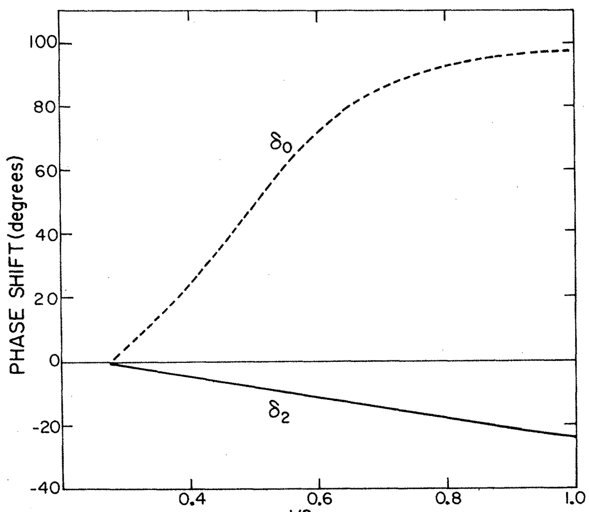

FIG. 2. Model phase shifts for the $T=0$ and $T=2$ S waves.

relations are sensitive to energies higher than those for which our parametrization can be expected to remain good.

This dependence on high-energy contributions means that the model we have presented cannot provide an accurate dynamical description of low-energy pion scattering. However, the amplitudes which appear from the model do have all of the qualitative features found in low-energy pion scattering, and we feel that this is evidence for the validity of the basic idea, that of a strictly low-energy bootstrap which is insensitive to the pion mass and which is driven by the current-algebra slope at zero energy.

To illustrate the nature of the amplitudes in our model, we present in Fig. 2 the $T=0$ and 2 S-wave phase shifts that correspond to the solution with the inclusion of the f^0 contribution. We now take account of the pion mass by computing the phase from that of Eq. (2) and use the values $M_0^2 = (0.755)^2$, $M_2^2 = -(0.685)^2$ that correspond to the shift $s=\nu+2m_\pi^2$. We have not included a graph of our P-wave phase shift since it was given in our previous paper[1] and agrees very well with experiment if the physical ρ mass is used.

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New Improved Bounds for K_{l3} Parameters*

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New exact bounds for derivatives of K_{l3} decay form factors $f_\pm(t)$ are obtained, provided that $\Delta(0)$, the propagator of the divergence of the strangeness-changing current at zero momentum, is known. Several estimates of $\Delta(0)$ are discussed, along with their experimental implications.

I. EXACT INEQUALITIES

Recently, some exact upper bounds for K_{l3} decay parameters have been obtained by several authors,[1-3] if $\Delta(0)$, the propagator of the divergence of the strangeness-changing current at zero momentum, is known. These bounds give rather stringent conditions on the decay parameters if we estimate $\Delta(0)$ from the $SW(3)$ model of Gell-Mann, Oakes, and Renner[4] and of Glashow and Weinberg.[5]

The purpose of this note is twofold. First, we will obtain new improved bounds for these quantities. Secondly, we will attempt to calculate $\Delta(0)$ in as model independent a way as possible. As we shall see in Sec. II, the resulting inequalities are stringent enough to test the chiral $SW(3)$ theory. In this section, we will state some simple mathematical theorems which will be useful in our analysis.

Let $F(\xi)$ be a real analytic function of a complex variable ξ with a cut on the real axis $t_0 \le \xi \le \infty$. The reality condition implies

$$F^*(\xi^*) = F(\xi) \tag{1}$$

in this cut plane. Because of this, $F(\xi)$ is real for real values of ξ less than t_0. Suppose now that the following integral over the cut satisfies an inequality

$$\frac{1}{\pi}\int_{t_0}^{\infty} dt\, w(t)|F(t)|^2 \le I, \tag{2}$$

where $w(t)$ is a given non-negative function of t. Then, we can ask whether we can obtain some bounds on $F(t)$, $F'(t)$, and $F''(t)$ for values of t less than t_0. The answer is affirmative, and we find the following inequalities for $t < t_0$:

$$|F(t)|^2 + |4(t_0 - t)F'(t) - BF(t)|^2 \le I^2A^2, \tag{3}$$

$$4[1+(B+2)^2]|F(t)|^2 + |16(t-t_0)^2F''(t) + DF(t)|^2 \le 4I^2A^2[1+(B+2)^2], \tag{4}$$

where A, B, and D are determined as functionals of $w(t)$ in the Appendix. For many physical applications, it is sufficient to consider a simple form for $w(t)$, given by

$$w(t) = N\prod_{j=1}^{n}[t - t_j]^{\alpha_j} \quad (t_j \le t_0), \tag{5}$$

where N, t_j, and α_j are real constants satisfying $N \ge 0$ and $t_0 \ge t_j$. In that case, A, B, and D are computed to be

$$A = [4N(t_0 - t)]^{-1/2}\prod_{j=1}^{n}(t_0 - t)^{-\alpha_j/2}(1+\beta_j)^{-\alpha_j}, \tag{6a}$$

