density and expression (19) the energy current density of the gravitational wave. These densities are identical with the C-energy densities introduced by Thorne.[7]

We shall deal in the following with a pure radiation field without sources on the axis of symmetry. The spatial geometry must be therefore locally Euclidean on the axis, i.e., the proper circumference of a small circle $R=$const, $T=$const, $z=$const must be the 2π multiple of its proper radius. This leads to the condition

$$\Gamma(T,0)=0\,. \tag{20}$$

Differentiating Eq. (20) with respect to T, we see that no energy can be absorbed or emitted on the axis and the energy current must vanish as $R\to 0$,

$$R\psi_{,T}\psi_{,R}\to 0 \quad \text{as } R\to 0\,.$$

We also assume that $\psi_{,T}$ and $\psi_{,R}$ fall off sufficiently rapidly at infinity,

$$\psi_{,T}=o(R^{-1})\,, \quad \psi_{,R}=o(R^{-1})\,,$$

in order that

$$\Gamma(T,\infty)=\int_0^\infty dR\, \tfrac{1}{2}R(\psi_{,T}{}^2+\psi_{,R}{}^2)$$

remains finite. We can interpret $2\pi\Gamma(T,\infty)$ as the total energy of the field contained between two parallel "planes" $z=z_0$ and $z=z_0+1$. Because $R\psi_{,T}\psi_{,R}=o(R^{-1})$, this energy remains constant:

$$2\pi\Gamma(T,\infty)=2\pi\Gamma_\infty=\text{const.}$$

If we keep T fixed and let R approach infinity, ψ approaches a constant value. The space-time is therefore locally Euclidean at spatial infinity. By rescaling z, we can even arrange to have $\psi\to 0$, so that the proper distances along the z lines coincide with the differences of z at spatial infinity. However, the spacetime is not globally Euclidean. The rate with which the proper circumference of the circle $R=\text{const}$, $T=\text{const}$, $z=\text{const}$ increases with its proper radius is not equal to 2π, but to $2\pi\exp(-\frac{1}{2}\Gamma_\infty)$. The constant Γ_∞ thus characterizes the conicality of space at spatial infinity.

The Einstein-Rosen form (16) of the line element is the best one for solving the Einstein field equations. On the other hand, the canonical formalism is naturally started from the general form (2). It is easy to return from (16) to (2), if we reintroduce the arbitrary coordinates t, r instead of the privileged Einstein-Rosen coordinates T, R,

$$\begin{aligned} &t=t(T,R)\,, \quad T=T(t,r)\,,\\ &r=r(T,R)\,, \quad R=R(t,R)\,. \end{aligned} \tag{21}$$

However, there are certain restrictions on the transformation (21). At first, we want t to be a timelike and r a spacelike coordinate. For this it is necessary that

$$R'^2>T'^2\,, \quad \dot{T}^2>\dot{R}^2\,. \tag{22}$$

Here, and in the following, a prime denotes differentiation with respect to a general radial coordinate r, and a dot denotes differentiation with respect to a general time coordinate t. Further, we want r to increase monotonically as we go away from the axis of symmetry towards infinity and the Einstein-Rosen time to increase monotonically as we proceed along a t line:

$$R'>0\,, \quad \dot{T}>0\,. \tag{23}$$

We also require that a hypersurface of constant t time have no conical singularity on the axis of symmetry. This means that no cusp can appear in the parametric equations (21) of this hypersurface on the line $r=0$:

$$T'\to 0 \quad \text{for } r\to 0\,. \tag{24}$$

We already know that under our conditions on ψ the spacetime geometry becomes asymptotically flat (though conical) far away from the axis. Therefore, whatever our system of coordinates is in the interior region, it is natural to require that it go over to an "asymptotically cylindrical" system of coordinates as $r\to\infty$, i.e.,

$$t\to T\,, \quad r\to R \quad \text{for } r\to\infty\,. \tag{25}$$

The hypersurface of constant t is therefore labeled by the value that T has on it at infinity. If we approach the axis, the spacetime becomes again locally Minkowskian. We already saw that the hypersurface of constant t must touch there the "plane" $T=\text{const}$, in order to eliminate the conical singularity in the spatial geometry [(condition (24)]. Now we require in addition that the radial coordinate r coincide with the Euclidean coordinate R also near the axis,

$$r\to R \quad \text{for } r\to 0\,. \tag{26}$$

Finally, we introduce the condition that our system of coordinates straighten itself into the Einstein-Rosen system of coordinates in the remote past and the remote future,

$$t\to T\,, \quad r\to R \quad \text{for } t\to\pm\infty\,. \tag{27}$$

Our boundary conditions are conveniently summarized in a schematic picture (Fig. 1). The reasons why we impose them will become clearer in Secs. VII and IX. Briefly, they justify the dropping of various boundary terms appearing in a canonical transformation from the intrinsic metric representation to the extrinsic time representation.

We are now in a position to write down the transformation equations from the Einstein-Rosen line element (16) to the general line element (2). We saw that R and ψ behave under this transformation as scalars. On the other hand,

$$\Gamma\to\gamma=\Gamma+\ln(R'^2-T'^2)\,, \tag{28}$$

and the lapse and shift functions are given by

$$N=e^{\frac{1}{2}(\Gamma-\psi)}(\dot{T}R'-T'\dot{R})(R'^2-T'^2)^{-1/2}\,, \tag{29}$$

$$N_1=e^{\Gamma-\psi}(R'\dot{R}-T'\dot{T})\,. \tag{30}$$

Let us note that γ is real because of the conditions (22). Moreover,

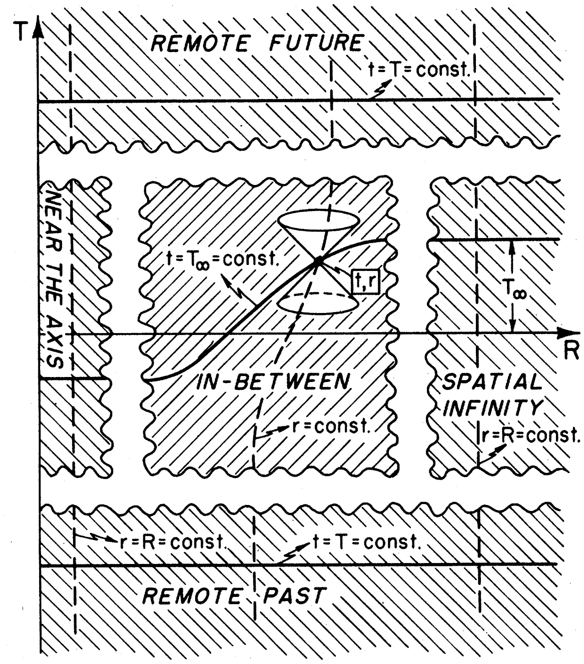

FIG. 1. The allowed slices and systems of coordinates for the Einstein-Rosen waves. The Einstein-Rosen coordinates are treated as Minkowskian coordinates. The slices t = const pass over to the T planes near the axis of symmetry, at spatial infinity, in the remote past and in the remote future. The radial label r coincides with R in these regions. The hypersurfaces t = const are spacelike and the lines r = const are timelike. The monotonic increase $\dot{T}\geq 0$, $R'\geq 0$ is assumed. The boundary conditions work in the marginal regions, the coordinate system is arbitrary in the central region. Note that the "plane" character of the slices t = const in the remote past, remote future, and at spatial infinity in our diagram does not imply that there are no gravitational waves there. The ψ function may still cause ripples in the intrinsic geometries of these "planes."

$$\gamma(t,0)=0\,, \tag{31}$$

by virtue of Eqs. (20), (24), and (26). At infinity,

$$\gamma(t,\infty)=\Gamma_\infty=\text{const} \tag{32}$$

because of the condition (25). The lower case γ is therefore subject to the same boundary conditions as the capital Γ. Of course, ψ is subject to the same boundary conditions in the general coordinates as in the Einstein-Rosen coordinates,

$$\psi\to 0\,,\quad \dot{\psi}=o(r^{-1})\,,\quad \psi'=o(r^{-1})\ \text{ for } r\to\infty\,,$$
$$r\dot{\psi}\psi'\to 0\ \text{ for } r\to 0\,. \tag{33}$$

The lapse and shift functions also behave as expected. The lapse function (29) is real and positive, because Eqs. (22) and (23) imply $\dot{T}R'>T'\dot{R}$. The shift function (30) vanishes on the axis of symmetry and at infinity, by virtue of Eqs. (24), (26), and (25).

IV. THE CANONICAL FORMALISM

ADM found an elegant way to cast the Einstein field equations into canonical form.[4] Their procedure starts by splitting the metric tensor ${}^4g_{\iota\kappa}$ into the spatial metric tensor g_{ik}, the lapse function N, and the shift functions N_i according to the schema

$$ {}^4g_{\iota\kappa}=\left\| \begin{matrix} -N^2+g^{lm}N_lN_m\,, & N_k \\ N_i & ,\ g_{ik} \end{matrix} \right\|,$$
$$N=(-{}^4g^{00})^{-1/2}\,,\quad N_i=g_{0i}\,. \tag{34}$$

If certain divergences are discarded in the gravitational Lagrangian density ${}^4\mathrm{R}(-{}^4g)^{1/2}$, the action functional S can be brought into the form

$$S=\int dt\int d^3x\, Ng^{1/2}(K_{ik}K^{ik}-K^2+\mathrm{R})\,. \tag{35}$$

Here, K_{ik} is the extrinsic curvature of the spacelike hypersurface t = const,

$$K_{ik}=\tfrac{1}{2}N^{-1}(-g_{ik,0}+N_{i|k}+N_{k|i})\,. \tag{36}$$

By varying (35) with respect to $g_{ik,0}$, the canonical momenta π^{ik} conjugate to g_{ik} are obtained,

$$\pi^{ik}=-g^{1/2}(K^{ik}-Kg^{ik})\,. \tag{37}$$

The action functional is then converted to the Hamiltonian form

$$S=\int dt\int d^3x\,(\pi^{ik}g_{ik,0}-N\mathcal{H}-N_i\mathcal{H}^i) \tag{38}$$

with the super-Hamiltonian $\mathcal{H}$ and the supermomentum $\mathcal{H}^i$ expressed as functions of the canonical variables g_{ik} and π^{ik},

$$\mathcal{H}\equiv g^{-1/2}(\pi_{ik}\pi^{ik}-\tfrac{1}{2}\pi^2)-g^{1/2}\mathrm{R}\,, \tag{39}$$

$$\mathcal{H}^i\equiv -2\pi^{ik}{}_{|k}=-2\pi^{ik}{}_{,k}-g^{il}(2g_{jl,k}-g_{jk,l})\pi^{jk}. \tag{40}$$

The variation of the action functional (38) with respect to the lapse and shift functions gives the initial value equations

$$\mathcal{H}=0\,,\quad \mathcal{H}^i=0\,. \tag{41}$$

The remaining Einstein vacuum equations are obtained by varying the action functional with respect to the dynamical variables g_{ik} and π^{ik},

$$g_{ik,0}=\frac{\delta H}{\delta\pi^{ik}}\,,\quad \pi^{ik}{}_{,0}=-\frac{\delta H}{\delta g_{ik}}\,, \tag{42}$$

$$H\equiv\int d^3x\,(N\mathcal{H}+N_i\mathcal{H}^i)\,. \tag{43}$$

In the Hamiltonian formalism, g_{ik} and π^{ik} are understood as independent variables given on a spacelike hypersurface t = const. However, as a consequence of the dynamical principle $\delta S=0$, these variables are subject to the constraints (41). If we know g_{ik} and π^{ik} satisfying these constraints on the initial hypersurface t, and want to know g_{ik} and π^{ik} on the neighboring hypersurface $t+\delta t$, we must first locate the new hypersurface with respect to the initial hypersurface, and choose a system of spatial coordinates on it. This is done by prescribing the lapse function N and shift functions N_i. We are completely free to do this, because the lapse and shift functions are not determined by the field equations. The lapse function gives the proper-time separation $\delta\tau(x^k)$ between the neighboring hypersurfaces t and $t+\delta t$ measured in the normal direction to the first hypersurface,

$$\delta\tau(x^k)=N(x^k)\delta t\,.$$

The shift functions N_i determine how the spatial system of coordinates on the hypersurface $t+\delta t$ is shifted with respect to the spatial system of coordinates on the hypersurface t. If the normal to the first hypersurface drawn at the point with the coordinates x^k intersects the second hypersurface at the point with the coordinates $x^k+\delta x^k$, then

$$\delta x^i=-N^i(x^k)\delta t\,.$$

Once the lapse and shift functions are prescribed, the dynamical variables $\tilde{g}_{ik}=g_{ik}+\delta g_{ik}$ and $\tilde{\pi}^{ik}=\pi^{ik}+\delta\pi^{ik}$ on the hypersurface $\tilde{t}=t+\delta t$ are obtained from the Hamilton equations (42),[21]

$$\begin{aligned}\delta g_{ik}&=\delta_0 g_{ik}+(N_{i|k}+N_{k|i})\delta t\,,\\ \delta_0 g_{ik}&\equiv 2Ng^{-1/2}(\pi_{ik}-\tfrac{1}{2}\pi g_{ik})\delta t\,,\end{aligned}\tag{44}$$

$$\begin{aligned}\delta\pi^{ik}&=\delta_0\pi^{ik}+[(\pi^{ik}N^m)_{|m}-\pi^{im}N^k{}_{|m}-\pi^{km}N^i{}_{|m}]\delta t\,,\\ \delta_0\pi^{ik}&\equiv[-Ng^{1/2}(\mathrm{R}^{ik}-\tfrac{1}{2}\mathrm{R}g^{ik})+\tfrac{1}{2}Ng^{-1/2}g^{ik}(\pi_{lm}\pi^{lm}-\tfrac{1}{2}\pi^2)-2Ng^{-1/2}(\pi^{im}\pi^k{}_m-\tfrac{1}{2}\pi\pi^{ik})+g^{1/2}(N^{|ik}-N_{|l}{}^l g^{ik})]\delta t\,.\end{aligned}\tag{45}$$

If the old variables g_{ik}, π^{ik} satisfy the constraints (41) on the hypersurface t, the new variables $g_{ik}+\delta g_{ik}$, $\pi^{ik}+\delta\pi^{ik}$ satisfy these constraints on the hypersurface $t+\delta t$. Equation (44) is the inversion of Eqs. (36) and (37).

The change δg_{ik} or $\delta\pi^{ik}$ of the dynamical variables consists of two parts. The first part, $\delta_0 g_{ik}$ or $\delta_0\pi^{ik}$, is independent of the shift functions. The second part contains the shift functions and vanishes if the shift functions vanish. Without changing the first part, we can easily eliminate the second part by making the transformation of coordinates

$$x^i\to x'^i=x^i-N^i\delta t$$

on the hypersurface $t+\delta t$. This trick, equivalent to putting $N_i=0$, simplifies many proofs. If we want to verify an equation which is covariant with respect to the spatial transformations and contains the quantities δg_{ik} and $\delta\pi^{ik}$, it is sufficient to verify the corresponding equation for the quantities $\delta_0 g_{ik}$ and $\delta_0\pi^{ik}$. We shall use this simplification in the Appendix.

