Closed-Fermion-Loop Calculations in Chiral Models of Electromagnetic Processes*

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The amplitudes involving two external γ's and an arbitrary number of soft π mesons emitted from a closed nucleon loop have been calculated in a model with a formally conserved (neutral) axial-vector current. The results violate the formal chiral symmetry of the underlying Lagrangian, but in a way that has the following simple group-theoretic interpretation stated in terms of an effective Lagrangian $\mathcal{L}_{\text{eff}}$: Instead of finding $\delta\mathcal{L}_{\text{eff}}=0$ under those chiral variations which leave the electromagnetic couplings invariant, we find $\delta\mathcal{L}_{\text{eff}}$ to be independent of the chirally transforming fields. If the group-theoretic principle is abstracted from the detailed calculation it leads to predictions for the processes $\gamma+\gamma\to$ (an odd number of pions) which should hold true in a wide class of chiral models. The generalization is given to a quark-loop calculation in a theory invariant under the neutral transformations from $SU(3)\otimes SU(3)\otimes U(1)$, and the results for $\gamma\gamma\to$ pions are unchanged.

I. INTRODUCTION

There has been considerable interest recently in the calculation of various closed-loop graphs in chiral models, that is, in models in which exact $SU(2)\otimes SU(2)$ symmetry is realized in the limit of vanishing pion mass.[1] These models provide at least a formal embodiment of the assumptions of current algebra and partial conservation of axial-vector current (PCAC), and it can easily be verified that in the tree-graph approximation all formal current-algebra results are sustained in an actual calculation based on these models.

When closed loops are calculated the situation is more confused. It has been verified that in the nonlinear $SU(2)\otimes SU(2)$ model of self-interacting pions, use of the correct Feynman rules leads formally to the soft-pion results of current algebra, although the integrals are highly divergent.[2,3] There is, however, a case in which a loop calculation gives a result at variance with one's expectations from a formal application of current algebra. This is the case of the process $\pi_0\to\gamma+\gamma$, as calculated from a graph involving a closed nucleon or quark loop in a chirally invariant theory.[4] According to formal current algebra this amplitude should vanish in the soft-pion limit; according to the best gauge-invariant calculation it does not.

This phenomenon is subject to several interpretations, the most widely accepted of which seems to be the following[5,6]: In the presence of electromagnetism the third component of the axial-vector current (formally conserved in the chiral models under consideration) has a divergence proportional to $\vec{E}\cdot\vec{B}$, where $\vec{E}$ and $\vec{B}$ are the electromagnetic field operators. The deviation from the formal result comes from defining the current carefully as the limit of the product of operators at different space-time points, or alternatively, by simply computing the matrix element for (axial-vector current $\to$ two photons) in a gauge-invariant way.[7] The $\vec{E}\cdot\vec{B}$ term in the divergence of the axial-vector current is generally referred to as the Adler anomaly.

The Adler anomaly is of some importance in assessing the successes of current algebra, since the actual rate of $\pi^0\to 2\gamma$ decay is roughly consistent with the loop calculation; while the naive application of PCAC would lead to a further suppression factor of $(m_\pi/M)^2$, where M is some typical inverse range for the strong interactions. Thus the anomalous term would seem to dominate the actual decay.

However, there are some unsatisfactory aspects to accepting this particular result (and a rather ambiguous one at that) from a calculation in a particular local field-theoretic model. The whole point of current algebra is to find results which could be derived without recourse to particular models. But if the fundamental equations of the current-algebraic approach must be altered in a different way in each detailed model, then the theory is obviously less useful. From direct calculation of the triangle graph in chiral models with massless pions it is known that in the equation for the divergence of the axial-vector current one obtains[5]

$$\partial_\mu J_3^{\mu(5)}=b(\vec{E}\cdot\vec{B}), \tag{1}$$

where the coefficient b depends on the charge of the spin-$\frac{1}{2}$ particles in the closed loop. It depends, for example, on whether the calculation is done with a nucleon or a quark loop. Unfortunately, to date, the coefficient b has been measured only in the $\pi_0\to 2\gamma$ decay, so that the theory has had no real test. It would be of the greatest interest to find independent determinations of b. A further question is whether, or how, b depends on the hadronic fields.