$$B = 2\sum_{j=1}^{n}\frac{\alpha_j}{1+\beta_j} + 2, \tag{6b}$$

$$D = 4\sum_{j=1}^{n}\alpha_j\beta_j(1+\beta_j)^{-2} + 1 - B^2 - 4B, \tag{6c}$$

where β_j is defined by

$$\beta_j = (t_0 - t_j)^{1/2}(t_0 - t)^{-1/2}. \tag{7}$$

We remark that Eq. (3) immediately leads to the special case

$$|F(t)| \le IA \quad (t < t_0). \tag{8}$$

This formula, together with the explicit form Eq. (6a), reproduces exactly the result given by Drell, Finn, and Hearn,[6] who derived it from Meiman's method.[7] Equation (3) also contains results of Refs. 1 and 3 as special cases. To see this, we take the minimum of the left-hand side of Eq. (3) with respect to $F(t)$. Since it is a simple quadratic form in $F(t)$, the solution is easily found to be

$$4(t_0 - t)|F'(t)| \le AI(1+B^2)^{1/2} \quad (t < t_0). \tag{9}$$

This is exactly the same as the result obtained in I. Analogously, taking the minimum of Eq. (4) with respect to $F(t)$, we find

$$16(t_0 - t)^2|F''(t)| \le 2AI[1+(B+2)^2+\tfrac{1}{4}D^2]^{1/2} \quad (t < t_0). \tag{10}$$

We can easily check that this inequality can also be obtained by the method of I.

To apply these theorems to the K_{l3} problem, we consider the propagator

$$\Delta(-q^2) = -\tfrac{1}{2}i\int d^4x\, e^{iqx}\langle 0|(\partial_\mu V_\mu^{(4-i5)}(x),\ \partial_\nu V_\nu^{(4+i5)}(0))_+|0\rangle, \tag{11}$$

where $V_\mu^{(4-i5)}(x)$ is the strangeness-changing vector current responsible for $K^+ \to \pi^0 l\bar{\nu}$ decays.

Defining the spectral weight $\rho(m^2)$ by

$$\rho(m^2) = \tfrac{1}{2}(2\pi)^3\sum_n |\langle 0|\partial_\mu V_\mu^{(4-i5)}(0)|n\rangle|^2\delta^{(4)}(p_n - k) \quad (k^2 = -m^2), \tag{12}$$

we may express $\Delta(-q^2)$ in Lehmann-Källén form:

$$\Delta(-q^2) = \int_{t_0}^{\infty} dt\,\frac{\rho(t)}{t+q^2}, \tag{13}$$

provided that the integral converges.

Now, as Li and Pagels[1] noted, the positivity of $\rho(t)$ implies

$$\rho(t) \ge \frac{3}{64\pi^2}t^{-1}(t - t_0)^{1/2}(t - t_1)^{1/2}|D(t)|^2 \quad (t > t_0), \tag{14}$$

where t_0 and t_1 are given by

$$t_0 = (m_K + m_\pi)^2, \quad t_1 = (m_K - m_\pi)^2. \tag{15}$$

Also, in Eq. (14) $D(t)$ is expressed as

$$D(t) = (m_K^2 - m_\pi^2)f_+(t) + tf_-(t), \tag{16}$$

where the K_{l3} decay form factors $f_\pm(t)$ are defined by

$$\langle \pi^0(p')|V_\mu^{(4-i5)}(0)|K^+(p)\rangle = -(\tfrac{1}{2})^{1/2}(4p_0p_0'V^2)^{-1/2}[(p_\mu+p_\mu')f_+(t)+(p_\mu-p_\mu')f_-(t)]\,, \tag{17}$$

with $t=-(p-p')^2$. The physical value of t is restricted to a region below the cut at t_0: $m_l{}^2 \leq t \leq t_1$, m_l being the lepton mass. Notice that $D(t)$ is a real analytic function of t with a cut on the real axis ($t_0 \leq t \leq \infty$) since $f_\pm(t)$ satisfies the same condition.

Let us now identify $F(t)$ in our theorem with $D(t)$ and choose $w(t)$ to be

$$w(t)=\frac{3}{64\pi^2}\,t^{-(n+1)}(t-t_0)^{1/2}(t-t_1)^{1/2}\,, \tag{18}$$

where n is an arbitrary positive number. Setting

$$\Delta_n=\int_{t_0}^{\infty} dt\, t^{-n}\rho(t)\,, \tag{19}$$

our inequalities then imply for $t<t_0$

$$|D(t)|\leq K\,, \tag{20a}$$

$$\left|\frac{4(t_0-t)}{D(t)}D'(t)+(2n-2)-(2n+2)\frac{(t_0)^{1/2}}{(t_0)^{1/2}+(t_0-t)^{1/2}}+\frac{(t_0-t_1)^{1/2}}{(t_0-t_1)^{1/2}+(t_0-t)^{1/2}}\right| \leq \left[\left(\frac{K}{D(t)}\right)^2-1\right]^{1/2}, \tag{20b}$$

where $K=AI$ is simply given by

$$K=4[\tfrac{1}{3}\pi\Delta_n]^{1/2}(t_0-t)^{(n-1)/2}\left[1+\left(\frac{t_0}{t_0-t}\right)^{1/2}\right]^{n+1}\left[1+\left(\frac{t_0-t_1}{t_0-t}\right)^{1/2}\right]^{-1/2}. \tag{21}$$