The formalism explained so far is common to the ADM and the Dirac methods. However, ADM proceed further to fix the coordinate labels t and x^i, i.e., to fix the slicing and the spatial system of coordinates. In effect, ADM choose four functionals $T(t,x^k)$ and $X^i(t,x^k)$ of the dynamical variables g_{ik}, π^{ik}, depending on t and x^i as parameters, and find the canonical transformation

$$g_{ik},\pi^{ik}\to T,X^i,\pi_T,\pi_{X^i};g_A,\pi^A,\quad A=1,2\tag{46}$$

after which T and X^i play the role of new canonical coordinates,

$$S=\int dt\int d^3x\,(\pi_T\dot{T}+\pi_{X^i}\dot{X}^i+\pi^A\dot{g}_A-N\mathcal{H}-N_i\mathcal{H}^i)\,.\tag{47}$$

Further, they solve the four initial value equations (41) explicitly for the four canonical momenta π_T, π_{X^i},

$$\begin{aligned}\pi_T&=\pi_T(g_A,\pi^A;T,X^i)\,,\\ \pi_{X^i}&=\pi_{X^i}(g_A,\pi^A;T,X^i)\,,\end{aligned}$$

and substitute these solutions into the action functional (47). In this way, the terms containing the lapse and shift functions disappear. Finally, they impose four coordinate conditions

$$\begin{aligned}T&=T(t,x^k)|_{t=T,\,x^k=X^k}\,,\\ X^i&=X^i(t,x^k)|_{t=T,\,x^k=X^k}\,,\end{aligned}\tag{48}$$

meaning that the functionals T and X^i are used as the privileged coordinate labels. After all these operations, the action functional (47) assumes the form

$$S=\int dT\int d^3X\,[\pi^A g_{A,T}-\underline{\mathcal{H}}_{\mathrm{ADM}}(T,X^i;g_A,\pi^A)]\,,\tag{49}$$

in which $\underline{\mathcal{H}}_{\mathrm{ADM}}\equiv-\pi_T$ represents the true Hamiltonian density, and g_A, π^A represent the true dynamical variables. The transition to the quantum theory is then straightforward. The dynamical variables g_A, π^A are replaced by operators satisfying the commutation relations

$$[g_A(X^i),\pi^B(\tilde{X}^i)]=i\delta_A^B\,\delta(X^i-\tilde{X}^i)\,.$$

If we decide to work in the Schrödinger picture and the g_A representation, the state of the gravitational field is described by a functional $\underline{\Psi}_{\mathrm{ADM}}$ of g_A, satisfying the Schrödinger equation

$$\begin{aligned}i\frac{\partial\underline{\Psi}_{\mathrm{ADM}}}{\partial T}&=\underline{H}_{\mathrm{ADM}}\underline{\Psi}_{\mathrm{ADM}}\,,\\ \underline{H}_{\mathrm{ADM}}&\equiv\int d^3X\,\underline{\mathcal{H}}_{\mathrm{ADM}}\,.\end{aligned}\tag{50}$$

The Dirac method of quantization is more direct. No coordinate conditions are imposed, the slicing and the spatial coordinates being left completely arbitrary. The dynamical variables g_{ik}, π^{ik} are replaced by operators satisfying the commutation relations

$$[g_{ik}(x^n), \pi^{lm}(\tilde{x}^n)] = \tfrac{1}{2}i(\delta_i^l\delta_k^m + \delta_i^m\delta_k^l)\delta(x^n - \tilde{x}^n).$$

If we decide to work in the "metric representation," the state of the gravitational field is described by a functional Ψ of the metric g_{ik}. The super-Hamiltonian and the supermomentum become operators and the initial value equations are imposed as constraints on the state functional:

$$\mathcal{H}\Psi = 0, \quad \mathcal{H}^i\Psi = 0. \tag{51}$$

Of course, other representations may be used instead of the metric representation. In particular, we can perform the canonical transformation (46), and decide to work in the T, X^i, g_A representation. The relation between the ADM state functional $\underline{\Psi}_{\rm ADM}$ and the Dirac state functional Ψ then becomes apparent:

$$\Psi[T(x^k), X^i(x^k), g_A(x^k)]\,|_{T(x^k)=T=\text{ const };\, X^i(x^k)=x^i} = \underline{\Psi}_{\rm ADM}[T, g_A(X^k)]. \tag{52}$$

The two methods are equivalent, but the Dirac method asks a broader set of questions. In the ADM method, we want to know the state of the gravitational field on a privileged but limited family of slices; in the Dirac method, we want to know the state of the gravitational field on an arbitrary slice.

V. MINI-SUPERSPACE, MINI-PHASE-SPACE, AND THE REDUCED CANONICAL FORMALISM

The configuration space of geometrodynamics is superspace: the set of all possible three-dimensional geometries. In the canonical formalism, we deal also with the geometrodynamical phase space. A point of the phase space is the class of couples g_{ik}, π^{ik}; two couples belong to the same class if they can be transformed into each other by a three-dimensional diffeomorphism. The dynamical evolution of geometry can start from an arbitrary initial geometry in superspace. On the other hand, the geometrodynamical momentum is restricted by the initial value equations (41). The dynamical trajectory in phase space is therefore necessarily confined to the constraint hypersurface, defined by the conditions (41).

Nobody knows how to find the general solution of the Einstein equations. All that we have today are various particular solutions, or classes of such solutions, characterized either by their symmetries or by other geometrical properties. Superspace is a bewilderingly large dynamical arena for such solutions; like animals adapted to their environment, they tend to keep themselves only in certain limited regions of superspace which are favorable to their geometrical properties. The same tendency can be observed also in the phase space; the symmetric solutions are to be found only in certain limited regions of the constraint hypersurface. It seems a real waste of energy to apply the formidable canonical formalism of the full geometrodynamics to such classes of solutions; one is tempted to think that a limited dynamical arena and the canonical formalism restricted to such a limited arena should suffice.

DeWitt and Misner found such limited arenas and restricted canonical formalisms for the specific problems they investigated – for the Friedmann universe and the mixmaster universe, respectively.[3,6] Misner invented the term "mini-superspace" to describe the limited region of superspace in which the dynamics takes place. The prefix "mini" emphasizes the drastic character of the reduction – out of the infinitely many degrees of freedom of the gravitational field, only a finite number are left. The reduction is based on the groups of motions of the investigated solutions. If we spell out the general outline of such a reduction – as we shall do later in this section – we find that the two reductions investigated by DeWitt and Misner are in fact very special. Their exclusiveness rests in the following: The minimal invariant varieties of the respective groups of motions are three-dimensional hypersurfaces fitted into a one-parameter family. The family provides a privileged slicing of spacetime, which is used in the reduced formalism. This simplifies the formalism enormously. But the very same simplification makes it impossible to illustrate certain aspects of the Dirac and the ADM methods within the DeWitt and Misner mini-superspaces. In fact, one of the most conspicuous features of the Dirac method is that no unique slicing of spacetime is given, every spacelike hypersurface being admissible. The Dirac formalism was invented just to cope with this general situation. A special slicing of spacetime is picked out in the ADM method, but it is picked out by arbitrarily imposed coordinate conditions, not by the symmetry of the problem. In the mini-superspaces mentioned so far the distinction between the Dirac and the ADM approaches is blurred out, the two approaches becoming virtually identical.

The most important motive for investigating various mini-superspaces is the belief that some characteristic features of quantum geometrodynamics are exhibited even by these extremely special (and therefore tractable) models. However, the more specialized the model we investigate, the less justified this belief appears to be. It is therefore important to investigate mini-superspaces of ever increasing generality. The mini-superspace corresponding to the cylindrical gravitational wave is a natural step in this program. In contradistinction to the symmetry groups of the Friedmann universe or the mixmaster universe, the minimal invariant varieties of the cylindrical wave are *two-dimensional* spatial surfaces. Any spacelike hypersurface containing these minimal invariant varieties is an admissible slice of the restricted canonical formalism. There are therefore many more slices than can be fitted into a one-parameter family, and the distinction between the Dirac and the ADM methods becomes nontrivial. Moreover, the cylindrical symmetry does not restrict the dynamical degrees of freedom on the admissible slices so radically as the groups of motions of the two previous models. Because of homogeneity, only a finite number of degrees of freedom is left in the Friedmann and in the mixmaster universe. The cylindrical wave is not homogeneous, and possesses ∞^1 degrees of freedom, described by one real function $\Psi(R)$. The corresponding mini-superspace is therefore much richer than the mini-superspaces of DeWitt and Misner. One is almost tempted to borrow once more a term from the world of fashion, and call it "midi-superspace." It is still not the full-length imperial robe that general geometrodynamics wears, but it certainly uses more fabric than the more youthful models. Curiously enough, truth is better revealed dressed than naked in geometrodynamics.

The Friedmann universe as treated by DeWitt, the mixmaster universe as treated by Misner, and the cylindrical gravitational wave as treated in this paper are all examples of the same general procedure. This procedure reduces the geometrodynamical phase space by allowing only such slices which respect the symmetry of spacetime. The intrinsic geometry and the extrinsic curvature of such slices are then symmetric, and also the lapse function between two neighboring slices is symmetric. This imposes a set of conditions on the dynamical variables g_{ik} and π^{ik}. If spacetime possesses reflection symmetries in addition to motions, the dynamical variables satisfy yet another set of conditions. All conditions share an important feature: They no longer explicitly refer to the imbedding spacetime. We can therefore forget that they were derived by using the four-dimensional picture. All we have to do is to accept them as definitions of our mini-superspace and the corresponding mini-phase-space.

What does it mean that a slice respects a spacetime symmetry? We always start with spacetime which admits an intransitive r-parameter group of motions ${}^4G_r$ with spacelike Killing vectors ${}^4\xi^{\iota}_{(A)}$, $A=1,2,\dots,r$. The minimal invariant variety of such a group is spacelike. We allow only such spatial slices which contain this minimal invariant variety and are therefore invariant varieties of the group. The Killing vectors ${}^4\xi^{\iota}_{(A)}$ are tangential to the slices and can be represented as three-dimensional vectors $\xi^i_{(A)}$ intrinsic to the slices. The vectors $\xi^i_{(A)}$ generate a group G_s, which is said to be induced by the group ${}^4G_r$. If there exist relations

$$\sum_{(A)} c^{(A)}\xi^i_{(A)}=0,\quad c^{(A)}=\text{const}$$

between the Killing vectors $\xi^i_{(A)}$ on the slices, the dimension s of the group G_s is smaller than the dimension r of the group ${}^4G_r$; otherwise, $s=r$.

By virtue of the spacetime symmetries

$$\mathcal{L}_{{}^4\xi^{\iota}_{(A)}}\,{}^4g_{\mu\nu}=0, \tag{53}$$

the dynamical variables g_{ik} and π^{ik} on the allowed slices have the induced symmetries

$$\mathcal{L}_{\xi^i_{(A)}}\,g_{mn}=0, \tag{54}$$

$$\mathcal{L}_{\xi^i_{(A)}}\,\pi^{mn}=0, \tag{55}$$

and also the lapse function between the neighboring slices is symmetric,

$$\mathcal{L}_{\xi^i_{(A)}}N=0. \tag{56}$$

An easy way to check Eqs. (54)–(56) is to choose a system of coordinates in which a one-parameter family of allowed slices is taken as the family of hypersurfaces of constant time. In this system,

$${}^4\xi^{\iota}_{(A)}=(0,\,\xi^i_{(A)}), \tag{57}$$

and Eq. (53) splits into the three equations

$$\begin{aligned}&\mathcal{L}_{\xi^i_{(A)}}\,g_{mn}=0,\\ &\mathcal{L}_{\xi^i_{(A)}}N_m+g_{mn}\xi^n_{(A),0}=0,\\ &\mathcal{L}_{\xi^i_{(A)}}\,g_{00}+2N_n\xi^n_{(A),0}=0.\end{aligned} \tag{58}$$

The first of them coincides with Eq. (54) which we wanted to prove. Equations (55) and (56) are simple consequences of Eqs. (58), if we recall the definitions (36), (37), and (34) of the momenta π^{ik} and the lapse function N.

As already mentioned, the dynamical variables are further restricted if the spacetime has reflection symmetries in hypersurfaces orthogonal to some of the Killing vectors. The Killing vector ${}^4\xi^{\iota}$ is hypersurface-orthogonal and the spacetime metric has the reflection symmetry, if

$$^4\epsilon^{\iota\kappa\lambda\mu}\,{}^4\xi_\kappa\,{}^4\xi_{\lambda,\mu}=0. \quad (59)$$

To derive the restrictions, we can again use the coordinate system in which Eq. (57) holds. The covariant components of $^4\xi^\iota$ are

$$^4\xi_\iota=(N_l\xi^l,\,\xi_i)$$

and Eq. (59) splits into two sets:

$$\epsilon^{klm}\xi_k\xi_{l,m}=0 \quad (60)$$

and

$$\epsilon^{ijk}\{\xi_j[\xi_{k,0}-(N_l\xi^l)_{|k}]+N_l\xi^l\xi_{j|k}\}=0, \quad (61)$$

corresponding to the choice $\iota=0$ and $\iota=i$. We take now the second equation of the system (58), and rearrange it by means of the definition (36) of the extrinsic curvature into the form

$$\xi_{k,0}-(N_l\xi^l)_{|k}=-2N^{-1}K_{kl}\xi^l-2N_l\xi^l{}_{|k}.$$

Substituting the last equation into Eq. (61), we obtain

$$\epsilon^{ijk}\xi_jK_{kl}\xi^l=0. \quad (62)$$

Returning to the definition (37) of the momenta, we see that they satisfy essentially the same equation as the extrinsic curvature, namely,

$$\epsilon_{ijk}\xi^j\pi^{kl}\xi_l=0. \quad (63)$$

Equations (62) and (63) are the additional restrictions imposed on the dynamical variables by the reflection symmetries.

Conditions (62) and (63) can be formulated yet in a slightly different way. According to Eq. (62), the vector product of the vectors ξ_j and $K_{kl}\xi^l$ vanishes. These two vectors are therefore collinear,

$$K_{kl}\xi^l=\alpha\xi_k,\quad \alpha=(\xi_m\xi^m)^{-1}K_{kl}\xi^k\xi^l, \quad (64)$$

which means that ξ_k is an eigenvector of K_{kl}. Similarly, Eq. (63) implies that ξ_k is an eigenvector of π^{kl},

$$\pi^{kl}\xi_l=\beta\xi^k,\quad \beta=(\xi_m\xi^m)^{-1}\pi^{kl}\xi_k\xi_l. \quad (65)$$

The spatial metric g_{ik} and the geometrodynamical momentum π^{ik} of the allowed slices are therefore restricted by conditions (54), (60) and (55), (63). Following our program, we take these restrictions as definitions of the mini-phase-space, and the first set of them, Eqs. (54), (60), as definitions of the mini-superspace. Let us describe their meaning in this new language, avoiding the notion of the imbedding spacetime.

Equation (54) tells us that from all geometries only those having the symmetries generated by the Killing vectors $\xi^i_{(A)}$ are allowed. Equation (60) means that some of the Killing vectors are surface-orthogonal, so that the spatial geometries have additional reflection symmetries. Fischer, analyzing the topological structure of superspace, came to the conclusion that the neighborhood of symmetric geometries in superspace has a different structure than the neighborhood of general geometries. He decomposed superspace into a system of manifolds of geometries, the strata, in such a way that the geometries of high symmetry are completely contained in the boundary of geometries of lower symmetry.[22] Our mini-superspace is therefore a union of Fischer's strata.

The restrictions (55) and (63) on phase space go hand in glove with the restrictions (54) and (60) on superspace. We wish the geometry to remain in mini-superspace throughout its dynamical evolution. The momenta must have the symmetries (55) and (63) to preserve the symmetries of the geometry. However, the symmetries of the momenta are insufficient to do the job alone. It is easy to see the reason, if we return for a while to the spacetime language. The symmetries could not be maintained if we decided to cut the next slice across the symmetrical spacetime in an arbitrary manner, disregarding the symmetries. How do we know that the next slice is a good one if the spacetime through which this slice is to be cut is not yet constructed, but is only to be built up step by step in the process of integrating the Hamilton equations? The answer is that we must proceed from one slice to the next in such a way that the proper time between the two slices measured in the direction normal to the first slice has the same value in all points of the slice that are equivalent under the group of motions G_s. The proper time between the two slices is proportional to the lapse function. We are thus led to the condition (56), which ensures that the symmetry will not be broken because of a bad choice of slicing.

While the symmetry (56) of the lapse function is necessary to keep the dynamical trajectory within the mini-superspace, no symmetry requirements are imposed on the shift functions. This is intuitively clear because the N_i fix only the spatial system of coordinates on a new hypersurface and have nothing to do with its intrinsic symmetries.

Let us present the same argument, but in different wording. Equations (36) and (37) defining the momenta π^{ik} in terms of the velocities $g_{ik,0}$ cannot be inverted to express the velocities by means of the momenta alone. There are always lapse and shift functions present in the inverse formula (44). As explained, the shift functions can be eliminated by a transformation of the spatial system of coordinates. On the other hand, the lapse function occurs in the geometrodynamical velocity $\delta_0 g_{ik}/\delta t$ as a multiplicative factor. The symmetry (56) of the lapse function is therefore necessary in addition to the symmetries (55) and

(63) of the momenta to make the geometrodynamical velocity symmetric, i.e., to make it tangential to mini-superspace as embedded in superspace.

There is still some question whether momenta may not lose their symmetries (55) and (63) a little while later and then push the geometry out of the mini-superspace. To safeguard ourselves against this chance, we prove in the Appendix the following basic theorem: If the dynamical variables g_{ik} and π^{ik} satisfy the conditions (54), (60), (55), and (63) on the initial hypersurface t, and if we pass to the new hypersurface by specifying a symmetric lapse function (56), then the dynamical variables $\tilde{g}_{ik}$ and $\tilde{\pi}^{ik}$ given on $\tilde{t}$ by the solutions (44), (45) of the Hamilton equations satisfy again the conditions (54), (60), (55), and (65) for any choice of the shift functions. This means that under the slicing (56) the whole dynamical trajectory lies in the mini-phase-space, if its initial point lies there.