In the present work we report some progress in understanding these two questions. Our considerations will be based on a calculation of a single nucleon (or quark) loop with two soft γ rays emitted or absorbed and an arbitrary number of soft neutral pseudoscalar mesons emitted or absorbed. This amplitude is calculated in a gauge-invariant way using a $SU(2)\otimes SU(2)$- (chiral) or $SU(3)\otimes SU(3)\otimes U(1)$-invariant Lagrangian and is found to break the formal symmetry of the theory, as explained above. However, the symmetry is broken in a particularly simple way. Summarizing the result of the loop calculation (in the soft-photon and soft-meson limit) in terms of an effective boson Lagrangian $\mathcal{L}_{\text{eff}}$, we find the following condition: $\delta_{\text{ch}}\mathcal{L}_{\text{eff}}$ is independent of the pseudoscalar meson fields. Here δ_{ch} means any of the (neutral) chiral variations which leave the electromagnetic coupling invariant. Alternatively, we could write

$$\partial_\mu J_a^{\mu(5)} = C_a(\vec{\mathrm{E}}\cdot\vec{\mathrm{B}}), \quad a=\text{neutral} \tag{2}$$

where the $J_a^{\mu(5)}$ are components of the neutral axial-vector currents and the coefficients C_a are independent of the neutral meson fields (the charged meson fields having been set to zero initially). The numerical values of the C_a will depend on which model is chosen (e.g., nucleons, quarks, integrally charged triplets). We conjecture that the result that the C_a's are independent of the neutral meson fields is a general one, rather than being specific to the models for which the calculation was done in detail. From this assumption we derive further physical predictions, most notably the soft-pion limit of the amplitude for $\gamma+\gamma\to\pi^+\pi^-\pi^0$ in terms of the measured $\pi^0\to 2\gamma$ decay rate.[8] In addition we find that a number of amplitudes involving two γ's and three pseudoscalar particles should vanish in the soft-meson limit.

II. SINGLE-LOOP CALCULATIONS

We shall consider two closely related models, the first with a nonlinear realization of $SU(2)\otimes SU(2)$, the second with a nonlinear realization of $SU(3)\otimes SU(3)\otimes U(1)$. In the $SU(2)\otimes SU(2)$ case we begin with massless pion $\vec{\pi}$ and couple to an isospin-$\frac{1}{2}$ Dirac field q in the Lagrangian for the familiar σ model,[1] $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_I$,

$$\begin{aligned}\mathcal{L}_0 &= \bar{q}i\gamma_\mu\partial^\mu q+\tfrac{1}{2}(\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi}+\partial_\mu\sigma\partial^\mu\sigma)-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\,,\\ \mathcal{L}_I &= -g\bar{q}(\sigma+i\gamma_5\vec{\tau}\cdot\vec{\pi})q+e\bar{q}\left(\frac{1+\tau_3}{2}\right)\gamma_\mu A^\mu q\,.\end{aligned} \tag{3}$$

Here σ is a shorthand for $(f^2-\vec{\pi}^2)^{1/2}$. We have left out of $\mathcal{L}_I$ the electromagnetic interactions of the charged π mesons which will play no role in our development.

All terms of the Lagrangian of Eq. (3), except for the electromagnetic coupling, are invariant under the chiral group $SU(2)\otimes SU(2)$, with the chiral transformation defined by

$$\begin{aligned}\delta q &= \frac{\delta\vec{\omega}\cdot\vec{\tau}}{2}\gamma_5 q\,,\\ \delta\vec{\pi} &= \delta\vec{\omega}(f^2-\vec{\pi}^2)^{1/2}=\delta\vec{\omega}\sigma\,.\end{aligned} \tag{4}$$

The electromagnetic term is invariant under the subgroup of $SU_2\otimes SU_2$ generated by $\tau_3\gamma_5$, τ_3 acting on the fields q.[4] We shall call this the restricted chiral group.

For the $SU(3)\otimes SU(3)\otimes U(1)$ case we take nine independent pseudoscalar fields π_i, $i=0,1,\ldots,8$, and nine dependent scalar fields σ_i, transforming in the $(3^*,3)\otimes(3,3^*)$ representation of the group. Using the 3×3 λ_i matrices of Gell-Mann,[9] we define

$$S=\sum_{\lambda=0}^{8}\lambda_i\sigma_i\,,\quad P=\sum_{\lambda=0}^{8}\lambda_i\pi_i\,. \tag{5}$$

The condition by which we can express the fields σ in terms of the fields π is taken as[10]

$$S^2+P^2=f^2\,, \tag{6}$$

which can be solved for the matrix S or the fields σ,

$$S=(f^2-P^2)^{1/2}\,,\quad \sigma_i=\tfrac{1}{2}\mathrm{Tr}\lambda_i(f^2-P^2)^{1/2}\,. \tag{7}$$

The variation of the pseudoscalar fields under the chiral part of the transformation is given in matrix form as

$$\delta P=\tfrac{1}{2}\{\lambda_a\delta\omega_a, S\}\,. \tag{8}$$

As the Lagrangian we choose

$$\begin{aligned}\mathcal{L}_0 &= \bar{q}i\gamma_\mu\partial^\mu q+\tfrac{1}{2}\mathrm{Tr}(\partial_\mu P\partial^\mu P+\partial_\mu S\partial^\mu S)-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu},\\ \mathcal{L}_I &= -g\bar{q}(S+i\gamma_5 P)q+e\bar{q}Q\gamma_\mu A^\mu q\,.\end{aligned} \tag{9}$$

