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Current Algebra and Analyticity: Bootstrapping the ρ and σ with the Pion Decay Constant Setting the Scale*

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Simple one-parameter forms for the S and P waves of pion scattering that obey current-algebra constraints and allow for a resonance are constructed. The parameters are completely determined by solving self-consistently three subtracted forward dispersion relations in a region up to 1 GeV. Resonances for the ρ and σ are produced with the mass scale set by the pion decay constant, but the "exotic" $T=2$ S wave is nonresonant.

Some time ago we proposed[1] a simple one-parameter extrapolation of Weinberg's current-algebra pion scattering amplitude[2] that is consistent with elastic unitarity. The essential idea is that the satisfaction of current-algebra constraints with the minimal momentum dependence to account for a possible resonant state provides a good approximation for the amplitude over a wide energy range, perhaps up to 1 GeV. The parametric form that we used corresponds to the infinite sum of "bubble" graphs with vertex functions having the lowest possible order of momentum dependence. This form automatically relates the ρ width to its mass and the current-algebra constraints, and gives a result[1] in excellent agreement with experiment. Our result has also been used[3] to explain the shape of the ρ as observed in e^+e^- annihilation. Here we extend the one-parameter form to the $T=0$ and 2 S waves as well as the $T=1$ P wave, and determine all three parameters by solving self-consistently three forward dispersion relations in a region up to 1 GeV. We find that it is the pion decay constant $4\pi F_\pi = 1.2$ GeV that sets the mass scale, not the pion mass. This is in accord with the notion of partial conservation of axial-vector current (PCAC), which asserts that it is a good approximation to neglect the pion mass. It may also have some connection with broken dilation invariance; for if the commutator of the axial charge with its divergences gives an isoscalar field σ, then $F_\pi = \langle\sigma\rangle$ and this isoscalar field may be connected with the trace of the energy-momentum tensor.

Current algebra determines[2] the first two terms of the expansion of the pion amplitude in powers of the squared center-of-mass energy s. The corresponding $T=0$ and 2 S-wave and the $T=1$ P-wave amplitudes are given[1] by

$$\begin{aligned} t_0^{(\mathrm{CA})}(s) &= (16\pi F_\pi^2)^{-1}(2s - m_\pi^2), \\ t_2^{(\mathrm{CA})}(s) &= -(16\pi F_\pi^2)^{-1}(s - 2m_\pi^2), \\ t_1^{(\mathrm{CA})}(s) &= (16\pi F_\pi^2)^{-1} \times \tfrac{1}{3}(s - 4m_\pi^2). \end{aligned} \tag{1}$$

Elastic unitarity is obeyed without spurious singularities in the left half plane if we write the partial-wave amplitude as

$$t_T(s) = t_T^{(\mathrm{CA})}(s)\left\{\frac{M_T^2 - s}{M_T^2 - s_{0T}} + \left[h_1(s) - h_1(|M_T^2|) - \frac{i}{2}\left(\frac{s-4m_\pi^2}{s}\right)^{1/2}\right] t_T^{(\mathrm{CA})}(s)\right\}^{-1}, \tag{2}$$

where

$$h_1(s) = \frac{1}{\pi}\left(\frac{s-4m_\pi^2}{s}\right)^{1/2} \ln\left[\frac{s^{1/2} + (s-4m_\pi^2)^{1/2}}{2m_\pi}\right]. \tag{3}$$

Here we choose s_{0T} to be the position of the zero of the current-algebra amplitude so that they are fitted at these points, and we have arranged the form so that M_T is the resonance mass. We shall neglect the pion mass for the most part in our work and use the very simple form which is the zero-mass limit of (2):

$$(2\pi)^{-1} t_T(s) = C_T s\left[\frac{M_T^2 - s}{M_T^2} + C_T s \ln\left(\frac{-s}{|M_T^2|}\right)\right]^{-1}, \tag{4}$$

where

$$C_T = (4\pi F_\pi)^{-2} \times \begin{cases} 1, & T=0 \\ \frac{1}{6}, & T=1 \\ -\frac{1}{2}, & T=2. \end{cases} \tag{5}$$