The Killing equations (54) and the conditions of surface orthogonality (60) reduce the number of independent canonical coordinates. The best way to get rid of the surplus variables is to choose a standard[23] system of space coordinates x^l reflecting the symmetries. In it, the canonical coordinates $g_{ik}(x^l)$ are expressed as functions of independent quantities g_A,

$$g_{ik}=g_{ik}(g_A). \tag{66}$$

In general, the g_A do not depend on certain of the coordinates x^l, and their number is less than the number of algebraically independent components of the metric tensor g_{ik}, i.e., less than six.

Of course, we require that the standard system of coordinates be introduced on each allowable hypersurface. Passing from one hypersurface to another during the dynamical evolution of geometry, we must adjust the shift functions in such a way that they preserve the standard system of coordinates. This subjects them to the restrictions of symmetry

$$\mathcal{L}_{\xi^i_{(A)}} N_k=0, \tag{67}$$

and also to the restriction

$$N_i\xi^i=0, \tag{68}$$

if the vector ξ^i is surface-orthogonal.

Equations (55) and (63) reduce the number of independent canonical momenta π^{ik} in the same way as Eqs. (54) and (60) reduce the number of independent canonical coordinates. If we write Eqs. (55) and (63) in the standard system of coordinates, the canonical momenta are expressible as functions of some independent quantities π^B,

$$\pi^{ik}=\pi^{ik}(g_A,\pi^B). \tag{69}$$

We do not exclude the possibility that these functions depend also on g_A. The main task is to choose the new quantities π^A in such a way that the variables g_A and π^A are canonically conjugate. This means that the action functional (38) expressed by means of the variables g_A and π^A assumes the canonical form

$$S=\int dt\int d^3x\,[\pi^A g_{A,0}-N\mathcal{H}(g_A,\pi^B)-N_i\mathcal{H}^i(g_A,\pi^B)]. \tag{70}$$

The super-Hamiltonian $\mathcal{H}$ and the supermomentum $\mathcal{H}^i$ are of course symmetric by virtue of their construction from the dynamcial variables g_{ik} and π^{ik},

$$\mathcal{L}_{\xi^i_{(A)}}\mathcal{H}=0, \tag{71}$$

$$\mathcal{L}_{\xi^k_{(A)}}\mathcal{H}^i=0. \tag{72}$$

If a Killing vector ξ^i is surface-orthogonal, it is also orthogonal to the supermomentum $\mathcal{H}^i$,

$$\mathcal{H}^i\xi_i=0. \tag{73}$$

We prove Eq. (73) by differentiating Eq. (65),

$$\pi^{ik}{}_{|k}\xi_i+\pi^{ik}\xi_{i|k}=\beta_{|k}\xi^k+\beta\xi^k{}_{|k}. \tag{74}$$

Because of the symmetries (54) and (55) of the metric and the momentum, we have

$$\mathcal{L}_{\xi^l}\beta=\beta_{|k}\xi^k=0. \tag{75}$$

Equation (75) and the Killing equation eliminate all the terms in Eq. (74) except the first one. By the definition (40) of the supermomentum, we obtain Eq. (73).

Equations (72) and (73) for the supermomentum have their counterparts in Eqs. (67) and (68) for the shift functions, just as Eq. (71) for the super-Hamiltonian has its counterpart in Eq. (56) for the lapse function. The role of this correlation is evident. In the standard system of coordinates, the super-Hamiltonian $\mathcal{H}$ and the supermomentum $\mathcal{H}^i$ do not depend on certain of the coordinates x^i by virtue of the conditions (71) and (72). Equations (56) and (67) ensure that also the lapse function N and the shift functions N_i do not depend on these x^i in the standard system of coordinates. The action functional can then be easily integrated over these coordinates. Also, by virtue of Eq. (73), some components of the supermomentum $\mathcal{H}^i$ may vanish in the standard system of coordinates. The condition (68) ensures that the corresponding shift functions also vanish in this system. This makes the reduction of the canonical formalism self-consistent.

VI. EINSTEIN-ROSEN WAVES IN THE REDUCED CANONICAL FORMALISM

In Sec. V, the general method of the reduction of superspace by the spatial groups of motions was developed. We shall see how this method works for the two-parameter Abelian group of motions ${}^4G_2$ with two hypersurface-orthogonal, spacelike Killing vectors ${}^4\xi^i_{(\varphi)}$, ${}^4\xi^i_{(z)}$, characterizing the Einstein-Rosen wave.

Let us recall that this group is intransitive and its minimal invariant varieties are the two-dimensional cylindrical surfaces. Any spacelike hypersurface containing these cylindrical surfaces is an allowed slice. In the Einstein-Rosen coordinates, any hypersurface $t(T, R) = \text{const}$ [subject to the conditions (22)–(27) discussed in Sec. III] represents such a slice. The group ${}^4G_2$ induces the Abelian group G_2 with two mutually orthogonal, surface-orthogonal Killing vectors $\xi^i_{(\varphi)}$ and $\xi^i_{(z)}$ on each of the allowed slices.

Let us begin by reducing the superspace by the group G_2. This means we impose the conditions (54) and (60) with the Killing vectors $\xi^i_{(\varphi)}$ and $\xi^i_{(z)}$ on the metric. Conditions (54) and (60) eliminate a number of the degrees of freedom of the gravitational field. To uncover the remaining independent degrees of freedom, we introduce a standard system of coordinates. It is easy to prove that if ξ^i is a surface-orthogonal Killing vector, then $\xi^i/\xi_k\xi^k$ is a gradient. Also, if $\xi^i_{(1)}$ and $\xi^i_{(2)}$ are two commuting (not necessarily surface-orthogonal) Killing vectors, and F is an arbitrary scalar invariant under the motions generated by $\xi^i_{(1)}$ and $\xi^i_{(2)}$,

$$\xi^i_{(1)} F_{,i} = \xi^i_{(2)} F_{,i} = 0, \tag{76}$$

then $F\epsilon_{ikl}\xi^k_{(1)}\xi^l_{(2)}$ is a gradient. Because the Killing vectors $\xi^i_{(\varphi)}$ and $\xi^i_{(z)}$ of G_2 are both commuting and surface-orthogonal, we have at our disposal three functions r, φ, z defined by the equations

$$r_{,i} = F\epsilon_{ikl}\xi^k_{(\varphi)}\xi^l_{(z)}, \tag{77}$$

$$\varphi_{,i} = \xi_{(\varphi)i}/\xi_{(\varphi)k}\xi^k_{(\varphi)}, \qquad z_{,i} = \xi_{(z)i}/\xi_{(z)k}\xi^k_{(z)}. \tag{78}$$

If we choose these functions as the standard coordinates $x^i = (r, \varphi, z)$, we have

$$r_{,i} = (1, 0, 0), \quad \varphi_{,i} = (0, 1, 0), \quad z_{,i} = (0, 0, 1).$$

Equations (78) determine the covariant components of the Killing vectors, $\xi_{(\varphi)i}$ and $\xi_{(z)i}$, by means of the components of the metric tensor in the standard system,

$$\xi_{(\varphi)i} = (0, (g^{22})^{-1}, 0), \quad \xi_{(z)i} = (0, 0, (g^{33})^{-1}). \tag{79}$$

The contravariant components of these vectors are

$$\xi^i_{(\varphi)} = (g^{22})^{-1}(g^{12}, g^{22}, g^{32}),$$
$$\xi^i_{(z)} = (g^{33})^{-1}(g^{13}, g^{23}, g^{33}).$$

Substituting them into Eq. (77), we find that the off-diagonal components of the metric tensor must vanish,

$$g^{12} = g^{13} = g^{23} = 0.$$

Returning to the restrictions (54) and (60), we see that Eqs. (60) are satisfied automatically as a consequence of Eqs. (78). On the other hand, the Killing equations (54) imply that the components of the metric tensor do not depend on the coordinates φ and z. We can therefore write

$$g_{11} = e^{\gamma-\psi}, \quad g_{22} = R^2 e^{-\psi}, \quad g_{33} = e^{\psi}, \qquad g_{12} = g_{13} = g_{23} = 0, \tag{80}$$

where γ, R and ψ are three arbitrary functions of the single coordinate r. In the dynamical theory, they must satisfy the boundary conditions (25), (26), (27), (31), and (33). The functions R and ψ can be given an invariant meaning by constructing them from the Killing vectors,

$$R = (\xi^i_{(\varphi)}\xi_{(\varphi)i}\xi^k_{(z)}\xi_{(z)k})^{1/2}, \tag{81}$$

$$\psi = \ln(\xi^i_{(z)}\xi_{(z)i}). \tag{82}$$

Equations (81) and (82) are the spatial counterparts of Eqs. (1) and (4).

The standard system of coordinates is not entirely unique. While φ and z may be changed only by the trivial transformations

$$\varphi \to \bar{\varphi} = \pm\varphi + \varphi_0, \quad z \to \bar{z} = \alpha z + z_0,$$

the radial coordinate r can be gauged completely arbitrarily,

$$r \to \bar{r} = f^{-1}(r), \tag{83}$$

without changing the general form of the metric (80). This is due to an arbitrary invariant factor F in the definition equation (7) for r. If r corresponds to the choice $F = 1$, then $\bar{r}$ corresponds to the choice $F = (f^{-1}(r))'$. Under the transformation (83), $R(r)$ and $\psi(r)$ behave as scalars, whereas

$$\gamma(r) \to \bar{\gamma}(\bar{r}) = \gamma(f(\bar{r})) + 2\ln f'(\bar{r}). \tag{84}$$

The three functions

$$\gamma(r), \quad R(r), \quad \psi(r) \tag{85}$$

of one real variable $r \in [0, \infty)$ specify completely the cylindrically symmetric geometry (80) and can be thought of as the coordinates of a point in a mini-superspace, with the understanding that the functions (85) and the functions

$$\gamma(f(r)) + 2\ln f'(r), \quad R(f(r)), \quad \psi(f(r)), \tag{86}$$

where $f(r)$ is an arbitrary function of r, represent the same point.

To preserve the symmetry of the slicing and the standard system of coordinates, the condition (56) on the lapse function, and the conditions (67) and (68) on the shift functions are imposed. If we write them in the standard system of coordinates, Eqs. (56) imply that the lapse function does not depend on φ and z, Eqs. (67) imply that the shift functions do not depend on φ and z, and Eqs. (68) imply that the azimuthal and axial shift functions N_2 and N_3 vanish. In short,

$$N = N(r), \quad N_i = (N_1(r), 0, 0). \tag{87}$$

Equations (87) together with the form of the reduced metric (80) bring us back, through the decomposition (34) of the spacetime metric, to the starting form (2) of the cylindrically symmetric spacetime line element.

The next task is the reduction of the geometrodynamical phase space. This is accomplished by writing down the reduction equations (55) and (63) for the momenta π^{ik} in the standard system of coordinates. Equations (55) imply that the momenta do not depend on φ and z, and Eqs. (63) imply that the off-diagonal components of the momentum tensor density π^{ik} vanish,

$$\begin{aligned} &\pi^{11} = \pi^{11}(r), \quad \pi^{22} = \pi^{22}(r), \quad \pi^{33} = \pi^{33}(r), \\ &\pi^{12} = \pi^{13} = \pi^{23} = 0. \end{aligned} \tag{88}$$

It remains to express the diagonal components of the momentum tensor density by the three quantities π_γ, π_R, and π_ψ canonically conjugate to the three functions γ, R, and ψ specifying the metric. The prescription

$$\begin{aligned} &\pi^{11} = \pi_\gamma e^{\psi-\gamma}, \quad \pi^{22} = \tfrac{1}{2}R\pi_R e^{\psi}, \\ &\pi^{33} = (\pi_\gamma + \tfrac{1}{2}R\pi_R + \pi_\psi)e^{-\psi} \end{aligned} \tag{89}$$

achieves this aim, bringing the expression $\pi^{ik}\dot{g}_{ik}$ to the canonical form $\pi_\gamma\dot{\gamma} + \pi_R\dot{R} + \pi_\psi\dot{\psi}$.

As already remarked, the standard system of coordinates is arbitrary up to the gauge transformation (84) of the radial coordinate r. From the fact that π^{ik} transforms as a tensor density of weight 1 we can deduce the transformation properties of the momenta π_γ, π_R, π_ψ. In contradistinction to the transformation properties of the variables γ, R, ψ themselves, the transformation properties of the momenta are uniform:

$$\begin{aligned} &\pi_\gamma(r) \to \bar{\pi}_\gamma(\bar{r}) = f'(\bar{r})\pi_\gamma(f(\bar{r})), \\ &\pi_R(r) \to \bar{\pi}_R(\bar{r}) = f'(\bar{r})\pi_R(f(\bar{r})), \\ &\pi_\psi(r) \to \bar{\pi}_\psi(\bar{r}) = f'(\bar{r})\pi_\psi(f(\bar{r})). \end{aligned} \tag{90}$$

We can now think of the six functions

$$\begin{aligned} &\gamma(r), \quad R(r), \quad \psi(r); \\ &\pi_\gamma(r), \quad \pi_R(r), \quad \pi_\psi(r) \end{aligned} \tag{91}$$

of one real variable $r \in [0, \infty)$ as the coordinates of a point in a mini-phase-space, with the understanding that the functions (91) and the functions

$$\begin{aligned} &\gamma(f(r)) + 2\ln f'(r), \quad R(f(r)), \quad \psi(f(r)); \\ &f'(r)\pi_\gamma(f(r)), \quad f'(r)\pi_R(f(r)), \quad f'(r)\pi_\psi(f(r)), \end{aligned} \tag{92}$$

where $f(r)$ is an arbitrary function of r, represent the same point.

To complete the reduction of the canonical formalism, we must express the super-Hamiltonian $\mathcal{H}$ and the supermomentum $\mathcal{H}^i$ as functions of the canonical variables. This is done by brute force, substituting the reduced metric (80) and the reduced momentum density (89) into the definitions (39) and (40). We obtain

$$\mathcal{H} = e^{\frac{1}{2}(\psi-\gamma)}(-\pi_\gamma\pi_R + \tfrac{1}{2}R^{-1}\pi_\psi{}^2 + 2R'' - \gamma'R' + \tfrac{1}{2}R\psi'^2), \tag{93}$$

$$\begin{aligned} &\mathcal{H}^1 = e^{\psi-\gamma}(-2\pi_\gamma{}' + \gamma'\pi_\gamma + R'\pi_R + \psi'\pi_\psi), \\ &\mathcal{H}^2 = \mathcal{H}^3 = 0. \end{aligned} \tag{94}$$

In accordance with Eqs. (71) and (72), the super-Hamiltonian $\mathcal{H}$ and the supermomentum $\mathcal{H}^i$ do not depend on the coordinates φ and z, and in accordance with Eqs. (73), the azimuthal and the axial components of the supermomentum vanish. This is in tune with the form (87) of the lapse and shift functions. There is a slight difficulty with the action functional (70), because the total action becomes infinite when integrating over z. We must limit the integration by two "planes," $z = z_0$, and $z = z_0 + 1$, which are at a unit distance apart at infinity. Performing the integrations over φ and z, we get the action in the form

$$S = 2\pi\int_{-\infty}^{\infty} dt \int_0^{\infty} dr\,(\pi_\gamma\dot{\gamma} + \pi_R\dot{R} + \pi_\psi\dot{\psi} - N\mathcal{H} - N_1\mathcal{H}^1). \tag{95}$$

There is yet one minor but convenient trick by which the action functional (95) can be simplified. We can get rid of the exponential factors in the expressions (93) and (94) for $\mathcal{H}$ and $\mathcal{H}^1$, if we rescale the super-Hamiltonian and the supermomentum and at the same time rescale the lapse and the shift functions,

$$\begin{aligned} &\tilde{\mathcal{H}} = e^{\frac{1}{2}(\gamma-\psi)}\mathcal{H}, \quad \tilde{N} = e^{\frac{1}{2}(\psi-\gamma)}N, \\ &\tilde{\mathcal{H}}^1 = e^{\gamma-\psi}\mathcal{H}^1, \quad \tilde{N}_1 = e^{\psi-\gamma}N_1. \end{aligned} \tag{96}$$

It is easy to check that the action functional

$$S = 2\pi\int_{-\infty}^{\infty} dt \int_0^{\infty} dr\,(\pi_\gamma\dot{\gamma} + \pi_R\dot{R} + \pi_\psi\dot{\psi} - \tilde{N}\tilde{\mathcal{H}} - \tilde{N}_1\tilde{\mathcal{H}}^1), \tag{97}$$

in which γ, R, ψ, π_γ, π_R, π_ψ and $\tilde{N}$, $\tilde{N}_1$ are varied as independent variables, leads to the same Hamilton equations as the action functional (95), primarily by virtue of the initial value equations.