Here the q are quark fields transforming the conventional way under the group $SU(3)\otimes SU(3)\otimes U(1)$.[11] Q is the charge matrix

$$Q=\tfrac{1}{2}[\lambda_3+(\tfrac{1}{3})^{1/2}\lambda_8]\,. \tag{10}$$

The Lagrangian of Eq. (9), leaving out the electromagnetic part, is invariant under the group $SU(3)\otimes SU(3)\otimes U(1)$. The electromagnetic coupling is invariant under the subgroup generated by

$$\gamma_5,\ \lambda_3,\ \lambda_3\gamma_5,\ \lambda_6,\ \lambda_6\gamma_5,\ \lambda_7,\ \lambda_7\gamma_5,\ \lambda_8,\ \lambda_8\gamma_5 \tag{11}$$

acting on the quark fields. We shall call this subgroup the restricted chiral $SU(3)\otimes SU(3)\otimes U(1)$ group.

As we shall be able trivially to specialize the final results of the $SU(3)$ case to the $SU(2)$ case, we deal henceforth with the Lagrangian (8) and make only occasional reference to the results in the $SU(2)\otimes SU(2)$ case.

We proceed by constructing effective Lagrange functions for low-energy processes involving two γ's and any number of neutral π_i's. This effective Lagrangian will be calculated from the contribution of a single, closed q loop with arbitrary numbers of π_i's attached, both through the $\gamma_5 P$ coupling and through the S coupling of Eq. (9).

The technique (inspired by Schwinger[7]) will be to calculate the vacuum expectation value of the $\mathcal{S}$ operator in a constant external electromagnetic field, $F_{\mu\nu}$, and in constant pseudoscalar fields π_i. The quantity

$$\langle 0|\mathcal{S}|0\rangle_{F_{\mu\nu},\,\pi_i} = i\int d^4x\,\mathcal{L}_{\text{eff}}(F_{\mu\nu},\pi_i) \tag{12}$$

defines an effective Lagrange function for low-energy boson reactions, which will give the same result as the closed-loop calculation in the limit of small meson momenta. We take only neutral external π_i fields ($\pi_0, \pi_3, \pi_8, \pi_6, \pi_7$), and consider the variation of $\langle \mathcal{S}\rangle$ induced through a variation of these fields,

$$\begin{aligned}\langle 0|\delta\mathcal{S}|0\rangle &= i\langle 0|\int \delta\mathcal{L}_{\text{eff}}(x)d^4x|0\rangle\\ &= -g\int d^4x\,\mathrm{Tr}[G(x,x)(\delta S + i\gamma_5\delta P)]\,.\end{aligned} \tag{13}$$

Here the trace is in the 3×3 $SU(3)$ space and in the Dirac spin space; $G(x,x)$ is the limiting form of the Green's function,

$$G(x',x)_{\alpha i,\beta j} = -i\langle 0|T(q_{\alpha i}(x')\bar{q}_{\beta j}(x))|0\rangle_{F^{\mu\nu},\pi}\,, \tag{14}$$

where α and β are spinor indices and i,j are $SU(3)$ indices. The Green's function G is to be calculated in the external π and electromagnetic fields. The variation δS in (13) is to be constructed from the variation of the pseudoscalar fields,

$$\delta S = \delta(f^2 - P^2)^{1/2}\,. \tag{15}$$

The Green's function, G, satisfies the equation

$$[-\gamma_\mu(i\partial^\mu + eQA^\mu) + g(S' + i\gamma_5 P')]G(x,y) = -\delta^4(x-y)\,, \tag{16}$$

where we have explicitly omitted all of the charged pseudoscalar fields by defining

$$P' = \sum_{0,3,6,7,8}\lambda_i\pi_i\,,\quad S' = \sum_{0,3,6,7,8}\lambda_i\sigma_i\,. \tag{17}$$

Equation (16) is invariant under the restricted chiral $SU(3)\otimes SU(3)\otimes U(1)$ group as defined earlier.

Defining

$$\mathfrak{M}^{\pm} = g(\pm S' + i\gamma_5 P')\,, \tag{18}$$

we can write the Green's function as

$$\begin{aligned}G(x',x) &= -\langle x'|[(-i\partial\!\!\!/ - eQA\!\!\!/) + \mathfrak{M}^{(+)}]^{-1}|x\rangle\\ &= -\langle x'|[(-i\partial\!\!\!/ - eQA\!\!\!/)^2 + \mathfrak{M}^{(-)}\mathfrak{M}^{(+)}]^{-1}(-i\partial\!\!\!/ - eQA\!\!\!/ + \mathfrak{M}^{(-)})|x\rangle\,.\end{aligned} \tag{19}$$