It follows from crossing symmetry that the combinations of forward scattering amplitudes T_T,

$$X_0 = T_0 - 2T_1, \quad X_2 = T_1 + T_2, \tag{6a}$$

are even in $\nu = s - 2m_\pi^2$, while

$$X_1 = 2T_0 - 5T_2 + 3T_1 \tag{6b}$$

is odd in ν. Hence we have the subtracted forward dispersion relations which, in the zero pion mass limit, are

$$X_{0,2}(\nu) = X_{0,2}(0) + \frac{2\nu^2}{\pi}\int_0^\infty \frac{d\nu'}{\nu'}\frac{\mathrm{Im}X_{0,2}(\nu')}{\nu'^2 - \nu^2}, \tag{7a}$$

$$X_1(\nu) = \frac{2\nu}{\pi}\int_0^\infty d\nu' \frac{\mathrm{Im}X_1(\nu')}{\nu'^2 - \nu^2}. \tag{7b}$$

Since these dispersion relations converge well at infinity, we assume that the entire forward amplitude can be approximated by the S- and P-wave amplitudes given by Eq. (4): $T_0 = t_0$, $T_2 = t_2$, and $T_1 = 3t_1$ (the factor 3 is $2l+1$). In this way we obtain constraints on the partial-wave amplitudes that convey the conditions of crossing symmetry and analyticity. These constraints are imposed only in the physical region, the region where our parametrization is valid, not in an unphysical region where the analytic continuation of an approximation that is good in the physical region can give a large error.

The dispersion integrals can be evaluated analytically by writing

$$\mathrm{Im}T(\nu') = \frac{1}{2i}[T(\nu' + i\epsilon) - T(\nu' - i\epsilon)],$$

opening up the contour, and using Cauchy's formula. The explicit poles in the integrands produce $T(\nu)$ and $T(-\nu)$. In addition, the amplitudes of Eq. (4) have poles on the negative real axis whose contribution we take into account.

We show in Fig. 1(a) the percentage by which the various partial-wave amplitudes evaluated from the dispersion relations (7) differ from their input values (4) for the best set of input parameters that we could find: $M_0^2 = (0.555)^2$, $M_1^2 = (0.560)^2$, $M_2^2 = -(0.565)^2$ GeV^2. We note that the dispersion relations are well satisfied out to an energy of about 1 GeV. A change of 10% in any combination of the masses makes the agreement worse by about a factor of 2; a change of 20% makes it worse by at least another factor of 2. Thus, the self-consistency requirements give a reasonably stringent determination of the three masses. The phase shifts corresponding to this solution show a narrow ($T=1$, $J=1$) ρ resonance and a broad ($T=0$, $J=0$) σ resonance with both at about the same mass; the small negative ($T=2$, $J=0$) phase requires the absence of "exotic" states. All of these features are consistent with experiment although the values of our masses are too small. In particular, it is noteworthy that the sign of the slope of the current-algebra amplitudes at zero energy is sufficient to determine that there be resonances in the $T=0$ and $T=1$ amplitudes and not in the exotic $T=2$ amplitude. We are intrigued by the near equality of all the masses that appear in the solution, $M_0^2 \simeq M_1^2 \simeq -M_2^2$. Perhaps there is some underlying symmetry which accounts for this.

In Fig. 1(b) we show the solution to a similar calculation in which, in order to estimate the sensitivity of our method to high-energy contributions, we have added to $(2\pi)^{-1}T_0$ a D-wave amplitude

$$5\nu^2\left[\left(\frac{\pi M_f}{2\Gamma_f} + \frac{1}{2}\right)(M_f^4 - \nu^2) + \nu^2 \ln\left(\frac{-\nu}{M_f^2}\right)\right]^{-1}, \tag{8}$$

which has the correct threshold behavior and gives an f^0 resonance of mass M_f and width Γ_f. We use $M_f = 1.2$ GeV and $\Gamma_f/M_f = \frac{1}{8}$. The quality of the solution and its sensitivity to changes in the parameters are not appreciably changed from the previous case. However, the masses are now raised by about 25% with the new values $M_0^2 = (0.730)^2$, $M_1^2 = (0.690)^2$, $M_2^2 = -(0.710)^2$. Thus our