If Δ_n is calculable, then these inequalities provide upper bounds for the physically observable quantities $D(t)$ and $D'(t)$ ($t<t_0$). Conversely, if $D(t)$ and $D'(t)$ ($t<t_0$) are experimentally known, then Eqs. (20) impose a lower bound on Δ_n which must be satisfied in any theoretical model. In this note we adopt the former view. In particular, we will consider only the case in which we set $n=1$ and $t=0$. Since Eqs. (13) and (19) imply $\Delta_1=\Delta(0)$, we find that Eqs. (20) lead to

$$|f_+(0)|\leq M\,, \tag{22a}$$

$$\left|4t_0(m_K{}^2-m_\pi{}^2)^{-1}\left(\frac{m_K{}^2-m_\pi{}^2}{m_\pi{}^2}\lambda_+ +\xi\right)-2+\frac{(4m_Km_\pi)^{1/2}}{m_K+m_\pi+(4m_Km_\pi)^{1/2}}\right| \leq \left[\left(\frac{M}{f_+(0)}\right)^2-1\right]^{1/2}, \tag{22b}$$

$$M=16[\tfrac{1}{3}\pi\Delta(0)]^{1/2}(m_K{}^2-m_\pi{}^2)^{-1}\left[1+\frac{(4m_Km_\pi)^{1/2}}{m_K+m_\pi}\right]^{-1/2}, \tag{22c}$$

where $\lambda_\pm$ and ξ are defined as usual by

$$\lambda_\pm=m_\pi{}^2f_\pm'(0)/f_\pm(0)\,,\quad \xi=f_-(0)/f_+(0)\,, \tag{23}$$

where the prime denotes a derivative. Also, we can derive an inequality involving $f_+''(0)$, but we will not do so here.

As we will see in Sec. II, we can estimate $\Delta(0)$ in several ways and thus make these inequalities experimentally testable.

II. EVALUATION OF $\Delta(0)$

First, integrating Eq. (11) by parts, we find that $\Delta(0)$ can be rewritten as an equal-time commutator;

$$\Delta(0)=\tfrac{1}{2}\int d^4x\langle 0|\delta(x_0-y_0)[V_4^{(4-i5)}(x),\partial_\nu V_\nu^{(4+i5)}(y)]|0\rangle\,. \tag{24}$$

Hence, we can calculate $\Delta(0)$ for any theory which gives some information on this quantity. One notable example is the chiral model[4,5] of Gell-Mann, Oakes, and Renner and of Glashow and Weinberg, in which the Hamiltonian density is specified as

$$H(x)=H_0(x)+\epsilon_0S_0(x)+\epsilon_8S_8(x)\,. \tag{25}$$

In the above, $H_0(x)$ is the $SW(3)$-invariant part, while $S_\alpha(x)$ ($\alpha=0,1,\ldots,8$) represents the scalar portion of the $(3,3^*)\oplus(3^*,3)$ representation of the $SW(3)$ group. It is convenient to set

$$\xi_\alpha=\langle 0|S_\alpha(0)|0\rangle \quad (\alpha=0,8) \tag{26}$$

and to define

$$a=(\tfrac{1}{2})^{1/2}\epsilon_8/\epsilon_0,\quad b=(\tfrac{1}{2})^{1/2}\xi_8/\xi_0,\quad \gamma=-\tfrac{2}{3}\epsilon_0\xi_0\,. \tag{27}$$

Moreover, consider the integral

$$A_{\alpha\beta}=-i\int d^4x\langle 0|(\partial_\mu A_\mu^{(\alpha)}(x),\partial_\nu A_\nu^{(\beta)}(0))_+|0\rangle \tag{28a}$$

for α, $\beta=0,1,2,\ldots,8$. Then, as has been noted elsewhere,[8] we obtain

$$\begin{aligned}&A_{33}=\gamma(1+a)(1+b)\,,\\&A_{44}=\gamma(1-\tfrac{1}{2}a)(1-\tfrac{1}{2}b)\,,\\&A_{88}=\gamma(1-a-b+3ab)\,,\\&V_{44}=\tfrac{9}{4}\gamma ab\,,\end{aligned}\tag{28b}$$

where we have set

$$\Delta(0)\equiv V_{44}\,, \tag{29}$$

in conformity with the notation of Eq. (28). The positivity of the Hilbert space implies that we have

$$A_{33}\geq 0,\quad A_{44}\geq 0,\quad A_{88}\geq 0,\quad V_{44}\geq 0\,. \tag{30}$$

The resulting restrictions on the values of a, b, and γ have been analyzed in Ref. 8.

Before going into details, we simply remark that ξ_8 and hence b are likely to be small numbers if the vacuum is nearly $SU(3)$-invariant, as is customarily assumed. Also, ordinarily the smallness of the pion mass is interpreted as an indication that the Hamiltonian is nearly $SW(2)$-invariant. The latter assumption implies that the parameter a must be approximately -1. Indeed, on the basis of such a philosophy, we made an estimate[8]

$$\begin{aligned}&a=-0.89,\quad b=-0.10,\\&\gamma=5.05m_\pi^2f_\pi^2,\quad f_K/f_\pi=1.08\,,\end{aligned}\tag{31}$$

which gives

$$[\Delta(0)]^{1/2}\approx 1.01m_\pi f_\pi\,. \tag{32}$$

If this value of $\Delta(0)$ is used, then Eq. (22a) leads to

$$|f_+(0)|\leq 1.01\,,$$

as has been already noted.[2,3] This is reasonable in view of the well-known Ademollo-Gatto theorem which states

$$f_+(0)=1+O(\epsilon_8^2),\quad f_-(0)=O(\epsilon_8)\,. \tag{33}$$

Also, this bound on $f_+(0)$ is consistent with the estimate $f_K/f_\pi=1.08$, given in Eq. (31), since experimentally we know that

$$f_K/f_\pi f_+(0)\sim 1.28\,. \tag{34}$$

Indeed, if we accept both Eqs. (31) and (34), we discover that $f_+(0)\sim 0.85$.

Similarly, the standard soft-pion theorem is consistent with our inequality, as has been noted in I.