VII. THE EXTRINSIC TIME AS A CANONICAL COORDINATE

The structure of the super-Hamiltonian (93) and the supermomentum (94) is far from being self-explanatory. The canonical variables γ, R, ψ and π_γ, π_R, π_ψ enter $\mathcal{H}$ and $\mathcal{H}^1$ in a highly nonsymmetrical fashion and it is difficult to see the physical meaning of the different terms. The complicated appearance of Eqs. (93) and (94) discourages us from attempting to impose the constraints in this form on the state functional, because it seems hopeless to solve the resulting functional differential equations. The vast simplifications achieved by reducing the canonical formalism are still insufficient to handle the problem of determining the dynamical evolution of the state functional.

This is not totally unexpected. The reduction of the metric (80) together with the reduction of the lapse and shift functions (87) corresponds to the general form (2) of the cylindrically symmetric line element in the standard four-dimensional treatment of the cylindrical gravitational fields. If we write the Einstein vacuum equations for the line element (2), they are also complicated and difficult to solve, unless we take advantage of the further simplification resulting from the introduction of the privileged Einstein-Rosen coordinates T and R. It is exactly this step we would like to repeat in the canonical formalism.

At first sight, this looks self-defeating for the Dirac approach to the canonical formalism. Is the introduction of the privileged coordinates not equivalent to taking only the privileged slices $T=\text{const}$ and the privileged worldlines $R=\text{const}$, $\varphi=\text{const}$, $z=\text{const}$, instead of admitting all slices and all congruences of worldlines compatible with the cylindrical symmetry? Not necessarily. When we introduce the privileged coordinates into the canonical formalism, we introduce them as *canonical variables*, not as the *coordinate labels* t and r, which may still remain arbitrary.

In this sense, one-half of the work is already done, because the Einstein-Rosen radial coordinate R is one of our canonical coordinates. But is it possible to accomplish the other half of the work? In the conventional approach, the Einstein-Rosen time T is introduced by the requirement $(N)^2=g_{11}$, implying that the cylindrical wavefront propagates with the unit velocity in the Einstein-Rosen coordinate frame. This condition makes no sense in the canonical formalism. We must stay within our own slice while finding out the Einstein-Rosen time corresponding to its different points. We do not pass to another slice, a step implied by prescribing the lapse function. We are given the canonical variables γ, R, ψ and π_γ, π_R, π_ψ on our slice, but no information on how this slice is cut out of the spacetime. Our task is to reconstruct the parametric equations $T=T(r)$, $R=R(r)$ of the slice in the Einstein-Rosen coordinate chart from the knowledge of the canonical variables.

The answer to our problem is surprisingly simple, being given by the formula

$$T(r)=T(\infty)+\int_\infty^r -\pi_\gamma dr. \tag{98}$$

We can check that $T(r)$ defined by Eq. (98) remains unchanged, if we change the coordinatization of the slice, by virtue of the transformation property (90) of the momentum π_γ under the change (83) of the coordinate label r. We could even guess that the Einstein-Rosen time is connected with the particular momentum π_γ. In the Einstein-Rosen coordinates, the Γ function plays the role of the energy superpotential, $2\pi\gamma_\infty$ being the total energy of the ψ field. Of course, time is canonically conjugate to energy in the canonical formalism. However, we must rely on a formal argument if we want to show that the connection between T and π_γ takes on the concrete form (98).

There are different ways of verifying Eq. (98), one of them starting by writing down the Hamilton equation for the canonical coordinate R, corresponding to the action (95),

$$\dot{R}=\frac{\delta}{\delta\pi_R}\int_0^\infty dr\,(N\mathcal{H}+N_1\mathcal{H}^1).$$

Because the super-Hamiltonian (93) and the supermomentum (94) do not contain the derivatives of the momentum π_R, the operation of taking the variational derivative is trivial. We can use the constraint equations $\mathcal{H}=\mathcal{H}^1=0$ and bring the result to the form

$$\dot{R}=-N\pi_\gamma e^{\frac{1}{2}(\psi-\gamma)}+N_1 e^{\psi-\gamma}. \tag{99}$$

The last equation determines the momentum π_γ by means of the canonical coordinates γ, ψ, the velocity $\dot{R}$, and the lapse and shift functions.

On the other hand, we already know from Eqs. (80) and (87) that the spacetime line element is in form (2). This means that the lapse and shift functions can be expressed through the Einstein-Rosen variables by Eqs. (29) and (30). Substituting these expressions for N and N_1 into Eq. (99), we initiate a great cancellation of terms leading to the result

$$\pi_\gamma=-T'. \tag{100}$$

Equation (98) is the integral form of Eq. (100). We must know the Einstein-Rosen time at one point of the slice to be able to determine it at other points of the slice. We choose the point at infinity, where the spacetime is locally Minkowskian, but any other point, e.g., the point at the axis of symmetry, would do as well. In open spaces, the intrinsic geometry or the extrinsic curvature need not carry complete information about time because the boundary conditions have their word to say. This explains why the knowledge of $T(r)$ at a boundary point of the slice is necessary, before the canonical variables propagate that knowledge to all other points of the slice.

The concept of time has many different meanings in the general theory of relativity. For the cylindrical fields, it is tempting to identify time with the Einstein-Rosen time, primarily because tremendous simplifications result from introducing it into the field equations. Equation (98) indicates that this time is a momentum variable and not a superspace coordinate. It is not the first time we have been taught this lesson. In a quite different context of the linearized theory of gravitation, it is also natural to identify time with a momentum variable.[4,17,18] Because momentum variables are connected rather with the extrinsic curvature of a slice than with its intrinsic geometry, we shall call this type of time the *extrinsic time*. The reasons why the extrinsic time works better than an intrinsic time for the almost flat spacetimes were discussed in Ref. 18. However, the prescription (98) works not only for weak cylindrical fields, but also for strong cylindrical fields. It would be important to inquire whether a natural extrinsic time exists in the other mini-phase-spaces and, even more important, if some privileged extrinsic time exists for the general geometrodynamics.

The proof that $T(r)$ given by Eq. (98) is identical with the Einstein-Rosen time as introduced in Secs. II and III rests on manipulations with the complete spacetime metric, and therefore reaches outside the narrow framework of the mini-phase-space. Not only that; in the quantized theory, the complete spacetime metric loses its meaning and it is impossible to introduce the Einstein-Rosen time as was done in Secs. II and III. However, in the canonical formalism we can *define* the Einstein-Rosen time by Eq. (98). Even more, we can perform a canonical transformation after which $T(r)$ becomes one of the canonical coordinates. This enables us to choose the quantum representation in which T as well as R and ψ are diagonal. The usefulness of this procedure is revealed by the resulting simplifications in the structure of the super-Hamiltonian and the supermomentum.

Wanting to find the canonical transformation to $T(r)$, we perform a double integration by parts in the action integral,

$$2\pi\int_{-\infty}^{\infty}dt\int_0^{\infty}dr\,\pi_\gamma\dot{\gamma}=2\pi\int_{-\infty}^{\infty}dt\int_0^{\infty}dr\,(-\gamma')\dot{T}+2\pi\int_{-\infty}^{\infty}dt\,[\dot{T}\gamma]_{r=0}^{r=\infty}-2\pi\int_0^{\infty}dr[T'\gamma]_{t=-\infty}^{t=\infty}. \qquad (101)$$

In this process we have picked up a number of boundary integrals. Except for one, all these boundary integrals vanish by virtue of the boundary conditions (24) – (27) and (31) that we have imposed on γ, T, and R. Indeed, $[T'\gamma]_{t=-\infty}^{t=\infty}=0$ by virtue of Eq. (27), and $[\dot{T}\gamma]_{r=0}=0$ by virtue of Eq. (31). The term $2\pi[\dot{T}\gamma]^{r=\infty}$ remains, but because of Eqs. (25) and (32) it equals a constant: the total energy $E_\infty=2\pi\Gamma_\infty$ of the ψ field. We can renormalize the action by subtracting E_∞ from the Hamiltonian (43). In fact, this renormalization is a characteristic feature of the formalism applied to open spaces, as remarked by DeWitt.[3]

The boundary terms having been eliminated from the relevant part (101) of the action functional, it is possible to identify $-\gamma'$ as a momentum canonically conjugate to the Einstein-Rosen time T,

$$\pi_T=-\gamma'. \qquad (102)$$

However, it pays to keep striving for the most natural choice of the canonical momenta. An unsatisfactory feature of the canonical momentum (102) is its behavior under the change (83) of the radial coordinate. Recalling the transformation properties (84) of the function γ, we see that

$$\overline{\pi}_T(\overline{r})=f'(\overline{r})\pi_T(f(\overline{r}))-2f''(\overline{r})/f'(\overline{r}).$$

On the other hand, the remaining momenta π_R, π_ψ transform simply as scalar densities, and the canonical coordinates T, R, ψ transform as scalars. The canonical momentum π_T is thus a black sheep in the flock of canonical variables.

We would expect that the momentum Π_T naturally conjugate to the Einstein-Rosen time T is the energy density $\Gamma_{,R}(T,R)$ (with the minus sign, and multiplied by the factor R', to account for the change of the coordinate cell): $\Pi_T=-\Gamma_{,R}R'$. Inspecting the relation (28) between Γ and γ, we see that π_T does not coincide with Π_T. In fact,

$$\Pi_T=\pi_T+[\ln(R'^2-T'^2)]'. \qquad (103)$$

By virtue of the transformation properties of T, R, and π_T, the momentum Π_T transforms as a scalar density,

$$\overline{\Pi}_T(\overline{r})=f'(\overline{r})\Pi_T(f(\overline{r})).$$

We would therefore like to introduce Π_T instead of π_T as a momentum canonically conjugate to T. This can be done only at the expense of introducing simultaneously a new momentum Π_R instead of the old momentum π_R,

$$\Pi_R = \pi_R + \{\ln[(R'+T')/(R'-T')]\}'. \tag{104}$$

The change (104) does not disturb the desired transformation properties of momenta. We can easily check that Π_R still transforms as a scalar density.

Equations (103) and (104) define a canonical transformation. Introducing Π_T, Π_R instead of π_T, π_R into the action functional, we pick up an additional integral:

$$2\pi\int_{-\infty}^{\infty} dt \int_0^{\infty} dr\,(\pi_T \dot{T} + \pi_R \dot{R}) = 2\pi\int_{-\infty}^{\infty} dt \int_0^{\infty} dr\,(\Pi_T \dot{T} + \Pi_R \dot{R}) - 2\pi\int_{-\infty}^{\infty} dt \int_0^{\infty} dr\left[(\ln(R'^2 - T'^2))'\dot{T} + \left(\ln\frac{R'+T'}{R'-T'}\right)'\dot{R}\right].$$

However, the expression enclosed in the large square brackets is a complete divergence

$$[(\dot{R}+\dot{T})\ln(R'+T') - (\dot{R}-\dot{T})\ln(R'-T')]' + [2T' - (R'+T')\ln(R'+T') + (R'-T')\ln(R'-T')]^{\cdot},$$

and the integral can be transformed into the sum of surface integrals at the remote past and future, at the axis of symmetry, and at spatial infinity. All these surface integrals vanish because of our boundary conditions (24) – (27). This proves that the change (103), (104) is really a canonical transformation.

We see there is a considerable freedom in the choice of the momenta canonically conjugate to a given set of canonical coordinates. This freedom has its source in the fundamental ambiguity existing in the specification of the Lagrangian density. We can add a complete divergence to the Lagrangian density without changing the Euler equations of the Hamilton variational principle. In particular, we can pick out either π_T, π_R or Π_T, Π_R as the momenta canonically conjugate to the coordinates T and R. For several reasons it is advantageous to work with the momenta Π_T and Π_R. First, they are susceptible to a simple physical interpretation. We have already seen that $-\Pi_T$ is the energy density (18) of the ψ field (contained in the unit cell of the coordinates r, φ, z). Similarly, we could show that Π_R is the momentum density (19) of the ψ field (contained in the unit cell of the coordinates r, φ, z). Second, when we use the momenta Π_T and Π_R instead of the momenta π_T, π_R, the canonical formalism for the gravitational wave becomes completely identical with the ordinary parametrized canonical formalism for a scalar wave in a flat Minkowskian background. (This will be proved in Sec. IX.) Third, the new coordinates of a point in mini-phase-space,

$$T(r), R(r), \psi(r);\quad \Pi_T(r), \Pi_R(r), \pi_\psi(r) \tag{105}$$

have simple transformation properties under the change (83) of the radial coordinate. The canonical coordinates T, R, ψ transform as scalars and the canonical momenta Π_T, Π_R, π_ψ as scalar densities. This means that the functions

$$T(f(r)),\ R(f(r)),\ \psi(f(r));$$

$$f'(r)\Pi_T(f(r)),\ f'(r)\Pi_R(f(r)),\ f'(r)\pi_\psi(f(r)),$$

where $f(r)$ is an arbitrary function of $r\in[0,\infty)$, represent the same point in mini-phase-space as the functions (105).

The purpose of the canonical transformations (98), (102), and (103), (104) was, of course, to simplify the structure of the super-Hamiltonian and supermomentum. We are now in a position to check that we have achieved this aim. The canonical transformations bring the action functional (97) into the form

$$S = 2\pi\int_{-\infty}^{\infty} dt \int_0^{\infty} dr\,(\Pi_T \dot{T} + \Pi_R \dot{R} + \pi_\psi \dot{\psi} - \tilde{N}\tilde{\mathcal{H}} - \tilde{N}_1 \tilde{\mathcal{H}}^1). \tag{106}$$

Introducing the new canonical variables into the rescaled super-Hamiltonian (93) and the rescaled supermomentum (94), we obtain

$$\tilde{\mathcal{H}} = R'\Pi_T + T'\Pi_R + \tfrac{1}{2}R^{-1}\pi_\psi{}^2 + \tfrac{1}{2}R\psi'^2, \tag{107}$$

$$\tilde{\mathcal{H}}^1 = T'\Pi_T + R'\Pi_R + \psi'\pi_\psi. \tag{108}$$

Even a casual comparison of Eq. (107) with Eq. (93) and of Eq. (108) with Eq. (94) reveals a remarkable gain of structure and simplicity. The variables T, R, Π_T, Π_R enter the super-Hamiltonian in almost the same way they enter the supermomentum so that we get the terms of $\tilde{\mathcal{H}}^1$ from the corresponding terms of $\tilde{\mathcal{H}}$ by the interchange $T \leftrightarrow R$, keeping the momenta Π_T and Π_R fixed. On the other hand, the truly dynamical variables ψ, π_ψ enter $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{H}}^1$ in characteristically different combinations. The dynamical term in $\tilde{\mathcal{H}}$ has the form of the energy density, and the dynamical term in $\tilde{\mathcal{H}}^1$ of the momentum density, of cylindrical scalar wave in Minkowskian spacetime.

The canonical transformation (98), (102) – (104) which proved itself so efficient in simplifying the constraints obviously mixes superspace with the momentum space. Our configuration space was thereby changed into the T, R, ψ space which no longer is a mini-superspace. This is perhaps the most important message from the cylindrical gravitational waves: superspace, though a dynamical arena of geometrodynamics, is not the only dynamical arena. A different configuration space may very well be much more convenient for a study of the time evolution of general gravitational fields. The principal question which remains unsolved is to uncover such a general dynamical arena.