In the last step we used the relations

$$[Q,S'] = [Q,P'] = 0$$

and

$$(-i\partial\!\!\!/ - eQA\!\!\!/)\mathfrak{M}^{(+)} = -\mathfrak{M}^{(-)}(-i\partial\!\!\!/ - eQA\!\!\!/)\,. \tag{20}$$

Substituting (19) into (13), recognizing that $G(x,x)$ (in constant external fields) is independent of x by translational invariance, and canceling out a factor of total space-time volume, we obtain

$$\delta\mathcal{L}_{\text{eff}} = -\mathrm{Tr}g\{\langle x'=0|[(-i\partial\!\!\!/ - eQA\!\!\!/)^2 + \tfrac{1}{2}eQ\sigma_{\mu\nu}F_{\mu\nu} + \mathfrak{M}^{(-)}\mathfrak{M}^{(+)}]^{-1}(-i\partial\!\!\!/ - eQA\!\!\!/ + \mathfrak{M}^{(-)})|x=0\rangle(\delta S' + i\gamma_5\delta P')\}\,. \tag{21}$$

The inverse operator in (21) is an even function of the γ matrices. Thus the $\gamma_\mu(\partial - eA)^\mu$ factor under the trace may be neglected. Substituting (18) into (21) we obtain

$$\delta\mathcal{L}_{\text{eff}} = -i\mathrm{Tr}\{\langle x'=0|[(-i\partial\!\!\!/ - eQA\!\!\!/)^2 + \tfrac{1}{2}eQ\sigma_{\mu\nu}F^{\mu\nu} + \mathfrak{M}^{(-)}\mathfrak{M}^{(+)}]^{-1}|x=0\rangle g^2[(-S'\delta S' - P'\delta P') + i\gamma_5(P'\delta S' - S'\delta P')]\}\,. \tag{22}$$

Now we can replace $\mathfrak{M}^{(-)}\mathfrak{M}^{(+)}$, using the constraint Eq. (7), by

$$\begin{aligned}\mathfrak{M}^{(-)}\mathfrak{M}^{(+)} &= -g^2(S'^2 + P'^2 + i\gamma_5[S',P'])\\ &= -g^2 f^2\,.\end{aligned} \tag{23}$$

In the first term of the second factor in (22) we can use the commutativity of Q with P' and $\delta P'$ and the properties of the trace to make the replacement

$$S'\delta S' + P'\delta P' \to \tfrac{1}{2}[(\delta S')S' + S'(\delta S') + (\delta P')P' + P'(\delta P')] = \tfrac{1}{2}\delta(S'^2 + P'^2) = 0\,. \tag{24}$$

In the second term of the second factor in (22) (the γ_5 term) we note that in order to obtain a nonvanishing spinor trace we need at least four γ matrices from the inverse operator. Thus, if we work to second order in a constant electromagnetic field, we can make the replacement

$$[(-i\partial - eA)^2 + \tfrac{1}{2}eQ\sigma_{\mu\nu}F^{\mu\nu} - g^2f^2]^{-1} \to \frac{1}{(-\partial^2 - g^2f^2)^3}(\tfrac{1}{2}eQ\sigma_{\mu\nu}F^{\mu\nu})^2 . \tag{25}$$

Making all of these substitutions in (22) and taking the spinor traces, we obtain

$$\delta\mathcal{L}_{\text{eff}} = 8ie^2g^2\vec{E}\cdot\vec{B}\,\mathrm{Tr}Q^2(S'\delta P' - P'\delta S')\frac{1}{(2\pi)^4}\int\frac{d^4k}{(k^2 - g^2f^2)^3}. \tag{26}$$

The trace here is only in the $SU(3)$ space.

The interpretation of Eq. (26) is as follows: In (22) the $(S'\delta S' + P'\delta P')$ term on the right-hand side would have multiplied the even-parity invariant formed from the electromagnetic fields, $(E^2 - B^2)$, but use of the constraint condition $S'^2 + P'^2 = f^2$ eliminated this term altogether. This reflects the cancellation of the pion emission from γ_5 vertices by those involving σ vertices, and in fact is an example of the Adler consistency condition at work. Since the $(E^2 - B^2)$ invariant multiplies all terms relevant to the processes $\gamma + \gamma \to$ (even number of pseudoscalar particles) in the soft-meson limit, we conclude that there are no anomalous terms in $\mathcal{L}_{\text{eff}}$ which contain even numbers of pseudoscalar fields.

The nonvanishing remainder in $\mathcal{L}_{\text{eff}}$, given by Eq. (26), is proportional to $\vec{E}\cdot\vec{B}$ and thus will contribute to processes with an odd number of external soft pseudoscalar particles. It breaks the chiral symmetry of the underlying theory and gives an escape from the Adler consistency condition (that is, from the vanishing of amplitudes in the soft-meson limit).

