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Sum Rules for Inclusive Cross Sections*

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In generalization of known results, we present rigorous sum rules for inclusive cross sections based upon conservation of momentum or of any additive discrete quantum number. Included are formulas which normalize inclusive cross sections in terms of σ_{total} independently of multiplicities. The sum rules could be useful in constraining theoretical models which construct inclusive cross sections without detailed reference to exclusive processes. Some applications are described and a generalization to the tensors governing weak and electromagnetic inclusive reactions is given.

For the inclusive process $a+b\to x+$(anything), it is well known that integrating the production cross section over all momenta of particle x yields the average multiplicity of particle x times the total cross section:

$$\int \frac{d^3\sigma}{d^3p_x}\, d^3p_x = \langle n_x\rangle\sigma\,. \tag{1}$$

It has been shown by Chou and Yang[1] that energy and momentum conservation imply

$$\sum_x \int \frac{d^3\sigma}{d^3p_x}\,(E_x \pm p_{x\parallel})d^3p_x = (E_a \pm p_{a\parallel} + E_b \pm p_{b\parallel})\sigma\,, \tag{2}$$

where the sum is taken over all stable species of particles x, and E_x, $p_{x\parallel}$, E_a, $p_{a\parallel}$, E_b, $p_{b\parallel}$ are the energies and longitudinal momenta of particles x, a, and b, respectively, in any frame.

We present here a collection of sum rules which generalize these results.

We write the exclusive cross section for producing n identical particles in the reaction $a+b\to 1+2+\cdots+n$ as

$$\frac{d\sigma}{dp_1dp_2\cdots dp_n} = \tfrac{1}{2}(2\pi)^4\lambda^{-1}(s, m_a^2, m_b^2)\delta^4(\textstyle\sum p_i - P)\langle p_a p_b|T^\dagger|p_1p_2\cdots p_n\rangle\langle p_1p_2\cdots p_n|T|p_ap_b\rangle\,, \tag{3}$$

where

$$\begin{aligned} &P = p_a + p_b\,,\\ &dp = d^3p/[2E(2\pi)^3]\,,\\ &\lambda^2(x,y,z) = x^2+y^2+z^2-2xy-2xz-2yz\,, \end{aligned} \tag{4}$$

and the normalization is specified by the completeness relation

$$\sum_{n=0}^{\infty}\frac{1}{n!}\int dq_1\cdots dq_n|q_1\cdots q_n\rangle\langle q_1\cdots q_n| = \underline{1}\,. \tag{5}$$

The cross section is invariant under permutations of identical particle labels.

The partial cross section for producing n identical particles is then

$$\sigma^{(n)} = \frac{1}{n!}\int dp_1\cdots dp_n\,\frac{d\sigma}{dp_1\cdots dp_n}\,, \tag{6}$$

so that the total cross section is

$$\sigma = \sum_n \sigma^{(n)}\,, \tag{7}$$

and the inclusive cross section for producing one particle with momentum p is

$$\frac{d\sigma}{dp} = \sum_{n=1}^{\infty}\frac{1}{(n-1)!}\int dp_2dp_3\cdots dp_n\,\frac{d\sigma}{dpdp_2\cdots dp_n}\,. \tag{8}$$

From Eq. (8) we easily find

$$\int \frac{d\sigma}{dp}\,dp = \sum_{n=1}^{\infty} n\sigma^{(n)} \equiv \langle n\rangle\sigma\,, \tag{9}$$

where the last equality defines the average multiplicity. If we multiply Eq. (8) by a component p_μ of the produced particle momentum and integrate over p as above, we obtain

$$\begin{aligned}\int \frac{d\sigma}{dp}\,p_\mu dp &= \sum_{n=1}^{\infty}\frac{1}{(n-1)!}\int dp\,dp_2\cdots dp_n\,\frac{d\sigma}{dp\,dp_2\cdots dp_n}\,p_\mu \\ &= \sum_{n=1}^{\infty}\frac{1}{n!}\int dp_1 dp_2\cdots dp_n\,\frac{d\sigma}{dp_1 dp_2\cdots dp_n}\left(\sum_{i=1}^{n} p_{i\mu}\right) \\ &= P_\mu\sigma\,,\end{aligned} \tag{10}$$

where the manipulations on the right-hand side use the symmetry properties of the exclusive cross section and momentum conservation.

The generalization of Eqs. (9) and (10) to the production of several species of stable hadrons is straightforward, and leads to the sum rules

$$\int \frac{d\sigma}{dp_x}\,dp_x = \langle n_x\rangle\sigma\,, \tag{A}$$

$$\sum_x \int \frac{d\sigma}{dp_x}\,p_{x\mu}dp_x = P_\mu\sigma\,, \tag{B}$$

where p_x is the four-momentum of particle x. Equation (B) expresses the momentum balance in hadron collisions, and provides a normalization of $d\sigma/dp_x$ in terms of the total cross section.