VIII. THE ADVANCED AND THE RETARDED TIME AS CANONICAL COORDINATES

The symmetry of the action functional (106) is exhibited even better if we introduce the advanced and retarded time coordinates U and V instead of the Einstein-Rosen coordinates T and R;

$$U=T+R, \qquad T=\tfrac{1}{2}(U+V), \qquad V=T-R, \qquad R=\tfrac{1}{2}(U-V). \tag{109}$$

We obtain a canonical transformation, if we accompany (109) by the transformation of momenta

$$\Pi_U=\tfrac{1}{2}(\Pi_T+\Pi_R), \quad \Pi_T=\Pi_U+\Pi_V, \qquad \Pi_V=\tfrac{1}{2}(\Pi_T-\Pi_R), \quad \Pi_R=\Pi_U-\Pi_V. \tag{110}$$

We also regroup the terms in the Hamiltonian

$$H=2\pi\int_0^\infty dr\,(\tilde{N}\tilde{\mathcal{H}}+\tilde{N}_1\tilde{\mathcal{H}}^1),$$

by introducing the new quantities $\mathcal{H}^U$, $\mathcal{H}^V$ and N_U, N_V,

$$\begin{aligned}
&\mathcal{H}^U=\tilde{\mathcal{H}}+\tilde{\mathcal{H}}^1, && \tilde{\mathcal{H}}=\tfrac{1}{2}(\mathcal{H}^U+\mathcal{H}^V),\\
&\mathcal{H}^V=\tilde{\mathcal{H}}-\tilde{\mathcal{H}}^1, && \tilde{\mathcal{H}}^1=\tfrac{1}{2}(\mathcal{H}^U-\mathcal{H}^V),\\
&N_U=\tfrac{1}{2}(\tilde{N}+\tilde{N}_1), && \tilde{N}=N_U+N_V,\\
&N_V=\tfrac{1}{2}(\tilde{N}-\tilde{N}_1), && \tilde{N}_1=N_U-N_V.
\end{aligned} \tag{111}$$

After the canonical transformation (109), (110) and the substitutions (111), the action functional (106) assumes the form

$$S=2\pi\int_{-\infty}^\infty dt\int_0^\infty dr\,(\Pi_U\dot{U}+\Pi_V\dot{V}+\pi_\psi\dot{\psi}-N_U\mathcal{H}^U-N_V\mathcal{H}^V). \tag{112}$$

Here we have

$$\mathcal{H}^U=\tfrac{1}{2}(R^{-1/2}\pi_\psi+R^{1/2}\psi')^2+2U'\Pi_U, \tag{113}$$

$$\mathcal{H}^V=\tfrac{1}{2}(R^{-1/2}\pi_\psi-R^{1/2}\psi')^2-2V'\Pi_V, \tag{114}$$

where R is an abbreviation for $\frac{1}{2}(U-V)$. The advantage of the form (112) is that the "coordinate part" of $\mathcal{H}^U$ contains only the advanced variables U, Π_U, and the "coordinate part" of $\mathcal{H}^V$ contains only the retarded variables V, Π_V. Of course, both U and V enter the dynamical terms $\frac{1}{2}(R^{-1/2}\pi_\psi\pm R^{1/2}\psi')^2$.

The use of the advanced and retarded times as canonical coordinates is not to be confused with the characteristic initial value approach to geometrodynamics.[24] This approach owes its name to the fact that the initial data are prescribed on the characteristic lightlike hypersurfaces rather than on the spacelike hypersurfaces. This is not what is done in this paper. Our initial data U, V, Π_U, Π_V, ψ, π_ψ are still given on the spacelike hypersurfaces $t=$const and not on the characteristic null hypersurfaces $U=$const or $V=$const.

IX. THE PARAMETRIZED CANONICAL FORMALISM FOR CYLINDRICAL SCALAR WAVES ON A MINKOWSKIAN BACKGROUND

We have reached the point where all preliminaries are prepared for the Dirac quantization of the Einstein-Rosen waves. We should only turn the super-Hamiltonian (107) and the supermomentum (108) into operators and impose the Dirac constraints on the state functional. However, we shall postpone the quantization of the Einstein-Rosen waves to Sec. XI, and begin instead quite a different line of investigation. It is well known from the work of ADM[4] that the general form (38) of the gravitational action functional has its counterpart in the "parametrized form" of the action functional of ordinary field theories. By "parametrization" we mean the introduction of curvilinear spacetime coordinates into the action functional $\int d^4X\,\mathcal{L}$, followed by a conversion of the Minkowskian coordinates into supplementary canonical variables. Now, the Einstein field equations for the cylindrical waves in the Einstein-Rosen coordinates reduce effectively to the cylindrical wave equation for a scalar field ψ in a Minkowskian spacetime. This strongly suggests that the parametrized action functional for the cylindrical scalar field in the Minkowskian background may be identical with the action functional (106) – (108) for the Einstein-Rosen wave. This is exactly what we shall prove in this section. For this purpose, we build the parametrized formalism for the scalar wave.

The cylindrical wave equation (17) may be derived from the action functional

$$S=2\pi\int_{-\infty}^\infty dT\int_0^\infty dR\,\mathcal{L}(\psi,\psi_{,T},\psi_{,R}), \tag{115}$$

with the Lagrangian density

$$\mathcal{L}=\tfrac{1}{2}R(\psi_{,T}{}^2-\psi_{,R}{}^2). \tag{116}$$

The time T is interpreted as the Minkowskian time, and the coordinate R as the radial distance from the axis of symmetry in a flat space. Let us now introduce the curvilinear coordinates t and r (still flat space) by the formulas (21). The functions $T(t,r)$ and $R(t,r)$ are assumed to be subject to the restrictions (22) – (27). Under the substitution (21), the action functional assumes the form

$$S=2\pi\int_{-\infty}^{\infty}dt\int_0^{\infty}dr\,\mathfrak{L}\,(\psi,\dot{\psi},\psi') \tag{117}$$

with

$$\mathfrak{L}=\tfrac{1}{2}R(\dot{T}R'-T'\dot{R})^{-1}\times[(R'^2-T'^2)\dot{\psi}^2-2(\dot{R}R'-\dot{T}T')\dot{\psi}\psi'-(\dot{T}^2-\dot{R}^2)\psi'^2]. \tag{118}$$

The canonical momentum π_ψ obtained from the Lagrangian density $\mathfrak{L}$ is

$$\pi_\psi\equiv\frac{\partial\mathfrak{L}}{\partial\dot{\psi}}=R(\dot{T}R'-T'\dot{R})^{-1}[(R'^2-T'^2)\dot{\psi}-(\dot{R}R'-\dot{T}T')\psi']. \tag{119}$$

Passing from the Lagrangian density $\mathfrak{L}$ to the Hamiltonian density $\mathfrak{H}=\pi_\psi\dot{\psi}-\mathfrak{L}$, we get

$$\mathfrak{H}=\dot{T}(R'^2-T'^2)^{-1}[\tfrac{1}{2}R'(R^{-1}\pi_\psi{}^2+R\psi'^2)-T'\psi'\pi_\psi]+\dot{R}(R'^2-T'^2)^{-1}[-\tfrac{1}{2}T'(R^{-1}\pi_\psi{}^2+R\psi'^2)+R'\psi'\pi_\psi]. \tag{120}$$

We can simplify the expression for $\mathfrak{H}$ even more if we again introduce the advanced and retarded time coordinates,

$$\mathfrak{H}=\dot{U}\times\tfrac{1}{4}U'^{-1}(R^{-1/2}\pi_\psi+R^{1/2}\psi')^2-\dot{V}\times\tfrac{1}{4}V'^{-1}(R^{-1/2}\pi_\psi-R^{1/2}\psi')^2. \tag{121}$$

The Hamiltonian (120) is linear in the "velocities" $\dot{T}$, $\dot{R}$, and the Hamiltonian (121) is linear in the "velocities" $\dot{U}$, $\dot{V}$. This allows us to write the action functional in a homogeneous form. If we introduce the abbreviations

$$\Pi_U\equiv-\tfrac{1}{4}U'^{-1}(R^{-1/2}\pi_\psi+R^{1/2}\psi')^2,\quad \Pi_V\equiv\tfrac{1}{4}V'^{-1}(R^{-1/2}\pi_\psi-R^{1/2}\psi')^2, \tag{122}$$

we have

$$\mathfrak{H}=-\Pi_U\dot{U}-\Pi_V\dot{V}$$

and

$$S=2\pi\int_{-\infty}^{\infty}dt\int_0^{\infty}dr\,(\Pi_U\dot{U}+\Pi_V\dot{V}+\pi_\psi\dot{\psi}). \tag{123}$$

The definitions (122) can be written in the form $\mathcal{K}^U=0$, $\mathcal{K}^V=0$, where $\mathcal{K}^U$, $\mathcal{K}^V$ have exactly the same structure as the expressions (113), (114) obtained for the gravitational wave. In the action functional (123), only the true dynamical variables ψ, π_ψ are varied, while U and V are thought of as some prescribed functions of t and r, and Π_U and Π_V as mere abbreviations (122). However, we may vary all the variables ψ, π_ψ, U, Π_U, V, Π_V freely if we add the constraint functions $\mathcal{K}^U$ and $\mathcal{K}^V$ to the action functional (123) by means of the Lagrange multipliers N_U, N_V. In this way, we arrive at the action functional which is identical with the gravitational action functional (112).

Of course, we can return from the advanced and retarded time coordinates to the original coordinates T and R by the canonical transformation (109), (110). After the substitutions (111), our action functional assumes the old form (106), with $\tilde{\mathcal{K}}$ and $\tilde{\mathcal{K}}^1$ defined by Eqs. (107), (108).

The ADM canonical formalism for the cylindrical gravitational wave is thus completely equivalent to the parametrized canonical formalism for the cylindrically symmetric massless scalar field on a Minkowskian spacetime background. This is a very important conclusion because it allows us to take all results we have about the classical and quantum behavior of the scalar field, and apply them to the cylindrical gravitational waves.

X. CYLINDRICAL SCALAR WAVES IN A HALF-PARAMETRIZED CANONICAL FORMALISM

The parametrization of the scalar waves in Sec. X was complete. Arbitrary cylindrically symmetric spacelike slices were admitted and an arbitrary radial coordinate r was used to label their points. While the arbitrariness of slicing is vital for the Dirac approach, the arbitrariness of the space coordinatization of such slices is much less important. Nothing physical is gained by maintaining the flexibility of the spatial system of coordinates, its only advantage being the preservation of manifest spatial covariance.

The canonical formalism is usually introduced only at the price of losing the *spacetime* covariance of the theory. The spatial covariance is the maximum which is retained. Thus the ADM super-Hamiltonian and supermomentum are manifestly covariant with respect to spatial transformations, but not with respect to spacetime transformations.[25] We can go one step further and play hide-and-seek with the spatial covariance itself. We can abandon it by adopting a special system of coordinates, and we can restore it again, starting from the canonical formalism in a special system of coordinates. We shall not try to present a general theory of these two complementary procedures, but we do illustrate how they work for the cylindrical waves. In this section, we deliberately abandon the spatial covariance of the scalar waves formalism. In Sec. XII, we quantize this noncovariant theory and show how to restore the spatial covariance of the quantum formalism.

To get rid of the spatial covariance is trivial. All we have to do is to pick out a special radial coordinate. The privileged radial coordinate with an invariant geometrical meaning is, of course, the radial distance R. Let us therefore put

$$r=R,\quad t=t(T,R), \tag{124}$$

which means that we parametrize even the *curved* slices $t=$const by the "straight" distance R from the axis of symmetry.

The formalism of Sec. IX is then largely simplified. Comparing Eqs. (124) with Eqs. (21), we see that only time is parametrized, the spatial coordinates being kept fixed. We shall therefore speak of a half-parametrized formalism. Because of Eq. (124), only the term proportional to $\dot{T}$ remains in the Hamiltonian density (120),

$$\mathfrak{H} = \dot{T}\underline{\mathcal{H}}(T(R), \psi(R), \pi_\psi(R)),$$

where $\underline{\mathcal{H}}$ assumes the form

$$\begin{aligned}\underline{\mathcal{H}} &= (1 - T_{,R}{}^2)^{-1}(\tfrac{1}{2}R^{-1}\pi_\psi{}^2 + \tfrac{1}{2}R\psi_{,R}{}^2 - T_{,R}\psi_{,R}\pi_\psi) \\ &= \tfrac{1}{2}(1 - T_{,R}{}^2)^{-1}(R^{-1/2}\pi_\psi - R^{1/2}T_{,R}\psi_{,R})^2 + \tfrac{1}{2}R\psi_{,R}{}^2.\end{aligned} \tag{125}$$

Calling the coefficient $\underline{\mathcal{H}}(T(R), \psi(R), \pi_\psi(R))$ of $\dot{T}$ in the Hamiltonian density $\mathfrak{H}$ by the new name $-\Pi_T$, we again cast the action functional to the homogeneous form. If we want to vary Π_T and T freely, we must take the constraint

$$\mathcal{H} = 0, \quad \mathcal{H} \equiv \Pi_T + \underline{\mathcal{H}}(T(R), \psi(R), \pi_\psi(R)), \tag{126}$$

multiply the constraint function $\mathcal{H}$ by a Lagrange multiplier N, and add it to the action functional. We thus end with the variational principle $\delta S = 0$, where

$$S = 2\pi\int_{-\infty}^{\infty} dt \int_0^\infty dR\,(\Pi_T\dot{T} + \pi_\psi\dot{\psi} - N\mathcal{H}), \tag{127}$$

and all variables T, Π_T, ψ, π_ψ, N are varied freely.

XI. THE DIRAC QUANTIZATION OF THE CYLINDRICAL WAVES

We can finally turn to the quantization of the cylindrical waves. Because the canonical formalism for the scalar waves in the Minkowskian spacetime is identical with the canonical formalism for the Einstein-Rosen cylindrical waves, and the prescribed Minkowskian geometry seems conceptually simpler than the quantized Riemannian geometry, we can concentrate our attention on the scalar waves when trying to grasp the meaning of quantum formalism. The interpretation of the identical formalism for the gravitational waves requires a few cautionary remarks which we postpone to Sec. XV.

To quantize the cylindrical waves, we turn the canonical variables T, R, ψ, Π_T, Π_R, Π_ψ into operators. All these operators commute, except

$$[T(r), \Pi_T(\tilde{r})] = [R(r), \Pi_R(\tilde{r})] = [\psi(r), \pi_\psi(\tilde{r})] = i\delta(r - \tilde{r}).$$

No ordering problems arise when we substitute the operators into the super-Hamiltonian (107) and supermomentum (108). The individual terms of the super-Hamiltonian contain only the manifestly commuting operators. In the supermomentum, products of the type $T'(r)\Pi_T(r)$ occur. Happily, in spite of the fact that $T(r)$ and $\Pi_T(\tilde{r})$ do not commute, $T'(r)$ and $\Pi_T(r)$ *taken at the same point* r do. This is a consequence of the differentiated commutation relations

$$[T'(r), \Pi_T(\tilde{r})] = i\delta'(r - \tilde{r})$$

and the antisymmetry of the differentiated δ function, by virtue of which $\delta'(0) = 0$.

We are now ready to impose the Dirac constraints (51) on the state functional Ψ. We choose the representation in which the canonical coordinates T, R, ψ are diagonal and

$$\Pi_T(r) = -i\frac{\delta}{\delta T(r)}, \quad \Pi_R(r) = -i\frac{\delta}{\delta R(r)}, \quad \pi_\psi(r) = -i\frac{\delta}{\delta\psi(r)}. \tag{128}$$

The Dirac constraints can then be written down explicitly:

$$\left[-iR'(r)\frac{\delta}{\delta T(r)} - iT'(r)\frac{\delta}{\delta R(r)} - \tfrac{1}{2}R^{-1}(r)\frac{\delta^2}{\delta\psi(r)^2} + \tfrac{1}{2}R(r)\psi'^2(r)\right]\Psi = 0, \tag{129}$$

$$\left[T'(r)\frac{\delta}{\delta T(r)} + R'(r)\frac{\delta}{\delta R(r)} + \psi'(r)\frac{\delta}{\delta\psi(r)}\right]\Psi = 0. \tag{130}$$

The state functional Ψ is a functional of the three functions $T(r)$, $R(r)$, $\psi(r)$. The first two functions $T(r)$, $R(r)$ define, in a parametric form, one spacelike hypersurface $T = T(r(R))$ in the Minkowskian spacetime. The functional $\Psi[T, R, \psi]$ is physically interpreted as the probability amplitude that the scalar field ψ has the definite distribution $\psi(r)$ on this hypersurface. Of course, the same hypersurface can be coordinatized in a number of different ways. If we change the coordinate label r to $\bar{r} = f^{-1}(r)$, the functions T, R, and ψ behave as scalars. In order that the interpretation of Ψ as the probability amplitude be consistent, Ψ must remain unchanged under the relabeling of the hypersurface,

$$\Psi[T(f(r)), R(f(r)), \psi(f(r))] = \Psi[T(r), R(r), \psi(r)]. \tag{131}$$

Because the finite transformation $f(r)$ may be accomplished in infinitesimal steps, it is sufficient to check that Eq. (131) holds for the infinitesimal transformation

$$f(r) = r + \epsilon(r). \tag{132}$$

To the first order in $\epsilon(r)$,

$$\Psi[T(r+\epsilon(r)),R(r+\epsilon(r)),\psi(r+\epsilon(r))]=\Psi[T(r),R(r),\psi(r)]+\int_0^\infty dr\left[T'\frac{\delta\Psi}{\delta T}+R'\frac{\delta\Psi}{\delta R}+\psi'\frac{\delta\Psi}{\delta\psi}\right]_{T(r),R(r),\psi(r)}\times\epsilon(r).$$

Because the infinitesimal quantity $\epsilon(r)$ is otherwise arbitrary, Eq. (131) reduces to Eq. (130). The supermomentum constraint (130) tells us merely that the state functional is independent of the coordinatization of the hypersurface. The only equation carrying dynamical content is the super-Hamiltonian constraint (129). We shall study it in detail in Secs. XII and XIII.