To determine $\mathcal{L}_{\text{eff}}$ from $\delta\mathcal{L}_{\text{eff}}$ we substitute in (26) the constraint condition for S and δS. For the case of the restricted $SU(2)\otimes SU(2)$ group we have only a π_3 and a single σ. In (26) we replace S by σ and P by π_0, $\mathrm{Tr}Q^2$ by unity and obtain

$$\delta\mathcal{L}_{\text{eff}} = \frac{-\alpha}{\pi f^2}(\vec{E}\cdot\vec{B})[(f^2 - \pi_3{}^2)^{1/2}\delta\pi_3 - \pi_3\delta(f^2 - \pi_3{}^2)^{1/2}]. \tag{27}$$

The integral of this is

$$\mathcal{L}_{\text{eff}} = \frac{-\alpha}{\pi}(\vec{E}\cdot\vec{B})\sin^{-1}\frac{\pi_3}{f} + \text{a term independent of } \pi_3. \tag{28}$$

As pointed out in Ref. 8, this agrees with the result of Schwinger, where we set $f = M_N/G$ and take only the first term in the expansion of the $\sin^{-1}$ function in powers of π_0.

In the $SU(3)\otimes SU(3)\otimes U(1)$ case in which $\delta\mathcal{L}_{\text{eff}}$ involves the trace of a product of 3×3 matrices, $\mathrm{Tr}Q^2(S'\delta P' - P'\delta S')$, we obtain in exactly the same way

$$\mathcal{L}_{\text{eff}} = \frac{-\alpha}{\pi}(\vec{E}\cdot\vec{B})\mathrm{Tr}[Q^2\sin^{-1}(P'/f)]. \tag{29}$$

In deriving (29) from (26) we used the commutativity of Q with P' and $\delta P'$ which resulted from deleting the charged fields in defining P'. Equation (29) is the main result of our loop calculation.

In the calculation described above, only the terms of second order in the electromagnetic field strengths were considered. However, following the methods of Schwinger,[7] it is easy to show that the terms of higher order in the electromagnetic field strengths vanish in the soft-meson limit. For the case of an even number of pseudoscalar mesons this is a consequence of (22) as it stands. For the case of an odd number the interested reader can verify the statement by consulting Eqs. (5.10), (5.11), and (3.43) of Ref. 7.

III. GROUP-THEORETIC INTERPRETATION

We used the expression (26) for the variation of $\mathcal{L}_{\text{eff}}$ in response to arbitrary variations of the neutral pseudoscalar fields δP as a determining equation for the effective Lagrangian. Let us now restrict the variation δP to be those due to an infinitesimal chiral transformation from the restricted $SU(3)\otimes SU(3)\otimes U(1)$ group. These transformations are given by

$$\begin{aligned}\delta_{\text{ch}}P' &= \tfrac{1}{2}\{\delta\omega_a\lambda_a, S'\},\\ \delta_{\text{ch}}S' &= -\tfrac{1}{2}\{\delta\omega_a\lambda_a, P'\},\end{aligned} \tag{30}$$

where $a = 0, 3, 6, 7, 8$ and S is given in terms of P by (7). Inserting these variations into Eq. (26) we find

$$\delta_{\text{ch}}\mathcal{L}_{\text{eff}} = -\frac{\alpha}{\pi}(\vec{E}\cdot\vec{B})\mathrm{Tr}(Q^2\lambda_a)\delta\omega_a, \tag{31}$$

where we have used the properties of the trace, commutativity of Q with the matrices S' and P', and the constraint equation $S^2 + P^2 = f^2$. There are two points of interest in Eq. (31):

(a) The variation of $\mathcal{L}_{\text{eff}}$ under the special chiral transformation is not zero. As remarked earlier, the gauge-invariant calculation does not preserve the chiral symmetry of the underlying Lagrangian.

(b) The chiral variation of the effective Lagrangian is independent of the neutral boson fields which enter into $\mathcal{L}_{\text{eff}}$ itself. Thus, if we construct the divergence of the 0, 3, 6, 7, 8 components of the axial-vector current, on the basis of $\mathcal{L}_{\text{eff}}$ as the destroyer of the symmetry we find

$$\partial_\mu J_a^{\mu(5)} = \frac{\alpha}{\pi f}\mathrm{Tr}(Q^2\lambda_a)(\vec{E}\cdot\vec{B}), \tag{32}$$

in close analogy to Adler's result[5] for the case of quantum electrodynamics that the divergence of the axial-vector current depends on the quantized fields only through the factor $\vec{E}\cdot\vec{B}$.