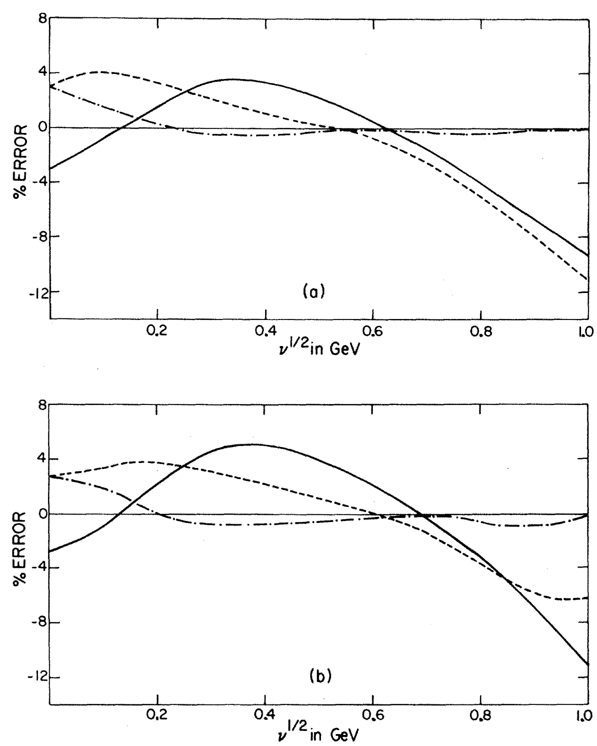

FIG. 1. Percentage error in our solution, $(T_T^{\mathrm{out}} - T_T^{\mathrm{in}})/|T_T^{\mathrm{in}}|$, as a function of the center-of-mass energy $\nu^{1/2}$. The dashed, broken-dashed, and solid lines give the error for the $T=0$, 1, and 2 amplitudes, respectively. The graph in (a) shows the solution for the pure bootstrap including only S and P waves; the graph (b) shows the solution when the f^0 D wave is included with fixed parameters.

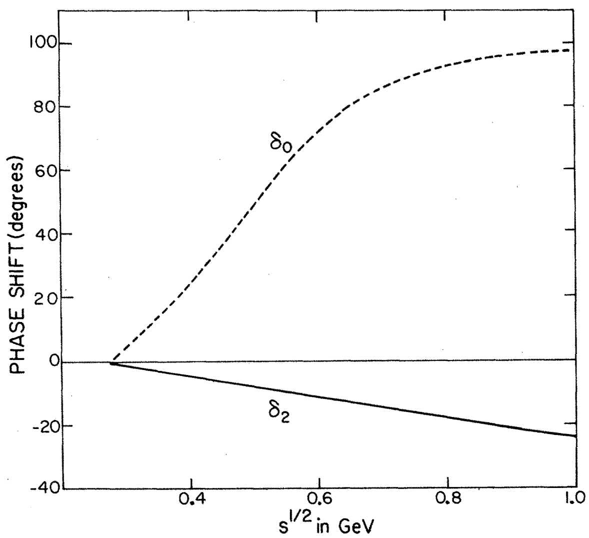

FIG. 2. Model phase shifts for the $T=0$ and $T=2$ S waves.

relations are sensitive to energies higher than those for which our parametrization can be expected to remain good.

This dependence on high-energy contributions means that the model we have presented cannot provide an accurate dynamical description of low-energy pion scattering. However, the amplitudes which appear from the model do have all of the qualitative features found in low-energy pion scattering, and we feel that this is evidence for the validity of the basic idea, that of a strictly low-energy bootstrap which is insensitive to the pion mass and which is driven by the current-algebra slope at zero energy.

To illustrate the nature of the amplitudes in our model, we present in Fig. 2 the $T=0$ and 2 S-wave phase shifts that correspond to the solution with the inclusion of the f^0 contribution. We now take account of the pion mass by computing the phase from that of Eq. (2) and use the values $M_0^2 = (0.755)^2$, $M_2^2 = -(0.685)^2$ that correspond to the shift $s = \nu + 2m_\pi^2$. We have not included a graph of our P-wave phase shift since it was given in our previous paper[1] and agrees very well with experiment if the physical ρ mass is used.

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New Improved Bounds for K_{l3} Parameters*

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New exact bounds for derivatives of K_{l3} decay form factors $f_\pm(t)$ are obtained, provided that $\Delta(0)$, the propagator of the divergence of the strangeness-changing current at zero momentum, is known. Several estimates of $\Delta(0)$ are discussed, along with their experimental implications.

I. EXACT INEQUALITIES

Recently, some exact upper bounds for K_{l3} decay parameters have been obtained by several authors,[1-3] if $\Delta(0)$, the propagator of the divergence of the strangeness-changing current at zero momentum, is known. These bounds give rather stringent conditions on the decay parameters if we estimate $\Delta(0)$ from the $SW(3)$ model of Gell-Mann, Oakes, and Renner[4] and of Glashow and Weinberg.[5]

The purpose of this note is twofold. First, we will obtain new improved bounds for these quantities. Secondly, we will attempt to calculate $\Delta(0)$ in as model independent a way as possible. As we shall see in Sec. II, the resulting inequalities are stringent enough to test the chiral $SW(3)$ theory. In this section, we will state some simple mathematical theorems which will be useful in our analysis.

Let $F(\xi)$ be a real analytic function of a complex