XII. THE DIRAC QUANTIZATION OF THE CYLINDRICAL WAVES IN THE HALF-PARAMETRIZED FORMALISM

Because the state functional is independent of the coordinatization, it appears sufficient to solve the constraint equations in one definite coordinatization, and to define the state functional for any other coordinatization by the requirement of coordinatization independence. The natural coordinatization is the choice $r=R$, leading to the half-parametrized formalism of Sec. X. In this formalism, there remain only two canonical coordinates, namely $T(R)$ and $\psi(R)$, and only one constraint, namely Eq. (126). In quantum theory, we replace the variables $T(R)$, $\psi(R)$, $\Pi_T(R)$, $\pi_\psi(R)$ by operators. All these operators commute, except

$$[T(R),\Pi_T(\tilde{R})]=[\psi(R),\pi_\psi(\tilde{R})]=i\delta(R-\tilde{R}). \qquad (133)$$

We now turn the functions $\underline{\mathcal{H}}$ and $\mathcal{H}$, defined by Eqs. (125) and (126), into operators. Repeating the argument of Sec. XI, we could show that no ordering problem arises. We choose the representation in which the canonical coordinates $T(R)$ and $\psi(R)$ are diagonal and the momenta assume the form of variational derivatives,

$$\Pi_T(R)=-i\frac{\delta}{\delta T(R)},\quad \pi_\psi(R)=-i\frac{\delta}{\delta\psi(R)}. \qquad (134)$$

In this representation, the state functional $\underline{\Psi}$ depends on two functions, $T(R)$ and $\psi(R)$. The classical constraint (126), $\mathcal{H}=0$, is replaced by the Dirac constraint $\mathcal{H}\underline{\Psi}=0$ imposed on the state functional $\underline{\Psi}$. The Dirac constraint can be written as a Schrödinger-type equation

$$i\frac{\delta\underline{\Psi}}{\delta T(R)}=\underline{\mathcal{H}}(T(R),\psi(R),\pi_\psi(R))\underline{\Psi}, \qquad (135)$$

with the Hamiltonian density operator $\underline{\mathcal{H}}$ defined by Eqs. (125), (134). Explicitly,

$$\underline{\mathcal{H}}=\tfrac{1}{2}(1-T_{,R}{}^2)^{-1}\left(-iR^{-1/2}\frac{\delta}{\delta\psi(R)}-R^{1/2}T_{,R}\psi_{,R}\right)^2+\tfrac{1}{2}R\psi_{,R}{}^2. \qquad (136)$$

The two functions $T(r)$, $R(r)$ specify in the parametric form the same hypersurface as one function, $T(r(R))$, specifies in the half-parametrized formalism. If $T(r)$, $R(r)$, $\psi(r)$ lie in the range of the functional Ψ, then $T(r(R))$, $\psi(r(R))$ lie in the range of the functional $\underline{\Psi}$. Let us have a $\underline{\Psi}$ that satisfies the constraint (135), and define the new functional Ψ by the requirement

$$\Psi[T(r),R(r),\psi(r)]=\underline{\Psi}[T(r(R)),\psi(r(R))]. \qquad (137)$$

We expect that the state functional Ψ satisfies the two constraints (129), (130).

To check our expectation formally, we must express the variational derivatives of Ψ by means of the variational derivatives of $\underline{\Psi}$. For this purpose, we vary the three independent functions $R(r)$, $T(r)$, and $\psi(r)$,

$$R(r)\to R(r)+\delta R(r), \qquad (138)$$

$$T(r)\to T(r)+\delta T(r),\quad \psi(r)\to\psi(r)+\delta\psi(r). \qquad (139)$$

The variation (138) of $R(r)$ induces the variation

$$\delta r(R)=-R'(r(R))^{-1}\delta R(r(R)) \qquad (140)$$

of the inverse function $r(R)$. The variations (139) of $T(r)$ and $\psi(r)$, together with the variation (140) of $r(R)$, induce the total variations

$$T(r(R))\to T(r(R))+\delta^*T(r(R)),\quad \psi(r(R))\to\psi(r(R))+\delta^*\psi(r(R)) \qquad (141)$$

of the composite functions $T(r(R))$ and $\psi(r(R))$. We have

$$\delta^*T(r)=\delta T(r)-(R'(r))^{-1}T'(r)\delta R(r),\quad \delta^*\psi(r)=\delta\psi(r)-(R'(r))^{-1}\psi'(r)\delta R(r). \qquad (142)$$

According to Eq. (137), the values of the functionals Ψ and $\underline{\Psi}$ must be the same even for the varied arguments,

$$\Psi[T(r)+\delta T(r),R(r)+\delta R(r),\psi(r)+\delta\psi(r)]=\underline{\Psi}[T(r(R))+\delta^*T(r(R)),\psi(r(R))+\delta^*\psi(r(R))]. \qquad (143)$$

If we subtract (137) from (143) and confine ourselves to terms linear in the variations, we obtain

$$\int_0^\infty dr\left[\frac{\delta\Psi}{\delta T(r)}\delta T(r)+\frac{\delta\Psi}{\delta R(r)}\delta R(r)+\frac{\delta\Psi}{\delta\psi(r)}\delta\psi(r)\right]=\int_0^\infty dR\left[\frac{\delta\underline{\Psi}}{\delta T(r(R))}\delta^*T(r(R))+\frac{\delta\underline{\Psi}}{\delta\psi(r(R))}\delta^*\psi(r(R))\right]. \qquad (144)$$

In the last integral, we can introduce r instead of R as the integration variable,

$$\int_0^\infty dr\, R'(r)\left(\frac{\delta\underline{\Psi}}{\delta T(r(R))}\bigg|_{R=R(r)}\delta^* T(r)+\frac{\delta\underline{\Psi}}{\delta\psi(r(R))}\bigg|_{R=R(r)}\delta^*\psi(r)\right), \tag{145}$$

and substitute the expressions (142) for $\delta^* T(r)$ and $\delta^*\psi(r)$. Comparing (145) with the first integral in (144), we see that

$$\begin{aligned}
\frac{\delta\Psi}{\delta T(r)}&=R'(r)\,\frac{\delta\underline{\Psi}}{\delta T(r(R))}\bigg|_{R=R(r)},\\
\frac{\delta\Psi}{\delta R(r)}&=-T'(r)\,\frac{\delta\underline{\Psi}}{\delta T(r(R))}\bigg|_{R=R(r)}-\psi'(r)\,\frac{\delta\underline{\Psi}}{\delta\psi(r(R))}\bigg|_{R=R(r)},\\
\frac{\delta\Psi}{\delta\psi(r)}&=R'(r)\,\frac{\delta\underline{\Psi}}{\delta\psi(r(R))}\bigg|_{R=R(r)}.
\end{aligned} \tag{146}$$

It is easy to verify that Ψ satisfies the super-Hamiltonian constraint (129), if $\underline{\Psi}$ satisfies the constraint (135). We take the left-hand side of Eq. (129), multiply it by $[1-T(r(R))_{,R}{}^2]^{-1}r_{,R}{}^2$, and replace the variational derivatives of Ψ by the variational derivatives of $\underline{\Psi}$ according to the schema (146). By this process, we get exactly Eq. (135). The supermomentum constraint (130) is satisfied automatically, as a consequence of Eqs. (146).

All quantum dynamics of the scalar cylindrical waves is thus concentrated in one functional differential equation (135) of the Schrödinger type. The Hamiltonian density operator (136) is a sum of two terms which are counterparts of the kinetic energy and the potential energy of the standard Hamiltonian. The kinetic energy term is rather complicated. The factor $(1-T_{,R}{}^2)^{-1}$ reminds us that the hypersurface $T=T(R)$ is not a hyperplane and that we observe the scalar field ψ from different local Lorentz frames at different points of the hypersurface. If the hypersurface is one of a constant Minkowski time, $T=T_0=\text{const}$, the factor $(1-T_{,R}{}^2)^{-1}$ reduces to unity. The rest of the kinetic energy term is reminiscent of the kinetic energy $\frac{1}{2}m^{-1}(p^k-eA^k)^2$ of a charged particle moving in the magnetic field described by the vector potential A^k. The correlation is $eA^k\longleftrightarrow R^{1/2}T_{,R}\psi_{,R}$. We shall see in Sec. XIII that the analogy continues if we compare the expressions for probability fluxes. The potential energy term is much simpler than the kinetic energy term. It has exactly the form of the potential energy density of the cylindrical scalar field on a constant Minkowski time hypersurface.

The most important aspect of the reduced super-Hamiltonian constraint (135), however, is the presence of the familiar energy operator $-i\delta/\delta T(R)$. This operator is brought in by our choice of the extrinsic time representation. In the metric representation, we would have obtained an equation of the Klein-Gordon type. The time variable $T(R)$ represents a Tomonaga-Schwinger many-fingered time. This is why Eq. (135) is a functional differential equation rather than a partial differential equation and why the kinetic and potential energy terms are to be interpreted as the energy densities, not as the total energies.

XIII. PATH INDEPENDENCE OF THE EVOLUTION OF STATE

The entire quantum dynamics of the cylindrical waves is contained in the functional differential equation (135). If we solve this single equation, we know how to generate the state functional Ψ which satisfies all Dirac's constraints. However, the single functional differential equation (135) still represents a system of ∞^1 equations, one equation for each value of the radius R. How do we solve such a system of equations? The answer is that these ∞^1 equations are not mutually independent. We can further reduce them to a single partial differential equation and solve this equation instead of the original system of ∞^1 equations. Let us show that such a reduction is both physically reasonable and mathematically consistent.

We expect, of course, that Eq. (135) governs the evolution of the state functional $\underline{\Psi}$, i.e., that it determines the state functional $\underline{\Psi}[T(R),\psi(R)]$ on an arbitrary hypersurface $T=T(R)$, if the initial value

$$\underline{\Psi}[T_0(R),\psi(R)]=\underline{\Psi}_0[\psi(R)]$$

of this functional on the initial hypersurface $T=T_0(R)$ is known. To obtain $\underline{\Psi}[T(R),\psi(R)]$ from $\underline{\Psi}[T_0(R),\psi(R)]$, we pass from the initial hypersurface $T=T_0(R)$ to the final hypersurface $T=T(R)$ through a continuous one-parameter family of hypersurfaces $T=T(R,t)$,

$$T(R,0)=T_0(R),\quad T(R,1)=T(R),\quad t\in[0,1], \tag{147}$$

labeled by the values of a time parameter t.

If we confine ourselves to such a family of hypersurfaces, the state functional $\underline{\Psi}$ becomes a functional of the single function $\psi(R)$, depending on t as a parameter,

$$\underline{\Psi}[T(R,t),\psi(R)]=\underline{\Psi}_t[\psi(R)]. \tag{148}$$

Its rate of change $\partial\underline{\Psi}_t/\partial t$ when passing from one hypersurface of the family to the next is

$$\frac{\partial\underline{\Psi}_t}{\partial t}=\int_0^\infty dR\left[\frac{\delta\underline{\Psi}}{\delta T(R)}\right]_{T(R)=T(R,t)}\frac{\partial T(R,t)}{\partial t}.$$

Using the Schrödinger equation (135), we get

$$i\frac{\partial\underline{\Psi}_t}{\partial t}=\underline{H}_t[\psi(R),\pi_\psi(R)]\underline{\Psi}_t, \tag{149}$$

where

$$\underline{H}_t\equiv\int_0^\infty dR\,\underline{\mathcal{H}}(T(R,t),\psi(R),\pi_\psi(R))\frac{\partial T(R,t)}{\partial t}. \tag{150}$$

In contrast to the Hamiltonian density operator $\underline{\mathcal{H}}$ which depends locally on the operators $T(R)$, $\psi(R)$ and $\pi_\psi(R)$, $\underline{H}_t$ is the total Hamiltonian defined on the hypersurfaces $t=\text{const}$, which is a *functional* of the operators $\psi(R)$, $\pi_\psi(R)$, depending on t as a parameter.

Equation (149) is a partial differential equation of the Schrödinger type which readily permits us to write down the formal solution of the evolution problem:

$$\underline{\Psi}[T(R),\psi(R)]=P\exp\left(-i\int_0^1 dt\,\underline{H}_t\right)\underline{\Psi}_0[\psi(R)]. \tag{151}$$

By P we denote the time ordering operator which, applied to a product $\underline{H}_{t1}\underline{H}_{t2}\cdots\underline{H}_{tn}$ of Hamiltonians, rearranges them with respect to the decreasing time parameter t,

$$P(\underline{H}_{t_1}\underline{H}_{t_2}\cdots\underline{H}_{t_n})=\underline{H}_{t_{i_1}}\underline{H}_{t_{i_2}}\cdots\underline{H}_{t_{i_n}},$$

(i_1, i_2, ..., i_n = a permutation of indices 1, 2, ..., n such that $t_{i_1}\geq t_{i_2}\geq\cdots\geq t_{i_n}$), where the Hamiltonian corresponding to the earliest t operates on the state functional $\underline{\Psi}_0$ first.

Writing down the solution (151), we come upon a serious consistency problem. We can pass from the initial hypersurface $T=T_0(R)$ to the final hypersurface $T=T(R)$ along different paths $T=T(R,t)$. If the final state functional $\underline{\Psi}[T(R),\psi(R)]$ should depend on the path chosen, the Schrödinger equation (135) would be clearly inconsistent. Because the finite deformation of the path can be achieved in infinitesimal steps, it is sufficient to ask if the change of the state functional $\underline{\Psi}$ induced by deforming the hypersurface $T=T(R)$ at first by a small amount $\delta T(R)$ and then by another small amount $\Delta T(R)$ is the same as the change of $\underline{\Psi}$ induced by the deformations performed in the reversed order $\Delta T(R)$, $\delta T(R)$. This question is finally equivalent to the formal question whether the system of equations (135), one equation for each value of R, is integrable. The variational derivatives taken at two points R and $\tilde{R}$ must commute,

$$\frac{\delta^2\underline{\Psi}}{\delta T(\tilde{R})\delta T(R)}=\frac{\delta^2\underline{\Psi}}{\delta T(R)\delta T(\tilde{R})}.$$

Differentiating Eq. (135) with respect to $T(\tilde{R})$ and then switching the labels R and $\tilde{R}$, we obtain the integrability conditions of this equation,

$$\frac{\delta\underline{\mathcal{H}}(T(R))}{\delta T(\tilde{R})}-\frac{\delta\underline{\mathcal{H}}(T(\tilde{R}))}{\delta T(R)}+i[\underline{\mathcal{H}}(T(\tilde{R})),\underline{\mathcal{H}}(T(R))]=0. \tag{152}$$

The evolution of the state functional is path-independent if and only if the integrability condition (152) is satisfied.

We can check directly that the Hamiltonian density operator (125) satisfies Eq. (152). By the repeated use of the commutation relations (133), we evaluate the commutator of the Hamiltonian density operators,

$$[\underline{\mathcal{H}}(T(\tilde{R})),\underline{\mathcal{H}}(T(R))]=2i\delta_{,R}(R-\tilde{R})(-(1-T_{,R}{}^2)^{-1}\psi_{,R}\pi_\psi+(1-T_{,R}{}^2)^{-2}T_{,R}(R^{-1}\pi_\psi{}^2-2T_{,R}\psi_{,R}\pi_\psi+R\psi_{,R}{}^2)).$$

The variational derivative of $\underline{\mathcal{H}}(T(\tilde{R}))$ with respect to $T(\tilde{R})$ yields the same expression multiplied by the factor $-\frac{1}{2}i$. The integrability conditions are therefore satisfied.[26] They tell us that the ∞^1 equations contained in the functional differential equation (135) are not all independent and can be replaced by the single partial differential equation (149).

XIV. THE INNER PRODUCT OF THE STATE FUNCTIONALS

The evolution of the state functionals is governed by the Schrödinger-type equation (135). As a consequence, the state functional $\underline{\Psi}$ may be normalized in the usual way,

$$\int D\psi\,\underline{\Psi}^*[T(R),\psi(R)]\underline{\Psi}[T(R),\psi(R)]=1, \tag{153}$$

and the expression

$$\underline{\Psi}^*[T(R),\psi(R)]\underline{\Psi}[T(R),\psi(R)],$$

interpreted as the probability density that the field ψ has the distribution $\psi(R)$ on the hypersurface $T=T(R)$. The symbol $\int D\psi$ denotes the functional integration with respect to the function variable ψ.