Another way of stating this result is in terms of the basic differential of the effective Lagrangian, Eq. (26), in which δP is an arbitrary field variation. If we now compute the change in $\delta\mathcal{L}_{\rm eff}$ under a special chiral operation $\delta_{\rm ch}$, we obtain

$$\delta_{\rm ch}\delta\mathcal{L}_{\rm eff} = 0, \tag{33}$$

since the chiral variation of $\mathcal{L}_{\rm eff}$ is independent of the neutral meson fields. This is the form of the group-theoretic principle used in Ref. 8, where it was noted that the gradients of $\mathcal{L}_{\rm eff}$ with respect to the external fields transform covariantly under the special chiral transformations, even though $\mathcal{L}_{\rm eff}$ is not invariant.

Although we have no compelling reason, outside of the model calculation, to assume that condition (33) is generally true, its simplicity alone makes it worth asking the question: What are the most general functions $\mathcal{L}_{\rm eff}$ obeying (33), and invariant under the restricted (neutral) transformations from ordinary $SU(3)$ [or $SU(2)$, as the case may be]?

In the $SU(2)\otimes SU(2)$ case the answer to the question is simply, an arbitrary constant times the $\mathcal{L}_{\rm eff}$ of Eq. (28). We prove this by demanding that

$$\delta_{\rm ch}\mathcal{L}_{\rm eff} = \frac{\partial\mathcal{L}_{\rm eff}}{\partial\pi_3}\delta_{\rm ch}\pi_3 = \frac{\partial\mathcal{L}_{\rm eff}}{\partial\pi_3}(\delta\omega)(f^2-\pi_3{}^2)^{1/2} \tag{34}$$

be independent of the π_3 field. Direct integration gives

$$\mathcal{L}_{\rm eff} = \text{const}\times\sin^{-1}(\pi_3/f) + \text{a term independent of } \pi_3\,. \tag{35}$$

In the $SU(3)\otimes SU(3)\otimes U(1)$ case the answer is: $\mathcal{L}_{\rm eff}$ is given by (29) with the matrix Q^2 replaced by $a + bQ^2$, where a and b are any constants. To prove this we begin by specializing to only three nonvanishing boson fields π_0, π_3, π_8 and only the three chiral operations generated by $\lambda_0\gamma_5$, $\lambda_3\gamma_5$, $\lambda_8\gamma_5$. We introduce linear combinations of the π fields π_A, π_B, π_C such that

$$\pi_A\lambda^A + \pi_B\lambda^B + \pi_C\lambda^C = \pi_0\lambda_0 + \pi_3\lambda_3 + \pi_8\lambda_8\,, \tag{36}$$

where

$$\lambda_A = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad \lambda_B = \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\quad \lambda_C = \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}. \tag{37}$$

Now, with all the other componets of the pion-field tensor set equal to zero, the transformation law plus the constraint condition yields

$$\begin{aligned}\delta\pi_A &= \delta\omega_A(f^2-\pi_A{}^2)^{1/2},\\ \delta\pi_B &= \delta\omega_B(f^2-\pi_B{}^2)^{1/2},\\ \delta\pi_C &= \delta\omega_C(f^2-\pi_C{}^2)^{1/2}.\end{aligned} \tag{38}$$

Thus the problem is reduced to three times the previously solved problem and the general solution to $\delta_{\rm ch}\delta\mathcal{L}_{\rm eff}=0$ is

$$\begin{aligned}\mathcal{L}_{\rm eff} &= \left(C_A\sin^{-1}\frac{\pi_A}{f} + C_B\sin^{-1}\frac{\pi_B}{f} + C_C\sin^{-1}\frac{\pi_C}{f}\right)(\vec{E}\cdot\vec{B})\\ &= C_A\sin^{-1}\left[\frac{(\frac{1}{3})^{1/2}U_0 - (\frac{2}{3})^{1/2}U_8}{f}\right](\vec{E}\cdot\vec{B})\\ &\quad + C_B\sin^{-1}\left[\frac{(\frac{1}{3})^{1/2}U_0 + (\frac{1}{6})^{1/2}U_8 + (\frac{1}{2})^{1/2}U_3}{f}\right](\vec{E}\cdot\vec{B})\\ &\quad + C_C\sin^{-1}\left[\frac{(\frac{1}{3})^{1/2}U_0 + (\frac{1}{6})^{1/2}U_8 - (\frac{1}{2})^{1/2}U_3}{f}\right](\vec{E}\cdot\vec{B}).\end{aligned} \tag{39}$$

In the second form we have introduced linear combinations of the π_0, π_3, and π_8 that transform simply under the U-spin subgroup of $SU(3)$; U_0 and U_8 being U-spin singlets and U_3 the third component of a vector. We are now in a position to reintroduce the other neutral fields π_6 and π_7 in such a way as to restore U-spin conservation to $\mathcal{L}_{\rm eff}$. It is clear that the field U_3 must enter only in the form of $\vec{U}^2$ (U_1 and U_2 are linear combinations of π_6 and π_7). This can be achieved if, and only if, $C_B = C_C$ in (39), by replacing U_3 by $(\vec{U}^2)^{1/2}$. When this is done, the new form for $\mathcal{L}_{\rm eff}$ is

$$\mathcal{L}_{\rm eff} = (\vec{E}\cdot\vec{B})\mathrm{Tr}\left[\left(a+bQ^2\right)\sin^{-1}\frac{P}{f}\right], \tag{40}$$

as can be verified by direct computation.