These techniques may be generalized to the two-particle inclusive cross section, which, in a simplified world of one stable species, is defined as follows:

$$\frac{d\sigma}{dp\,dq} = \sum_{n=2}^{\infty}\frac{1}{(n-2)!}\int dp_3\cdots dp_n\,\frac{d\sigma}{dp\,dq\,dp_3\cdots dp_n}\,. \tag{11}$$

Proceeding as in Eqs. (9) and (10), we find the sum rules

$$\int \frac{d\sigma}{dp\,dq}\,dp\,dq = \sum_{n=2}^{\infty} n(n-1)\sigma^{(n)} \equiv \langle n(n-1)\rangle\sigma\,, \tag{12}$$

$$\int p_\mu \frac{d\sigma}{dp\,dq}\,dp = (P-q)_\mu \frac{d\sigma}{dq}\,. \tag{13}$$

In the real world with several kinds of stable hadrons, the general results for the two-particle distribution $d\sigma/dp_x dp_y$ in the reaction $a+b\to x+y$ + (anything) are

$$\int \frac{d\sigma}{dp_x dp_y}\,dp_x dp_y = \langle n_x n_y - n_x\delta_{xy}\rangle\sigma\,, \tag{C}$$

$$\sum_x \int p_{x\mu}\frac{d\sigma}{dp_x dp_y}\,dp_x = (P-p_y)_\mu \frac{d\sigma}{dp_y}\,. \tag{D}$$

Combining (A), (B), and (D), we obtain

$$\sum_{x,y}\int p_{x\mu}\frac{d\sigma}{dp_x dp_y}\,dp_x dp_y = P_\mu\left(\sum_y \langle n_y\rangle - 1\right)\sigma\,, \tag{E}$$

while (B) and (D), taken together, yield the second-moment sum rule

$$\sum_{x,y}\int p_{x\mu}p_{y\nu}\frac{d\sigma}{dp_x dp_y}\,dp_x dp_y + \sum_y \int p_{y\mu}p_{y\nu}\frac{d\sigma}{dp_y}\,dp_y = P_\mu P_\nu\sigma\,. \tag{F}$$

Contracting tensor indices and using the definition $s=(p_a+p_b)^2=P^2$, we get the Lorentz-invariant sum rule

$$\sum_{x,y}\int p_x\cdot p_y\,\frac{d\sigma}{dp_x dp_y}\,dp_x dp_y + \sigma\sum_y m_y^2\langle n_y\rangle = s\sigma\,. \tag{G}$$

Further generalization of these techniques to n-particle inclusive cross sections is possible, and leads to sum rules relating integrals over m-particle distributions, for $m \le n$.

A new class of sum rules can be derived, based on the conservation of any discrete additive quantum number such as charge, baryon number, etc. The conservation law demands that in any exclusive process initiated by particles a and b,

$$\sum_x Q_x n_x = Q = Q_a + Q_b\,, \tag{14}$$

where n_x is the multiplicity and Q_x is the charge (or baryon number, etc.) of the stable hadronic species x produced in the final state. If we form the integrated inclusive distribution for production of species x, and use Eqs. (14) and (A), the sum rule

$$\sum_x Q_x\langle n_x\rangle = Q \tag{H}$$

follows immediately. It is also straightforward to show that

$$\sum_x Q_x \langle n_x n_y \rangle = Q\langle n_y \rangle \ . \qquad \text{(I)}$$

The sum rules (A)–(I) must be satisfied in all theoretical models for inclusive reactions. Of course, if one constructs inclusive distributions as integrals of exclusive cross sections, they would be satisfied trivially. However, in theoretical approaches which construct 2-2 and 3-3 amplitudes without detailed use of unitarity, the sum rules become nontrivial constraints expressing part of the unitarity requirement. As an example, using Mueller's[2] approach, the asymptotic behavior of the 3-3 amplitude is described by a set of Regge parameters which must be constrained by sum rule (B) if unitarity is to be satisfied. Curiously, sum rule (B) is a consequence of exact unitarity, valid at all energies, which explicitly involves only amplitudes with six or fewer external lines.

The possibility of using inclusive reactions and optical theorems for deriving bootstrap conditions has been proposed recently,[3] with special emphasis on constraining the coupling constant of dual models. The sum rule (B) is indeed an exact constraint which can be used to that purpose.

Because they are based solely on energy conservation, sum rules (B), (D), and (F) might be of some use in checking the normalization or production cross sections in counter experiments; however, their usefulness is limited, since it is necessary to sum over all particle species, including neutral species, which are difficult to detect. Neutral particles often carry off appreciable fractions of the total energy. The sum over cross sections for produced charge species is subject to an inequality constraint, but this would be easily satisfied by data unless large normalization errors were present.

At high energy it is possible to isolate regions of phase space which separately saturate sum rules. From sum rule (B), applied in the c.m. frame, we can deduce approximately

$$\sum_y \int_{x>0} dp_y\, x\, \frac{d\sigma}{dp_y} \approx \sigma \ , \qquad (15)$$

where $p_{y\parallel} = \sqrt{s}\, x/2$ and the integral covers half the phase space. In sum rule (G), the dominant contribution comes from the first term on the left side, integrated only in the double-fragmentation region where $p_x \cdot p_y \sim s$.