Next, let us consider Eq. (22b). With $f_+(0)=0.85$, we calculate

$$|D'(0)-0.18|\leq 0.07 \tag{35}$$

or equivalently

$$0.12\leq\xi+12.3\lambda_+\leq 0.30\,. \tag{36}$$

Although the upper bound in Eq. (36) is essentially the same as the one obtained earlier,[1,3] the more interesting lower bound is greatly improved. For values of λ_+ equal to 0.02, 0.03, and 0.06, this gives

$$\begin{aligned}&-0.12\leq\xi\leq 0.06\quad(\lambda_+=0.02)\,,\\&-0.24\leq\xi\leq-0.06\quad(\lambda_+=0.03)\,,\\&-0.61\leq\xi\leq-0.43\quad(\lambda_+=0.06)\,.\end{aligned}\tag{37}$$

It is interesting to remark that Eq. (35) suggests $D'(0)\approx 0.18$. This fact is in rough accord with the Dashen-Weinstein sum rule[9]

$$D'(0)=\tfrac{1}{2}[f_K/f_\pi)-(f_\pi/f_K)]+O(\epsilon_0^2)\,. \tag{38}$$

Analogously, we can estimate an upper bound for the second-order derivative $D''(0)$ by means of Eq. (4); we obtain

$$0.006\leq m_\pi^2D''(0)\leq 0.018\,. \tag{39}$$

Hence, if $f_+''(0)=0$, as is commonly assumed, then we must have

$$0.003\leq\lambda_-\xi\leq 0.011\,.$$

Our result mentioned above depends critically on the value of $\Delta(0)$ calculated in Eq. (32). Hence, it will be interesting to see whether we can find other ways of estimating $\Delta(0)$. First, we notice that the estimate in Eq. (32) was obtained from the mass formulas for m_π^2, m_K^2, and m_η^2, derived in Ref. 8. If we give up the mass formula for m_η^2 and, instead, use κ (kappa) dominance for V_{44} and assume the validity of asymptotic $SW(2)$ symmetry, we can compute[10]

$$a=-0.89,\quad b=-0.15,\quad \gamma=5.3m_\pi^2f_\pi^2,\quad f_K/f_\pi=1.13\,, \tag{40}$$

which gives

$$[\Delta(0)]^{1/2}=1.26m_\pi f_\pi,\quad f_+(0)=0.88\,. \tag{41}$$

If we use this new value, the previous inequalities change only slightly.

However, a far more troublesome question concerns the validity of the estimate $a\approx-0.89$. Recently Cheng and Dashen[11] reevaluated the old calculation of von Hippel and Kim and reached the conclusion that, in contrast to the currently accepted idea, $SW(2)$ is perhaps not such a good symmetry. If this conclusion is accepted, then either the chiral $SW(3)$ theory is incorrect or the value of a must be near the $SU(3)$ point $a=0$ rather than the $SW(2)$ point $a=-1$. In the first instance, we have to abandon entirely our estimate of $\Delta(0)$, and we must find some other means to calculate $\Delta(0)$. We shall come back to this problem at the end of this paper. Given the second alternative, we can no longer use our estimate Eq. (32). However, within the framework of our Hamiltonian Eq. (25), we can make a rather model-independent estimate of the upper bounds of $\Delta(0)$ as follows. From Eq. (28), it is possible to obtain an identity:

$$\{\Delta(0)-[(A_{33})^{1/2}+(A_{44})^{1/2}]^2\}\times\{\Delta(0)-[(A_{44})^{1/2}-(A_{33})^{1/2}]^2\}=\tfrac{9}{4}\gamma^2(a-b)^2. \quad (42)$$

Since the right-hand side of Eq. (42) is non-negative, we must have either

$$[\Delta(0)]^{1/2}\le|(A_{44})^{1/2}-(A_{33})^{1/2}| \quad (43)$$

or

$$[\Delta(0)]^{1/2}\ge(A_{44})^{1/2}+(A_{33})^{1/2}. \quad (44)$$

We note in passing that, if we saturate A_{33}, A_{44}, and V_{44} with only pion, kaon, and κ intermediate states, respectively, then we find

$$A_{33}=\tfrac{1}{2}m_\pi^2 f_\pi^2,\quad A_{44}=\tfrac{1}{2}m_K^2 f_K^2, \quad (45)$$

as well as

$$V_{44}=\tfrac{1}{2}m_\kappa^2 f_\kappa^2. \quad (46)$$

In this approximation, Eqs. (43) and (44) exactly reproduce inequalities found by Glashow and Weinberg.[5] Also, in the terminology of Ref. 8, the validity of Eq. (43) is restricted to the domains (II)-(III) and (IV) while the inequality Eq. (44) is satisfied in the remaining regions (I), (V), (VI), and (VII).

At any rate, we reject Eq. (44) since it is inconsistent with the ordinary $SU(3)$ limit.[12] Then, we can find an upper limit on $\Delta(0)$ if the right-hand side of Eq. (43) is known. We can use the pole-dominance approximation for A_{33} and A_{44} [Eq. (45)] to obtain

$$[\Delta(0)]^{1/2}\le(\tfrac{1}{2})^{1/2}(m_K f_K-m_\pi f_\pi). \quad (47)$$

It should be emphasized that in deriving this inequality we need not assume κ dominance [Eq. (46)] for V_{44}. Equation (47) enables us to compute

$$\begin{aligned}&[\Delta(0)]^{1/2}\le 2.11\,m_\pi f_\pi \text{ for } f_+(0)=0.85\\&[\Delta(0)]^{1/2}\le 2.27\,m_\pi f_\pi \text{ for } f_+(0)=0.90.\end{aligned} \quad (48)$$

Unfortunately, these values are a bit large (by a factor of 2) in comparison to the value in Eq. (32). Consequently, the bounds for $D'(0)$ and $D''(0)$ take the slightly weakened form

$$\begin{aligned}&-0.10\le\xi+12.3\lambda_+\le0.53,\\&-0.007\le m_\pi^2 D''(0)\le0.033\end{aligned} \quad (49)$$

in practice for both cases.