IV. APPLICATIONS

In Sec. III it was shown how a group-theoretic principle could be extraced from the result of the closed-loop calculation with neutral-meson attachments, leading to Eq. (40) for the case of $SU(3)$ symmetry and to

$$\mathcal{L}_{\rm eff} = C(\vec{E}\cdot\vec{B})\sin^{-1}\frac{\pi_3}{f} \tag{41}$$

for the case when only pions are considered. The latter formula was derived by considering the $SU(2)\otimes SU(2)$ group, but can be arrived at also by setting to zero all the nonet fields except π_3 in the $SU(3)\otimes SU(3)$ result (40). The trick used to calculate the loop graphs fails when charged meson fields are introduced, and the group-theoretic principle (33) leads to inconsistencies as well in the presence of charged fields. So, at the moment, we can make predictions only for amplitudes involving two photons and any number of neutral mesons.

The basis for these predictions will be a boson Lagrangian consisting of the sum of $\mathcal{L}_{\text{eff}}$ of Eq. (40) or (41) plus the mesonic part of the $\mathcal{L}_0$ introduced in Eq. (9) or (3), respectively.

Now, when we expand the $\sin^{-1}$ function in (41) in powers of the π_3 field, the first term is responsible for $\pi^0 \to \gamma+\gamma$, the rate of which determines the constant C. When we expand (40) in the same way, we obtain as well the terms responsible for $\eta \to 2\gamma$ and $X^0 \to 2\gamma$. Our constants a and b can be determined from an analysis of the widths, as discussed by Glashow, Jackiw, and Shei.

In the cubic terms we encounter amplitudes for, e.g., $\eta \to 2\pi^0 + 2\gamma$ or $\gamma+\gamma \to 3\pi^0$. However, the complete contribution to these processes also includes a contribution from the pole graphs of Fig. 1(b), where the four-meson vertex is taken from the nonlinear terms in $\mathcal{L}_0$ of Eq. (3) or (9). These contribute a nonvanishing term in the soft limit, as long as the intermediate meson is massless.

When these graphs are included, we obtain exact cancellation between the graphs of Figs. 1(a) and 1(b). In fact, all amplitudes for nonstrange mesons except those for $\gamma+\gamma \to$ (one pseudoscalar) will vanish in the sum of all tree graphs based on the Lagrangian $\mathcal{L}_0 + \mathcal{L}_{\text{eff}}$. The reason for this is easy to see in the restricted $SU(2)\otimes SU(2)$ case. The nonlinear σ model with only a neutral pion is equivalent to a free field, and it happens that $\varphi = f\sin^{-1}(\pi_3/f)$ is precisely that redefinition of the π^0 field which removes the nonlinear pion-pion interaction terms from the Lagrangian. Similar considerations rule out multi-nonstrange-neutral-meson processes in the $SU(3)\otimes SU(3)$ case.

This cancellation, of course, depends on the masslessness of the intermediate meson in Fig. 1(b), which will be a very poor approximation in the case of η or X^0. It would be tempting simply to put in a mass for the intermediate meson in order to destroy the cancellation. However, such a calculation would be completely meaningless in the sense that the answer would depend completely on which definition of the pseudoscalar fields was chosen before the mass term, $\mu^2\varphi^2$, was inserted into the Lagrangian.

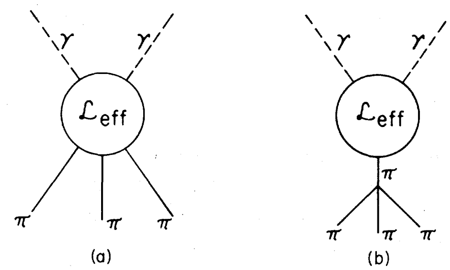

FIG. 1. (a) The three-pion amplitude arising from the cubic term in $\mathcal{L}_{\text{eff}}$, Eq. (28). (b) The three-pion pole term which comes from the linear term in $\mathcal{L}_{\text{eff}}$, combined with the four-pion coupling in $\mathcal{L}_0$.

Thus, the only prediction for the neutral five particle amplitudes (γ, γ, 3 pseudoscalar) is that they will be suppressed (relative to f^{-2} times the measured amplitude for $\pi^0 \to \gamma+\gamma$, $\eta \to \gamma+\gamma$, etc.). This prediction should be much more reliable for the particular case $\gamma+\gamma \to 3\pi^0$ than for the reactions involving η and X^0, owing to the smallness of the pion mass.