One can also use sum rule (B) to deduce, without supplemental counting arguments, that the triple Pomeranchukon vertex must vanish[4] at zero momentum transfer in any theory with a Pomeranchukon Regge pole at $\alpha_V(0) = 1$.

It should be pointed out that sum rule (B) for the energy component, in conjunction with the Froissart bound[5] for σ, gives the inequality

$$\int dp_x (E_x/E_T) \frac{d\sigma}{dp_x} < c(\ln s)^2 \ . \qquad (16)$$

Further, by restriction to the integration region $(E_x/E_T) > f$ where f is a positive fraction and E_T is the total energy, one obtains

$$\int_{E_x > fE_T} dp_x \frac{d\sigma}{dp_x} < c f^{-1}(\ln s)^2 \ . \qquad (17)$$

These trivial but rigorous inequalities[6] are of interest only because integrals of the inclusive cross section are not, *a priori*, bounded from above by σ.

The techniques of this paper can be applied to discontinuities of arbitrary S-matrix elements, or to weak processes such as deep-inelastic electroproduction, lepton-pair annihilation, and others. With normalization specified in Eq. (5) and for a world with one stable hadron species, we write the discontinuity[7] in the variable P^2 in the $k \to j$ scattering amplitude as

$$\text{Disc}\langle p_1' \cdots p_k' | T | p_1 \cdots p_j \rangle = \sum_{n=1}^{\infty} \frac{1}{n!} \int dq_1 \cdots dq_n \langle p_1' \cdots p_k' | T^\dagger | q_1 \cdots q_n \rangle \langle q_1 \cdots q_n | T | p_1 \cdots p_j \rangle (2\pi)^4 \delta^4\Big(P - \sum_{i=1}^{n} q_i \Big) \qquad (18)$$

and, in the variable $(P-q)^2$ in the $(k+1) \to (j+1)$ amplitude, as

$$\text{Disc}\langle p_1' \cdots p_k', q | T | p_1 \cdots p_j, q \rangle = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int dq_2 \cdots dq_n \langle p_1' \cdots p_k' | T^\dagger | q q_2 \cdots q_n \rangle \langle q q_2 \cdots q_n | T | p_1 \cdots p_j \rangle (2\pi)^4 \delta^4\Big(P - q - \sum_{i=1}^{n} q_i \Big) . \qquad (19)$$

Then, using momentum conservation and identical-particle symmetry, we find

$$\int dq\, q_\mu \mathrm{Disc}\langle p_1' \cdots p_k', q|T|p_1 \cdots p_j, q\rangle = P_\mu \mathrm{Disc}\langle p_1' \cdots p_k'|T|p_1 \cdots p_j\rangle\,, \tag{20}$$

where $P_\mu = \sum_{i=1}^k p'_{i\mu} = \sum_{i=1}^j p_{j\mu}$. In the real world, the left-hand side of Eq. (20) would include a sum over stable hadron species.

Finally, we write the tensors which describe the electroproduction processes $e+a \to e+$(anything) and $e+a\to e+x+$(anything) in a one-species world as

$$W_{\mu\nu}(p,q) = \sum_{n=1}^{\infty} \frac{1}{n!} \int dq_1 \cdots dq_n \langle p|J_\mu(0)|q_1 \cdots q_n\rangle \langle q_1 \cdots q_n|J_\nu(0)|p\rangle (2\pi)^4 \delta^4\Big(p+q-\sum_{i=1}^{n} q_i\Big)\,, \tag{21}$$

and

$$\overline{W}_{\mu\nu}(p,q,k) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int dq_2 \cdots dq_n \langle p|J_\mu(0)|k, q_2 \cdots q_n\rangle \langle k, q_2 \cdots q_n|J_\nu(0)|p\rangle (2\pi)^4 \delta^4\Big(p+q-k-\sum_{i=2}^{n} q_i\Big)\,, \tag{22}$$

where p, q, and k are the momenta of incident hadron, incident virtual photon, and detected final hadron, respectively. One then easily derives the identity

$$\sum_x \int dk_x\, k_{x\lambda} \overline{W}_{\mu\nu}(p,q,k_x) = (p+q)_\lambda W_{\mu\nu}(p,q)\,, \tag{23}$$

valid in the real world, where it perhaps can be used to constrain theoretical models for electroproduction based on partons, light-cone expansions, and approximations to field theory.

Note added in proof. An unpublished report describing very similar work has recently come to our attention. It is by Dr. K.-J. Biebl and Dr. J. Wolf of the Institut für Hochenergiephysik, Deutsche Akademie der Wissenschaften zu Berlin, Berlin-Zeuthen, DDR.

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[6]Sharper results can be derived from sum rule (A) in conjunction with the counting argument that there are at most $n-1$ particles produced with $E_x > n^{-1}E_T$. One then obtains Eq. (17) with $f=n^{-1}$ on the left, and f^{-1} replaced by $n-1$ on the right.

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