Nevertheless, this gives

$$\begin{aligned}&-0.35\le\xi\le0.29\quad(\lambda_+=0.02),\\&-0.49\le\xi\le0.15\quad(\lambda_+=0.03),\\&-0.83\le\xi\le-0.20\quad(\lambda_+=0.06),\end{aligned} \quad (50)$$

which is still useful. So far, the experimental determination of ξ and λ_+ appears to be subject to large uncertainties.[13] We may remark that our bounds, Eq. (47) and hence Eq. (49), are perhaps optimal within the framework of the conventional theory. For example, using κ dominance for V_{44}, Glashow and Weinberg[5] derive a relation

$$f_K/f_\pi f_+(0)=2f_\kappa^2[f_\pi^2+f_K^2-f_\kappa^2]^{-1}=1.28.$$

On the basis of this formula, Weinberg suggests[14]

$$|f_\kappa/f_\pi|\approx0.58,\quad m_\kappa\le670\text{ MeV},$$

which leads to

$$[\Delta(0)]^{1/2}\le1.96\,m_\pi f_\pi. \quad (51)$$

On the other hand, Brandt and Preparata,[15] who advocate a small value for a, find

$$a=-0.2,\quad f_K/f_\pi=1.20.$$

However, unfortunately, the value of b is not explicitly given in their paper. Nevertheless, if we use the pion- and kaon-dominance approximation, Eq. (45), then we compute

$$b=-0.89,\quad[\Delta(0)]^{1/2}\approx1.50\,m_\pi f_\pi. \quad (52)$$

Both estimates, Eqs. (51) and (52), are still better than Eq. (48). Notice that the roles of a and b are practically interchanged in the latter case.

So far all calculations have been performed within the framework of the chiral $SW(3)$ Hamiltonian model, Eq. (25). We can make some estimates of $\Delta(0)$, independent of any specific form assumed for the Hamiltonian. For example, if we assume the validity of asymptotic $SU(6)_W$ symmetry, we find[16]

$$[\Delta(0)]^{1/2}\simeq1.02\,m_\pi f_\pi,\quad f_K/f_\pi=1.07, \quad (53)$$

where we assumed the κ mass $m_\kappa\approx1100$ MeV. This value is extremely near the first estimate, Eq. (32). Similarly, on the basis of asymptotic $SW(3)$ symmetry, Matsuda and Oneda[17] derive a formula

$$f_\kappa G_{\kappa^0K^-\pi^+}=[(f_K/f_\pi)-1](m_\kappa^2-m_K^2).$$

Assuming $m_\kappa=1100$ MeV, $\Gamma(\kappa\to K\pi)=300$ MeV, and $f_K/f_\pi=1.20$, we compute

$$[\Delta(0)]^{1/2}=1.60\,m_\pi f_\pi \quad (54)$$

from this formula.

All these results suggest that $[\Delta(0)]^{1/2}$ may be a bit larger than the value in Eq. (32) but perhaps smaller than the number in Eq. (48). If we use the median value, $[\Delta(0)]^{1/2}=1.50\,m_\pi f_\pi$ and $f_+(0)\doteq0.90$, then our inequalities become

$$|f_+(0)| \leqslant 1.50\,,$$
$$0.03 \leqslant \xi + 12.3\lambda_+ \leqslant 0.40\,, \tag{55}$$
$$-0.002 \leqslant m_\pi{}^2 D''(0) \leqslant 0.029\,.$$

Finally, as another application of our method, let us consider the electromagnetic form factor of the pion.

Defining

$$-i\int d^4x\, e^{iqx}\langle 0|(j_\mu(x), j_\nu(0))_+|0\rangle = \int_{t_0}^\infty dt\, \frac{\rho(t)}{t+q^2}\left(\delta_{\mu\nu} - \frac{1}{t}q_\mu q_\nu\right), \tag{56}$$

where $j_\mu(x)$ is the hadronic electromagnetic current, and $t_0 = 4m_\pi{}^2$, we know that[3,6]

$$\rho(t) \geqslant \frac{1}{48\pi^2}\, t^{-1/2}(t-t_0)^{3/2}|F_\pi(t)|^2 \quad (t \geqslant t_0)\,. \tag{57}$$

$F_\pi(t)$ is the pion electromagnetic form factor normalized to $F_\pi(0) = 1$. Then, as noted in I, we find for $t < t_0$

$$|F_\pi(t)| \leqslant M \equiv 2\left[3\pi\int_{t_0}^\infty dt\, t^{-n}\rho(t)\right]^{1/2}(t_0-t)^{(n-1)/2}\left[1+\left(\frac{t_0}{t_0-t}\right)^{1/2}\right]^{n+1/2}, \tag{58}$$

where n is a non-negative number. We find also the following inequality for the derivative:

$$\left|\frac{4(t_0-t)F'_\pi(t)}{F_\pi(t)} + (2n+1)\frac{(t_0-t)^{1/2}}{(t_0)^{1/2}+(t_0-t)^{1/2}} - 5\right| \leqslant \left[\left(\frac{M}{F_\pi(t)}\right)^2 - 1\right]^{1/2} \quad (t<t_0)\,. \tag{59}$$