We can easily prove that the normalization (153) of the state functional is preserved under the deformation of the initial hypersurface. The first step is to derive, by exactly the same procedure as in particle dynamics, the continuity equation for probability. The time variation of the probability density is determined by the Schrödinger equation (135),

$$\frac{\delta(\underline{\Psi}^*\underline{\Psi})}{\delta T(R)}=-i((\underline{\mathcal{H}}\underline{\Psi})\underline{\Psi}^*-(\underline{\mathcal{H}}\underline{\Psi})^*\underline{\Psi}).$$

The right-hand side of the last relation may be expressed as the variational derivative with respect to $\psi(R)$ of a functional $-\Sigma_R[T(R),\psi(R)]$. In fact,

$$\Sigma_R=\tfrac{1}{2}i(1-T_{,R}{}^2)^{-1}\left\{R^{-1}\left(\underline{\Psi}\frac{\delta\underline{\Psi}^*}{\delta\psi(R)}-\underline{\Psi}^*\frac{\delta\underline{\Psi}}{\delta\psi(R)}\right)+2iT_{,R}(R)\psi_{,R}(R)\underline{\Psi}\,\underline{\Psi}^*\right\}. \quad (154)$$

This leaves us with the functional differential equation

$$\frac{\delta(\underline{\Psi}\,\underline{\Psi}^*)}{\delta T(R)}+\frac{\delta\Sigma_R}{\delta\psi(R)}=0\,. \quad (155)$$

The structure of Σ_R is strikingly similar to the structure of the probability flux for one particle in an external electromagnetic field. The discrete index k labeling the Cartesian coordinates x^k of the particle is replaced by the continuous parameter R, the coordinates x^k are replaced by the field variables $\psi(R)$, the momentum operator $-i\partial/\partial x^k$ is replaced by the field momentum operator $-i\delta/\delta\psi(R)$, and the vector potential term $em^{-1}A^k$ is analogous to the factor $T_{,R}(R)\psi_{,R}(R)$. The functional Σ_R depends on the continuous parameter R; similarly, the probability flux for one particle depends on the discrete index k' labeling the components of the flux vector. However, Eq. (155) still does not have the form of the equation of continuity. There is no integration implied over the label R, so that the expression $\delta\Sigma_R/\delta\psi(R)$ is not analogous to the divergence of the probability flux. In fact, Eq. (155) is not a single equation, as the equation of continuity should be, but represents an infinite system of equations, one equation for each value of the parameter R.

To get a true equation of continuity, we again pick out a one-parameter family of hypersurfaces $T=T(R,t)$ and ask how $\underline{\Psi}_t^*\underline{\Psi}_t$ changes when we pass from one hypersurface of the family to the next one. We get

$$\frac{\partial(\underline{\Psi}_t^*\underline{\Psi}_t)}{\partial t}=\int_0^\infty dR\,\frac{\delta(\underline{\Psi}^*\underline{\Psi})}{\delta T(R)}\bigg|_{T(R)=T(R,t)}\frac{\partial T(R,t)}{\partial t}$$
$$=-\int_0^\infty dR\,\frac{\delta}{\delta\psi(R)}\left(\Sigma_R|_{T(R)=T(R,t)}\frac{\partial T(R,t)}{\partial t}\right). \quad (156)$$

The last integral represents a divergence of the "flux vector" $\Sigma_R\partial T(R,t)/\partial t$ in the infinite-dimensional space of functions $\psi(R)$. We can now integrate both sides of Eq. (156) with respect to the function variable ψ. The functional integral $\int D\psi$ of the divergence on the right-hand side of Eq. (156) can be transformed by an infinite-dimensional version of the Gauss theorem into a functional surface integral over the boundary of the space of functions ψ. Under the appropriate boundary conditions for the state functional $\underline{\Psi}$, this functional surface integral vanishes. Therefore, the normalization integral $\int D\psi\,\underline{\Psi}_t^*\underline{\Psi}_t$ does not depend on t. Because our slicing $T=T(R,t)$ is completely arbitrary, we can conclude that

$$\frac{\delta}{\delta T(R)}\int D\psi\,\underline{\Psi}^*\underline{\Psi}=0\,. \quad (157)$$

More generally, let us have any two functionals $\underline{\Psi}_1[T(R),\psi(R)]$ and $\underline{\Psi}_2[T(R),\psi(R)]$ which satisfy the Schrödinger equation (135). On any hypersurface $T=T(R)$, we can define the inner product of these functionals by the formula

$$\langle\underline{\Psi}_1,\underline{\Psi}_2\rangle_{T(R)}=\int D\psi\,\underline{\Psi}_1^*[T(R),\psi(R)]\,\underline{\Psi}_2[T(R),\psi(R)]. \quad (158)$$

By virtue of the Schrödinger equation, this inner product does not depend on the choice of the hypersurface $T=T(R)$. The normalization condition (153) can be written as $\langle\underline{\Psi},\underline{\Psi}\rangle=1$ in the new notation. The conjugate operators $\psi(R)$ and $\pi_\psi(R)=-i\delta/\delta\psi(R)$ are Hermitian with respect to the inner product (158). So is the Hamiltonian density operator $\underline{\mathcal{H}}$. The inner product (158) turns the space of the state functionals into a Hilbert space.

XV. THE REALIZATION OF THE EXTRINSIC TIME REPRESENTATION

To obtain knowledge of the good and evil of quantized geometry, we have committed the original sin of freezing the extracylindrical degrees of freedom. This sin is transmitted from one generation of our results to another and corrupts their strict validity. However, within the necessarily imperfect world of the model, the results are as perfect as they can be. The quantum formalism for the cylindrical gravitational waves looks as simple as the quantum formalism for the scalar waves on the unquantized background of the Minkowskian geometry. There are, however, some subtle points in the interpretation of the quantized gravitational field we would like to touch briefly at this moment.

We want to work in the "extrinsic time representation," in which the operators $T(r)$, $R(r)$, and

$\psi(r)$ are diagonal. But what does this mean in terms of the theory of measurement? The representation is physically realized by a complete set of measuring apparatuses, one subset of apparatuses designed to measure $T(r)$, another subset designed to measure $R(r)$, and still another subset designed to measure $\psi(r)$. Because the operators $T(r)$, $R(r)$, and $\psi(r)$ commute, the apparatuses can be constructed in such a way that the measurement of any one of these quantities does not disturb in an unpredictable way the measurement of any other of these quantities. What is the nature of the apparatuses? Things are easy for a scalar field in the Minkowskian spacetime. There we know exactly what the apparatuses measuring $T(r)$ and $R(r)$ look like. The spacetime is not quantized, and the measurement of time intervals and distances does not disturb the measurement of other quantities, like the field variable $\psi(r)$. We have a rigid inertial structure of reference points, each point carrying a label indicating its coordinates r, φ, z, and equipped with a coordinate clock showing the coordinate time t. Moreover, at each reference point there is a standard clock showing the standard time T, different standard clocks being synchronized by light signals, and a net of standard measuring rods, connecting the reference point with all neighboring points. The hypersurface $t = \text{const}$ is determined by the readings of the coordinate clocks. The standard clocks installed on the reference points measure the time T, and the rods stretching radially from the axis of symmetry of the ψ field to the point in question measure R.

Measurements of the quantized Riemannian geometry are much more sophisticated. They again presuppose a structure of reference points defining our system of coordinates and the hypersurfaces of constant coordinate time. But the net of standard measuring rods may be set up only in the azimuthal and the axial directions. If the rods are placed also in the radial direction, they would measure the g_{11} component of the metric, and with it the coefficient γ. But $-\gamma'$ is canonically conjugate to T, so that γ and T cannot be measured simultaneously. Laying the rods in all possible directions would be equivalent to fixing a representation quite different from the extrinsic time representation T, R, ψ – namely the metric representation γ, R, ψ. That is why we can lay the rods only in the azimuthal and axial directions. These rods serve as the apparatuses measuring $R(r)$ and $\psi(r)$. We measure by them the width and the circumference of the portion of the cylindrical surface $r = \text{const}$ lying between the circles $z = z_0$, $z = z_0 + 1$. According to Eq. (80), the surface area of this portion divided by 2π is the radial coordinate R and the natural logarithm of the width is ψ.

The apparatuses for measuring the Einstein-Rosen time T are also much trickier than the standard clocks synchronized by light. Some of the components of the metric tensor ${}^4g_{\iota\kappa}$ become operators, and it thus makes no sense to speak about the classical propagation equations for light signals and consequently about light synchronization. Moreover, the proper time along the worldline of a reference point is also an operator, and the standard clocks moving along the worldline do not show a definite time. All this indicates that our "T clocks" must be very queer clocks indeed. What we require is a device measuring a peculiar combination $\pi_\gamma = \pi_1^1 = g^{1/2}(K_2^2 + K_3^3)$ of the components of the extrinsic curvature, without ever attempting to measure these components or the determinant g individually, plus a standard clock at spatial infinity, where the spacetime is asymptotically Minkowskian. This set of instruments permits us to measure T as defined by Eq. (98).

These few remarks suffice to point out the degree of sophistication needed to interpret the quantum gravitational formalism. Let us stress once more that the subject of this interpretation – namely, the formalism itself – is the same as the Dirac formalism for the cylindrical massless scalar field on an unquantized Minkowskian spacetime background.

XVI. THE ADM QUANTIZATION OF THE EINSTEIN-ROSEN WAVES

After all the intricacies of the Dirac method, it is a relief to follow the ADM procedure. The most difficult point is, of course, a clever choice of the functionals T and X^i, which later become the privileged coordinates of the theory. By clever choice we mean a choice which makes it easy to solve the initial value equations (41) explicitly for the momenta π_T and π_{X^i}, and which makes the "true" Hamiltonian density $\mathcal{H}_{\text{ADM}}$ as simple as possible. But we have already seen that the maximum simplification of the formalism is achieved by the Einstein-Rosen variables T, R. We therefore take the Einstein-Rosen time (98) as our T, and R as our X^i. Equations (98) and (103) realize the canonical transformation (46), and the action functional (106) has the form (47). The next step is to solve the initial value equations (41). Here we see the advantage of the Einstein-Rosen variables: The super-Hamiltonian and (what is more remarkable) the supermomentum do not contain the derivatives of the momenta Π_T and Π_R, and the momenta themselves appear linearly in these expressions. It is thus trivial to solve the initial value equations for Π_T and Π_R, and substitute these solutions back

TABLE I. The quantization of the cylindrical gravitational waves.

The standard form of the cylindrical metric	$g_{11}=e^{\gamma(r)-\psi(r)},\ g_{22}=R^2(r)e^{-\psi(r)},$ $g_{33}=e^{\psi(r)},\ g_{12}=g_{13}=g_{23}=0$
The standard form of the cylindrical momenta	$\pi^{11}=\pi_\gamma(r)e^{\psi-\gamma},\ \pi^{22}=\frac{1}{2}R\pi_R(r)e^{\psi},$ $\pi^{33}=[\pi_\gamma(r)+\frac{1}{2}R\pi_R(r)+\pi_\psi(r)]e^{-\psi},\ \pi^{12}=\pi^{13}=\pi^{23}=0$
The standard form of the lapse and shift functions	$N=N(r),\ N_i=(N_1(r),0,0)$
The super-Hamiltonian and supermomentum	$\mathcal{H}=e^{\frac{1}{2}(\psi-\gamma)}(-\pi_\gamma\pi_R+\frac{1}{2}R^{-1}\pi_\psi{}^2+2R''-\gamma'R'+\frac{1}{2}R\psi'^2),$ $\mathcal{H}^1=e^{\psi-\gamma}(-2\pi'_\gamma+\gamma'\pi_\gamma+R'\pi_R+\psi'\pi_\psi),$ $\mathcal{H}^2=\mathcal{H}^3=0$
The canonical transformation to the extrinsic time representation T,R. The coordinates T,R are the Einstein-Rosen time and cylindrical radius, and the momenta Π_T, Π_R are the C-energy density and C-energy flux.	$T(r)=T(\infty)+\int_\infty^r -\pi_\gamma(r)dr,$ $\Pi_T(r)=-\gamma'(r)+\{\ln[R'^2(r)-\pi_\gamma{}^2(r)]\}',$ $R(r)=R(r),$ $\Pi_R(r)=\pi_R(r)+\left(\ln\frac{R'(r)-\pi_\gamma(r)}{R'(r)+\pi_\gamma(r)}\right)'$
The standard coordinate system determined up to the gauge transformations	$r\rightarrow f^{-1}(r),$ $\varphi\rightarrow\pm\varphi+\varphi_0,\ z\rightarrow\alpha z+z_0$
Gauge transformation of canonical variables	$T(r),R(r),\psi(r)\rightarrow T(f(r)),R(f(r)),\psi(f(r)),$ $\Pi_T(r),\Pi_R(r),\pi_\psi(r)\rightarrow f'(r)\Pi_T(f(r)),f'(r)\Pi_R(f(r)),f'(r)\pi_\psi(f(r))$
The rescaled super-Hamiltonian and supermomentum in the extrinsic time representation	$\tilde{\mathcal{H}}=R'\Pi_T+T'\Pi_R+\frac{1}{2}R^{-1}\pi_\psi{}^2+\frac{1}{2}R\psi'^2,$ $\tilde{\mathcal{H}}^1=T'\Pi_T+R'\Pi_R+\psi'\pi_\psi$
The reduced action functional	$S=2\pi\int_{-\infty}^{\infty}dt\int_0^\infty dr(\Pi_T\dot{T}+\Pi_R\dot{R}+\pi_\psi\dot{\psi}-\tilde{N}\tilde{\mathcal{H}}-\tilde{N}_1\tilde{\mathcal{H}}^1)$
The Dirac constraints in the fully parametrized formalism	$\left(-iR'\frac{\delta}{\delta T}-iT'\frac{\delta}{\delta R}-\frac{1}{2}R^{-1}\frac{\delta^2}{\delta\psi^2}+\frac{1}{2}R\psi'^2\right)\Psi=0,$ $\left(T'\frac{\delta}{\delta T}+R'\frac{\delta}{\delta R}+\psi'\frac{\delta}{\delta\psi}\right)\Psi=0$
The Einstein-Schrödinger equation in the half-parametrized formalism	$i\frac{\delta\underline{\Psi}}{\delta T(R)}=\underline{\mathcal{H}}\,\underline{\Psi},$ $\underline{\mathcal{H}}=\frac{1}{2}(1-T_{,R}{}^2)^{-1}\left(-iR^{-1/2}\frac{\delta}{\delta\psi(R)}-R^{1/2}T_{,R}\psi_{,R}\right)^2+\frac{1}{2}R\psi_{,R}{}^2$
The connection between the functionals Ψ and $\underline{\Psi}$	$\Psi[T(r),R(r),\psi(r)]=\underline{\Psi}[T(r(R)),\psi(r(R))]$
The integrability conditions of the Einstein-Schrödinger equation	$\frac{\delta\underline{\mathcal{H}}(T(R))}{\delta T(\tilde{R})}-\frac{\delta\underline{\mathcal{H}}(T(\tilde{R}))}{\delta T(R)}+i[\underline{\mathcal{H}}(T(\tilde{R})),\underline{\mathcal{H}}(T(R))]=0$
Solution of the evolution problem	$\underline{\Psi}[T(R),\psi(R)]=P\exp\left(-i\int_0^1 dt\,\underline{H}_t[\psi(R),\pi_\psi(R)]\right)\underline{\Psi}_0[\psi(R)],$ $\underline{H}_t\equiv\int_0^\infty dR\,\underline{\mathcal{H}}(T(R,t),\psi(R),\pi_\psi(R))\frac{\partial T(R,t)}{\partial t}$