This is a generalization of the inequality given by Cooper and Pagels.[18] An especially interesting case is obtained when we set $t=0$. Also, we note that it is likely that both $\int_{t_0}^\infty dt\, t^{-1}\rho(t)$ and $\int_{t_0}^\infty dt\, t^{-2}\times\rho(t)$ are divergent; on the other hand $\int_{t_0}^\infty dt\, t^{-3}\rho(t)$ should be finite, as has been emphasized by Drell, Finn, and Hearn.[6] Therefore, we set $n=3$. Also, we evaluate $\int_{t_0}^\infty dt\, t^{-3}\rho(t)$ by the ρ dominance. When we set

$$\langle 0|j_\mu(0)|\rho(k)\rangle = (2k_0V)^{-1/2}\epsilon_\mu(k)G_\rho\,, \tag{60}$$

the present experimental data for the decay rate of $\rho^0 \to e\bar{e}$ suggest the approximate validity[19] of the familiar Kawarabayashi-Suzuki-Riazuddin-Fayazuddin relation $G_\rho = m_\rho f_\pi$, which we shall use here. Then we find that Eqs. (58) and (59) give the numerical result

$$|F_\pi(0)| \leqslant 9.86\,,$$
$$-5.0 \leqslant m_\rho{}^2F_\pi{}'(0) \leqslant 11.0\,. \tag{61}$$

The first relation is trivially satisfied since $F_\pi(0) = 1$. The second relation gives

$$-5.0 \leqslant \langle r_\pi\rangle^2/\langle r_\rho\rangle^2 \leqslant 11.0\,, \tag{62}$$

where $\langle r_\pi\rangle$ is the actual electromagnetic radius of the pion and $\langle r_\rho\rangle$ is that radius computed by means of the ρ-dominance model for $F_\pi(t)$. If we demand $\langle r_\pi\rangle^2 \geqslant 0$, then Eq. (62) gives an upper bound for the radius which is slightly larger than the ρ-dominance model. We may remark that we could have used an experimentally measurable cross section for electron-positron annihilations in the evaluation of $\int_{t_0}^\infty dt\, t^{-3}\rho(t)$. However, in view of the large cross section for ρ production, our approximation will be sufficient. At any rate, it should be emphasized that our use of the ρ dominance is only for two-point functions but not for three-point functions. Perhaps the approximation is better for the former, although it could be worse for the latter.

However, even for the former, the ρ dominance is incompatible with Eqs. (58) and (59) for an arbitrary large value of n. This is understandable since, as $n\to\infty$, the low-mass region will be more important in the evaluation of the integral $\int_{t_0}^\infty dt\, t^{-n}\times\rho(t)$. This is the reason why we adopted $n \doteq 3$ as the possible lowest value, where the integral may converge.

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APPENDIX

Here we shall prove inequalities used in the text. Although this can be done by the method used in I, we present a simpler proof.

Let $h(z)$ be analytic inside the unit circle, and, moreover, suppose that $h(z)$ belongs to the class H^2, i.e., it satisfies

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta|h(\gamma e^{i\theta})|^2 \leqslant M < \infty$$

for all $0 \leqslant \gamma < 1$. Then, we must have the inequalities

$$|h(0)|^2 \leqslant I^2, \tag{A1}$$

$$|h(0)|^2 + |h'(0)|^2 \leqslant I^2, \tag{A2}$$

$$|h(0)|^2 + |h'(0)|^2 + \tfrac{1}{4}|h''(0)|^2 \leqslant I^2, \tag{A3}$$

where I is defined by

$$I = \left[\frac{1}{2\pi}\int_0^{2\pi} d\theta |h(e^{i\theta})|^2\right]^{1/2}. \tag{A4}$$

Of course, the validity of Eq. (A3) automatically implies the validity of Eqs. (A1) and (A2), but we wrote the three expressions separately for convenience. The proof of these equations is simple. Suppose first that $h(z)$ is a polynomial in z, i.e.,

$$h(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n.$$

Then, an elementary calculation gives

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta |h(e^{i\theta})|^2 = |a_0|^2 + |a_1|^2 + \cdots + |a_n|^2,$$

with

$$a_n = \frac{1}{n!} h^{(n)}(0).$$

Therefore, for this case, the validity of Eqs. (1), (2), and (3) is obvious. But since all polynomials form a dense set in the space of H^2, the inequalities are also valid for all functions $h(z)$ belonging to the class H^2. If we wish, we can give a more stringent inequality involving higher derivatives, but this is not necessary for the present purpose.

Let $W(\theta)$ $(0 \leqslant \theta \leqslant 2\pi)$ be a given non-negative summable function of θ on the unit circle. Moreover, we assume that $\ln W(\theta)$ is also summable. Then a function defined[20] by

$$\varphi(z) = \exp\left[\frac{1}{4\pi}\int_0^{2\pi} d\theta \frac{e^{i\theta}+z}{e^{i\theta}-z} \ln W(\theta)\right] \tag{A5}$$

belongs to H^2, and, in addition, it is outer. Also, it satisfies[20]

$$|\varphi(e^{i\theta})|^2 = W(\theta) \tag{A6}$$

almost everywhere on the unit circle.

Hence if $f(z)$ is analytic for $|z| < 1$ and is such that its boundary value exists almost everywhere[21] and the integral

$$I = \left[\frac{1}{2\pi}\int_0^{2\pi} d\theta W(\theta) |f(e^{i\theta})|^2\right]^{1/2} \tag{A7}$$

is finite, then the product function

$$h(z) = f(z)\varphi(z) \tag{A8}$$

belongs to the class H^2 with the condition

$$I^2 = \frac{1}{2\pi}\int_0^{2\pi} d\theta |h(e^{i\theta})|^2 = \frac{1}{2\pi}\int_0^{2\pi} d\theta W(\theta) |f(e^{i\theta})|^2. \tag{A9}$$

Thus, we can apply[22] our theorem, Eqs. (1), (2), and (3) to obtain

$$|f(0)|^2 \leqslant I^2 A^2, \tag{A10}$$

$$|f(0)|^2 + |f'(0) + Bf(0)|^2 \leqslant I^2 A^2, \tag{A11}$$

$$|f(0)|^2 + |f'(0) + Bf(0)|^2 + \tfrac{1}{4}|f''(0) + 2Bf'(0) + Cf(0)|^2 \leqslant I^2 A^2, \tag{A12}$$

where A, B, and C are defined by

$$A = \exp\left(-\frac{1}{4\pi}\int_0^{2\pi} d\theta \ln W(\theta)\right),$$
$$B = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{-i\theta} \ln W(\theta), \tag{A13}$$
$$C = \frac{1}{\pi}\int_0^{2\pi} d\theta\, e^{-2i\theta} \ln W(\theta) + B^2.$$

If we wish to derive an inequality involving only $f'(0)$, then we have to take a minimum with respect to $f(0)$ on the left-hand side of Eq. (A11). This is easily done since it is a quadratic form in $f(0)$. In this way, we obtain

$$|f'(0)| \leqslant IA[1 + |B|^2]^{1/2}. \tag{A14}$$

More generally, if γ is an arbitrary real number, and if we want to derive a bound for $f'(0) + \gamma f(0)$, then a similar method gives

$$|f'(0) + \gamma f(0)| \leqslant IA[1 + |B - \gamma|^2]^{1/2}. \tag{A15}$$

This result exactly agrees with the inequality obtained in I. Conversely, we can derive Eq. (A11) from Eq. (A15) as follows. We can take, respectively, the minimum and maximum of the upper and lower bounds for $f'(0)$ with respect to γ in Eq. (A15). After a simple calculation, this approach leads to Eq. (A11).

Analogously, we can obtain a bound for the combination $f''(0) + \gamma f'(0)$ by taking a minimum with respect to $f'(0)$ in Eq. (A12):

$$|f''(0) + \gamma f'(0) + Df(0)|^2 + 4[1 + (B - \tfrac{1}{2}\gamma)^2]|f(0)|^2 \leqslant 4I^2A^2[1 + (B - \tfrac{1}{2}\gamma)^2], \tag{A16}$$

$$D = C - 2B(B - \tfrac{1}{2}\gamma)$$
$$= \frac{1}{\pi}\int_0^{2\pi} d\theta\, e^{-2i\theta} \ln W(\theta) - B^2 + \gamma B.$$

In deriving this formula, we assumed for simplicity that B and C are real; this is certainly correct if $W(2\pi - \theta) = W(\theta)$, as in our applications. Finally, taking a minimum with respect to $f(0)$ in Eq. (16), we find

$$|f''(0) + \gamma f'(0)|^2 \leqslant 2IA[1 + (B - \tfrac{1}{2}\gamma)^2 + \tfrac{1}{4}D^2]^{1/2}. \tag{A17}$$

This formula can also be derived by the method of (I).

In deriving these formulas, we have assumed that $W(\theta)$ is summable. In some applications, it happens that $W(\theta)$ may have singularities at $\theta=0$ and $\theta=\pi$. In that case, we set

$$\begin{aligned} f(z)&=(1-z)^{\alpha}(1+z)^{\beta}\tilde{f}(z),\\ \tilde{W}(\theta)&=W(\theta)(2\sin\tfrac{1}{2}\theta)^{2\alpha}(2\cos\tfrac{1}{2}\theta)^{2\beta}\end{aligned} \tag{A18}$$

for some positive numbers α and β. If $\tilde{W}(\theta)$ becomes summable for some choices of α and β, then we use $\tilde{W}(\theta)$ and $\tilde{f}(z)$ instead of $W(\theta)$ and $f(z)$ and take the minimum with respect to the allowable ranges of α and β. However, this procedure does not, in general, affect the final result, as we see from results of I; we will not go into the details of this case.

In real applications, we consider an analytic function $F(t)$, which is analytic everywhere except for a cut on the real axis ($t_0 \le t < \infty$). In that case, we reduce the problem to the previous one with the mapping

$$(t-t_0)^{1/2}=i(t_0-a)^{1/2}(1+z)(1-z)^{-1}, \tag{A19}$$

where a is an arbitrary real point satisfying the condition $a<t_0$. Under this transformation, it is easy to see that the upper and lower cuts in the t plane are mapped, respectively, into lower and upper semicircles of the unit circle, $|z|=1$. Also, the three points $t=\infty$, a, and t_0 are mapped into $z=1$, 0, and -1, respectively. On the boundary, we set $z=e^{i\theta}$; then

$$t=t_0+(t_0-a)\cot^2(\tfrac{1}{2}\theta) \quad (t\ge t_0>a). \tag{A20}$$

For a given non-negative weight function $w(t)$, defined on the cut $t=t_0$, we set