TABLE I. *(Continued)*

The "continuity equation" for probability	$\frac{\delta(\underline{\Psi}\underline{\Psi}^*)}{\delta T(R)}+\frac{\delta\Sigma_R}{\delta\psi(R)}=0$, $\Sigma_R=\frac{1}{2}i(1-T_{,R}{}^2)^{-1}\left[R^{-1}\left(\underline{\Psi}\frac{\delta\underline{\Psi}^*}{\delta\psi(R)}-\underline{\Psi}^*\frac{\delta\underline{\Psi}}{\delta\psi(R)}\right)+2iT_{,R}(R)\psi_{,R}(R)\underline{\Psi}\underline{\Psi}^*\right]$
The inner product and the normalization of the state functionals	$\langle\underline{\Psi}_1,\underline{\Psi}_2\rangle_{T(R)}=\int D\psi\,\underline{\Psi}_1^*[T(R),\psi(R)]\underline{\Psi}_2[T(R),\psi(R)]$, $\langle\underline{\Psi},\underline{\Psi}\rangle_{T(R)}=1$
The ADM coordinate conditions	$T=T(t,r)\|_{t=T,r=R}$, $R=R(t,r)\|_{t=T,r=R}$
The ADM state functional	$\underline{\Psi}_{\rm ADM}(T)=\underline{\Psi}[T(R),\psi(R)]\|_{T(R)=T=\rm const}$
The ADM Hamiltonian	$\underline{H}_{\rm ADM}=\int_0^\infty dR\left(-\frac{1}{2}R^{-1}\frac{\delta^2}{\delta\psi(R)^2}+R\psi_{,R}{}^2\right)$
The ADM Schrödinger equation	$i\frac{\partial\underline{\Psi}_{\rm ADM}}{\partial T}=\underline{H}_{\rm ADM}\underline{\Psi}_{\rm ADM}$

into the action functional (106). We arrive at the action functional

$$S=\int_{-\infty}^{\infty}dt\int_0^\infty dr(\pi_\psi\dot{\psi}-\mathfrak{H})\,, \tag{159}$$

where $\mathfrak{H}$ is identical with the expression (120). Finally, and this is a decisive step, we impose the coordinate conditions

$$T=T(t,r)|_{t=T,r=R},\quad R=R(t,r)|_{t=T,\,r=R}\,,$$

as prescribed by Eq. (48). These coordinate conditions reduce the action functional (159) to the form

$$S=\int_{-\infty}^{\infty}dT\int_0^\infty dR[\pi_\psi\psi_{,T}-\tfrac{1}{2}(R^{-1}\pi_\psi{}^2+R\psi_{,R}{}^2)]\,. \tag{160}$$

We easily recognize $\frac{1}{2}(R^{-1}\pi_\psi{}^2+R\psi_{,R}{}^2)$ as the Hamiltonian density corresponding to the cylindrical massless scalar field ψ propagating in a Minkowskian spacetime. The quantization then proceeds along the familiar lines. We may choose the field representation, in which $\underline{\Psi}_{\rm ADM}=\underline{\Psi}_{\rm ADM}[\psi(R)]$, and

$$\pi_\psi(R)=-i\frac{\delta}{\delta\psi(R)}\,,$$

and write the Schrödinger equation (50),

$$i\frac{\partial\underline{\Psi}_{\rm ADM}}{\partial T}=\underline{H}_{\rm ADM}\underline{\Psi}_{\rm ADM}\,,$$

$$\underline{H}_{\rm ADM}=\int_0^\infty dR\left(-\tfrac{1}{2}R^{-1}\frac{\delta^2}{\delta\psi(R)^2}+\tfrac{1}{2}R\psi_{,R}(R)^2\right)\,.$$

The ADM state functional $\underline{\Psi}_{\rm ADM}$ is the Dirac state functional $\Psi[T(r),R(r),\psi(r)]$ evaluated at the point $T(r)=T=\text{const}$, $R(r)=r$, in accordance with the general formula (52).

Other representations may be used instead of the ψ-field representation. For example, we can take a complete set of solutions of the cylindrical wave equation (17), construct the corresponding creation and annihilation operators, and use the occupation number representation. In any case, the gravitational formalism is equivalent to that of a standard field theory.

XVII. SUMMARY

To summarize our discussion of the quantization of the cylindrical gravitational waves, we present the main results in the form of a table.

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APPENDIX

We give here a formal proof of the statement that the whole dynamical trajectory lies in the mini-phase-space, if its initial point lies there.

We simplify the argument by imposing the coordinate conditions $N_i = 0$ as discussed in Sec. IV. We proceed in several stages. In the first stage we check that, under the conditions (55) on the momenta and the condition (56) on the lapse function, the original symmetries (54) of the spatial geometry are preserved. The central formula is Eq. (44), determining the geometry $\tilde{g}_{ik} = g_{ik} + \delta g_{ik}$ on the spacelike hypersurface with the time label $\tilde{t} = t + \delta t$, if the geometry g_{ik} on the initial spacelike hypersurface with the time label t is known. We show that the new geometry $\tilde{g}_{ik}$ has the Killing vectors $\tilde{\xi}^i_{(A)} = \xi^i_{(A)}$, if the old geometry has the Killing vectors $\xi^i_{(A)}$. We use the identity

$$\mathcal{L}_{\xi^l}(g_{ik} + \delta g_{ik}) = \mathcal{L}_{\xi^l} g_{ik} + \mathcal{L}_{\xi^l} \delta g_{ik} . \tag{A1}$$

The first Lie derivative on the right-hand side of Eq. (A1) vanishes because the old geometry admits ξ^l as a Killing vector. Under the coordinate conditions $N_i = 0$, we have $\delta g_{ik} = \delta_0 g_{ik}$, and the second Lie derivative vanishes because of Eqs. (54), (55), and (56). The new geometry therefore admits $\tilde{\xi}^l = \xi^l$ as a Killing vector. In the coordinate system fixed by the coordinate condition $N_i = 0$, the new Killing vectors have the same components as the old Killing vectors and the group structure of the generators is therefore preserved.

In the second stage of the proof we show that the new Killing vector $\tilde{\xi}^i$ is surface-orthogonal and Eqs. (60) and (63) are satisfied. We again assume that $N_i = 0$. Then

$$\delta g_{ik} = 2NK_{ik}\,\delta t$$

and, as we have just proved, $\tilde{\xi}^i = \xi^i$. On the other hand,

$$\tilde{\xi}_i = \tilde{g}_{ik}\tilde{\xi}^k = \xi_i + \delta g_{ik}\xi^k .$$

We must show that

$$\tilde{\epsilon}^{ikl}\tilde{\xi}_i\,\tilde{\xi}_{k|l} \sim \epsilon^{ikl}\xi_{i|k}\xi_l + 2\epsilon^{ikl}[(NK_{im}\xi^m\xi_l)_{|k} - 2NK_{im}\xi^m\xi_{l|k}]\,\delta t = 0 .$$

Because we assume that Eqs. (60) and (63) [and therefore also Eq. (62)] hold, the only relation to be checked is

$$\epsilon^{ikl}K_{im}\xi^m\xi_{l,k} = 0 . \tag{A2}$$

This is easy. We already know that Eq. (62) implies Eq. (64), telling us that ξ^l is an eigenvector of K_{kl}. Because of this, Eq. (A2) reduces back to Eq. (62). This completes the second stage of our proof.

In the third stage of the proof, we check that the new momenta $\tilde{\pi}^{ik}$ have the symmetries (55), if the old momenta have the symmetries (55), the old geometry has the symmetries (54), and the lapse function has the symmetries (56). We again put $N_i = 0$, and look at the variation $\delta_0\pi^{ik}$ given by Eq. (45). If the geometry has the symmetry (54), then $\mathcal{L}_{\xi^l} R^{ik} = 0$. Under the rules of how to apply the Lie derivative to a product of tensor densities, we see that $\mathcal{L}_{\xi^l}\delta_0\pi^{ik} = 0$. Using the identity

$$\mathcal{L}_{\xi^l}(\pi^{ik} + \delta_0\pi^{ik}) = \mathcal{L}_{\xi^l}\pi^{ik} + \mathcal{L}_{\xi^l}\delta_0\pi^{ik},$$

we see that under our assumptions we arrive at the desired result

$$\mathcal{L}_{\xi^l}(\pi^{ik} + \delta_0\pi^{ik}) = 0 .$$

In the fourth and last stage of the proof, we show that the new momenta satisfy the condition (63), if the old momenta satisfy it, the old geometry has the symmetries (54) and (60), and the lapse function has the symmetry (56). We need only to check that Eq. (63) is fulfilled, if $\delta_0\pi^{ik}$ is substituted for π^{ik}. This means that we must prove that ξ_k is an eigenvector of $\delta_0\pi^{ik}$. The condition (63) is equivalent to Eq. (65) telling us that ξ_k is an eigenvector of π^{ik}. It is therefore also an eigenvector of $\pi^{im}\pi^k_m$. Of course, ξ_k is automatically an eigenvector of g^{ik}. Looking at the structure (45) of the variation $\delta_0\pi^{ik}$, we see that it remains to be proved that ξ_k is an eigenvector of R^{ik} and $N^{|ik}$. The whole proof then reduces to the proof of the following two statements:

(1) Every surface-orthogonal Killing vector is an eigenvector of the Ricci tensor, i.e., $R_{ik}\xi^k = \rho\xi_i$.

(2) If a scalar function N is symmetric with respect to a Killing vector ξ^i, i.e., if

$$\mathcal{L}_{\xi^i} N = N_{|i}\xi^i = 0 , \tag{A3}$$

then ξ^k is an eigenvector of the matrix $N_{|ik}$,

$$N_{|ik}\xi^k = \nu\xi_i .$$

The condition that the Killing vector ξ^i is surface-orthogonal can be written in the form

$$\xi_i\,\xi_{k|l} + \xi_k\,\xi_{l|i} + \xi_l\,\xi_{i|k} = 0 . \tag{A4}$$

If we differentiate it with respect to x^m, and use the integrability condition

$$\xi_{i|kl} = -R^m{}_{lik}\,\xi_m$$

of the Killing equation, we get

$$\xi_{i|m}\xi_{k|l} + \xi_{k|m}\,\xi_{l|i} + \xi_{l|m}\,\xi_{i|k} - \xi_n(\xi_i R^n{}_{mkl} + \xi_k R^n{}_{mli} + \xi_l R^n{}_{mik}) = 0 .$$

We contract the last equation in the indices m and l. Because the Killing vectors are divergenceless, we get

$$-\xi_i\,\xi_n R^n{}_k + \xi_k\,\xi_n R^n{}_i = 0 .$$

Multiplying this equation by ξ^k, we get the result

$$R^n{}_i \xi_n = (\xi_k \xi^k)^{-1} R_{mn} \xi^m \xi^n \xi_i ,$$

which completes the proof of the first statement.

We now multiply Eq. (A4) by $N^{|i}$ and use Eq. (A3) together with the Killing equation. We obtain the relation

$$-\xi_k \xi_{i|l} N^{|i} + \xi_l \xi_{i|k} N^{|i} = 0 ,$$

which can be rearranged by a repeated use of Eq. (A3) into the form

$$\xi_k N_{|il} \xi^i - \xi_l N_{|ik} \xi^i = 0 .$$

Multiplication of the last equation by ξ^k shows that ξ^i is an eigenvector of $N_{|ik}$. This completes the proof of the second statement, and with it the proof of the whole theorem.

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[1]C. W. Misner, Rev. Mod. Phys. 29, 497 (1957).

[2]S. Mandelstam, Ann. Phys. (N.Y.) 19, 25 (1962).

[3]B. S. DeWitt, Phys. Rev. 160, 1113 (1967).

[4]See R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research,* edited by L. Witten (Wiley, New York, 1962), Chap. 7, and the series of papers by the same authors quoted here.

[5]P. A. M. Dirac, Can. J. Math. 2, 129 (1950); Proc. Roy. Soc. (London) A246, 326 (1958); A246, 333 (1958); Phys. Rev. 114, 924 (1959).

[6]C. W. Misner, Phys. Rev. Letters 22, 1071 (1969); Phys. Rev. 186, 1319 (1969); 186, 1328 (1969).

[7]A. Einstein and N. Rosen, J. Franklin Inst. 223, 43 (1937); N. Rosen, Bull. Res. Council Israel 3, 328 (1954); J. Weber and J. A. Wheeler, Rev. Mod. Phys. 29, 509 (1957); L. Marder, Proc. Roy. Soc. (London) A244, 524 (1958); A246, 133 (1958); J. Ehlers and W. Kundt, in *Gravitation: An Introduction to Current Research,* edited by L. Witten (Wiley, New York, 1962), Chap. 2; K. S. Thorne, Phys. Rev. 138, B251 (1965).

[8]N. Rosen, Phys. Z. Sowjetunion 124, 366 (1937); J. Hély, Compt. Rend. 249, 1867 (1959); A. Peres, Phys. Rev. Letters 3, 571 (1959); H. Bondi, Nature 179, 1072 (1957); H. Bondi, F. A. E. Pirani, and I. Robinson, Proc. Roy. Soc. (London) A251, 519 (1959); P. Jordan, J. Ehlers, and W. Kundt, Akad. Wiss. Lit. (Mainz), Abhandl. Math. Nat. Kl. 1960, Nr. 2.

[9]*The Confessions of St. Augustine,* translated by E. B. Pusey (Library of the Fathers of the Holy Catholic Church, Oxford, England, 1838) Book XI, Chap. 15.

[10]Reference 9, Aug. de Civ. Dei, Book XI, Chap. 6.

[11]Reference 9, Aug. de Gen. ad Lit. Book V, Chap. 5.

[12]"In the first place, it is of course obvious to anyone that fire, earth, water, and air are bodies; ... Now, the question to be determined is this: What are the most perfect bodies that can be constructed, four in number, unlike one another, but such that some can be generated out of one another by resolution.... To earth let us assign the cubical figure; for of the four kinds earth is the most immobile and the most plastic of bodies. The figure whose bases are the most stable must best answer that description; and as a base, if we take the triangles we assumed at the outset, the face of the triangle with equal sides is by nature more stable than that of the triangle whose sides are unequal; and further, of the two equilateral surfaces, respectively, composed of the two triangles, the square is necessarily a more stable base than the triangle, both in its parts and as a whole. Accordingly we shall preserve the probability of our account, if we assign this figure to earth; and of the remainder the least mobile to water, the most mobile to fire, and the intermediate figure to air." *Plato's Cosmology, The Timaeus of Plato,* translated by F. M. Cornford (Kegan Paul, London, 1937).

[13]"I hold in fact

(1) That small portions of space *are* in fact of a nature analogous to little hills on a surface which is on the average flat; namely, that the ordinary laws of geometry are not valid in them.

(2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave.

(3) That this variation of the curvature of space is what really happens in that phenomenon which we call the *motion of matter,* whether ponderable or ethereal.

(4) That in the physical world nothing else takes place but this variation, subject (possibly) to the law of continuity." W. K. Clifford, Proc. Phil. Soc. (Cambridge) 2, 158 (1876). Reprinted in *Mathematical Papers by W. K. Clifford,* edited by R. Tucker (Macmillan, London, 1882).

[14]"The 'unitary field theory', which represents itself as a mathematically independent extension of the general theory of relativity, attempts to fulfill this last postulate of the field theory. The formal problem should be put as follows: – Is there a theory of the continuum in which a new structural element appears side by side with the metric such that it forms a single whole together with the metric? If so, what are the simplest field laws to which such a continuum can be made subject? And finally, are these field laws well fitted to represent the properties of the gravitational field and the electromagnetic field? Then there is the further question whether the corpuscles (electrons and protons) can be regarded as positions of particularly dense fields, whose movements are determined by the field equations." A. Einstein, "The Problem of Space," in *The World As I See It,* translated by A. Harris (Querido, Amsterdam, 1933).

[15]"From the standpoint of geometrodynamics the primordial entity is not one particle, nor an intercoupled family of particle fields, but the geometry of empty space itself. On this view a particle is not itself a 10^{-33}-cm fluctuation in the geometry; instead, it is a fantastically weak alteration in the pattern of these fluctuations, extending over a zone containing very many such 10^{-33} regions. In brief, a particle is a quantum state of excitation of the geometry; it is a *geometrodynamical exciton.*" J. A. Wheeler, in *Battelle Rencontres 1967,* edited by C. M. DeWitt and J. A. Wheeler (Benjamin, New York, 1968), Chap. 9.

[16]R. F. Baierlein, D. H. Sharp, and J. A. Wheeler, Phys. Rev. 126, 1864 (1962).

[17]A. Peres, Phys. Rev. 171, 1335 (1968).

[18]K. Kuchař, J. Math. Phys. 11, 3322 (1970).

[19]A group of finite motions is said to act freely on space if the only transformation leaving all points fixed is the identity transformation.

[20]See J. Ehlers and W. Kundt, or K. S. Thorne, Ref. 7.

[21]S. Deser, Ann. Inst. Henri Poincaré 7, 149 (1967).

[22]A. Fischer, in *Relativity*, edited by L. Witten (Plenum, New York, 1970).

[23]Such a system of coordinates is often called canonical [see, e.g., J. Ehlers and W. Kundt, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), Chap. 2], but we shall avoid this term, because the adjective "canonical" has a different meaning in the Hamiltonian formalism.

[24]See, e.g., R. K. Sachs's report to the Warsaw conference, published in *Relativistic Theories of Gravitation*, edited by L. Infeld (Pergamon, Oxford, England, 1964).

[25]However, in the course of a study of the general relativistic initial value problem (unpublished), J. W. York has rederived the ADM canonical formalism, including its extension to the case of nonholonomic slicing of space-time, by four-dimensionally invariant methods (private communication).

[26]*Note added in proof.* Quite generally, such integrability conditions are satisfied as a consequence of the commutation relations of the super-Hamiltonian and supermomentum. See the forthcoming paper by the author.