$$W(\theta)=(t-t_0)^{1/2}(t_0-a)^{-1/2}(t-a)w(t). \tag{A21}$$

Similarly, defining $f(z)$ by

$$f(z)\equiv F(t), \tag{A22}$$

we find then

$$\frac{1}{\pi}\int_{t_0}^{\infty} dt\,w(t)|F(t)|^2=\frac{1}{2\pi}\int_0^{2\pi} d\theta W(\theta)|f(e^{i\theta})|^2, \tag{A23}$$

where we assumed that $F(t)$ is real, i.e.,

$$F^*(t^*)=F(t). \tag{A24}$$

Moreover, we note the formulas

$$\begin{aligned} &\int_0^{2\pi} d\theta \ln[1+\lambda^2\tan^2\tfrac{1}{2}\theta]=4\pi\ln(1+\lambda),\\ &\int_0^{2\pi} d\theta\, e^{-i\theta}\ln[1+\lambda^2\tan^2\tfrac{1}{2}\theta]=-4\pi\lambda(1+\lambda)^{-1},\\ &\int_0^{2\pi} d\theta\, e^{-2i\theta}\ln[1+\lambda^2\tan^2\tfrac{1}{2}\theta]=4\pi\lambda(1+\lambda)^{-2}\end{aligned} \tag{A25}$$

for a non-negative number λ. Then, for the special choice

$$w(t)=N\prod_{j=1}^{n}(t-t_j)^{\alpha_j} \quad (t_j\le t_0),$$

we can easily calculate integrals A, B, and D as in the text.

Also, we notice

$$\begin{aligned} f(0)&=F(a),\\ f'(0)&=-4(t_0-a)F'(a),\\ f''(0)-4f'(0)&=16(t_0-a)^2F''(a).\end{aligned} \tag{A26}$$

Hence we choose $\gamma=-4$ in Eqs. (A16) and (A17). Then, our formulas give upper bounds for $F(a)$, $F'(a)$, and $F''(a)$. Replacing a by t, these expressions reproduce the results quoted in the text.

It should be emphasized that our bounds are optimal. Given no further information about $D(t)$, it is easy to show that the bounds can be saturated and hence they are the best bounds within the constraints given.

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$\Delta I = 1$ Mass Differences and $\eta \to 3\pi$ Decay*

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The octet baryon $\Delta I = 1$ mass shifts, kaon electromagnetic mass shift, and $\eta \to 3\pi$ decay amplitude are all correlated in the framework of the linear $SU(3)$ σ model. It is shown how this model provides a natural realization of the "tadpole" idea of Coleman and Glashow. An attempt is made to resolve the long-standing mystery associated with the $\eta \to 3\pi$ process, and the rare mode $\eta' \to 3\pi$ is also calculated. Finally, it is shown that the chiral-symmetry-breaking and isospin-nonconserving coefficients in the Lagrangian have the same order of magnitude.

I. INTRODUCTION

In this paper, we study $\Delta I = 1$ electromagnetic effects in the linear $SU(3)$ σ model of pseudoscalar and scalar mesons. The formalism to be used was developed in several previous papers[1-3] for the purpose of studying chiral-symmetry breaking and $SU(3)$ breaking in this model. However, we are attempting to make the present paper self-contained enough that the reader may get the *general idea* without constantly having to refer back.

We shall limit the mesons in the model to the spin-0 ones to avoid the usual maze of alternatives that present themselves when many additional kinds of particles are treated together with symmetry breaking. It was found that in the isospin limit with the simplest "$[(3, 3^*) + (3^*, 3)]$" type of symmetry-breaking term,[4] the present model gave quite a reasonable amount of the mass spectrum. Here we shall see that the model is able to correlate successfully the baryon electromagnetic mass shifts with the kaon electromagnetic mass shift and the $\eta \to 3\pi$ decay process. Furthermore, the mass of the isovector scalar meson is constrained (from electromagnetic considerations) to be around a physically reasonable value. It turns out, in addition, that the fit to the mass spectrum requires the coefficient in the Lagrangian which violates isospin invariance to be of the same order of magnitude as the chiral-symmetry-breaking coefficient. This might indicate a fundamental connection between electromagnetism and chiral-symmetry breaking.[5]

Before going on to the formalism, we shall briefly discuss the historical background. In the problem of electromagnetic mass splittings and associated second-order processes (such as $\eta \to 3\pi$), it has been recognized that it is necessary to treat the $\Delta I = 1$ and $\Delta I = 2$ effects separately. From a dispersion-theory viewpoint, as pointed out by Harari[6] and others, this amounts to a statement that we can calculate the $\Delta I = 2$ processes by ordinary Feynman diagrams involving photon exchange, since the amplitudes for these processes satisfy *unsubtracted* dispersion relations, implying that high-energy contributions can be safely neglected. On the other hand, the $\Delta I = 1$ processes are said to satisfy *subtracted* dispersion relations so that a knowledge of the subtraction constant (which does not come from the lowest-order Feynman diagram) is required. This general argument is borne out by the fact that the $\Delta I = 2$ π^+-π^0 mass difference has been calculated successfully from essentially second-order perturbation theory by many workers.[7] On the other hand, all the $\Delta I = 1$ mass differences and the $\Delta I = 1$ $\eta \to 3\pi$ processes have not been explained by analogous calculations.

One way to explain the $\Delta I = 1$ mass differences is to use the "tadpole" approach of Coleman and Glashow.[8] By postulating scalar mesons having