We turn next to the emission of charged pions. Although these are not included in the effective Lagrangians, (40) and (41), we can draw some conclusions nevertheless. The isospin selection rule of second-order electromagnetism is $\Delta I < 3$. Now let us consider the effective Lagrangian (41) for neutral pions alone. Every power of π_3 higher than the first, ${\pi_3}^3$, ${\pi_3}^5, \ldots$, violates the $\Delta I < 3$ rule. In fact, in each order there exists only one amplitude involving undifferentiated pion fields which obeys the $\Delta I < 3$ rule, and which agrees with (41) when π_1 and π_2 are set equal to zero, i.e., that amplitude in which $(\pi_3)^{2n+1}$ is replaced by $\pi_3(\vec{\pi}\cdot\vec{\pi})^n$.

This follows from the fact that of all the possible isospin states for $2n+1$ pions there exists a single completely symmetrical state for each odd value of I. What we have written above is the $I_3 = 0$ component of the $I=1$ completely symmetrical state. The other completely symmetrical states are inaccessible according to the $\Delta I < 3$ rule.

Thus we can write a generalized meson Lagrangian, including the charged pion fields by making the appropriate modification in (41),

$$\mathcal{L}_{\text{meson+photon}} = \tfrac{1}{2}(\partial_\mu\vec{\pi})(\partial^\mu\vec{\pi}) + \tfrac{1}{2}[\partial_\mu(f^2-\pi^2)^{1/2}][\partial^\mu(f^2-\pi^2)^{1/2}] - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + C(\vec{E}\cdot\vec{B})\frac{\pi_3}{\sqrt{\pi^2}}\sin^{-1}\frac{\sqrt{\pi^2}}{f}. \qquad (42)$$

Using this Lagrangian we can now calculate the amplitude for $\gamma+\gamma \to \pi^+\pi^-\pi^0$ as the sum of contributions of graphs of Figs. 1(a) and 1(b), obtaining

$$M(\gamma\gamma \to \pi_+\pi_-\pi_0) = \left(\frac{4\pi\Gamma^0}{\mu}\right)^{1/2} \frac{4}{\mu f^2}\left[\frac{1}{3} - \frac{(k_+ + k_-)^2}{(k_+ + k_- + k_0)^2}\right](\vec{E}\cdot\vec{B}) . \tag{43}$$

Here k_+, k_-, k_0 are the pion four-momenta and Γ^0 is the $\pi^0 \to \gamma\gamma$ width [which determined the constant C in (42)]. The numerical value of f is approximately equal to M_N/G_π. The result (43) was discussed in Ref. 8.

Note that the contribution of the pole term in Eq. (43) leads to anything but a completely symmetrical orbital state. This is what gave the possibility of having a $\pi^+\pi^-\pi^0$ matrix element which does not vanish in the soft limit, while the $3\pi^0$ amplitude does. According to the isotopics of $I=1$ three-pion amplitudes, we should symmetrize (43) with respect to k_+, k_-, k_0 in order to obtain the $3\pi^0$ amplitude, and the resulting amplitude does indeed vanish.

V. DISCUSSIONS

The group-theoretic interpretation of our closed-loop calculation was quite simple; instead of finding $\delta\mathcal{L}_{\text{eff}}=0$ under those chiral variations which formally leave the electromagnetic couplings invariant, we found $\delta\mathcal{L}_{\text{eff}}$ to be independent of the chirally transforming fields. We could state this result as $\delta\delta\mathcal{L}_{\text{eff}}=0$. The strict symmetry statement $\delta\mathcal{L}_{\text{eff}}=0$ would have forbidden the reactions $\gamma+\gamma\to$ (an odd number of pions) in the soft-pion limit; the weakened statement $\delta\delta\mathcal{L}_{\text{eff}}=0$, instead, gave relations between processes with different numbers of pions.

The only conceivably verifiable numerical prediction that we have found is the amplitude for $\gamma+\gamma\to\pi^+\pi^-\pi^0$ calculated in terms of the π^0 width. In addition, we found that amplitudes with three neutral pseudoscalar particles, e.g., $\gamma+\gamma\to 3\pi^0$ and $\eta\to\pi^0+\pi^0+\gamma+\gamma$ vanish in the soft-meson limit. We find no anomalous terms for reactions involving an even number of pseudoscalar particles, such as $\eta\to\pi^0+\gamma+\gamma$ or $\eta\to 3\pi+\gamma+\gamma$. Thus we see no way in which the present considerations can bear on another outstanding problem of current algebra, the decay $\eta\to 3\pi$ (regarded as $\eta\to 3\pi+2\gamma\to 3\pi$).

It would be interesting to apply the $\delta\delta\mathcal{L}_{\text{eff}}=0$ approach to other forms of symmetry breakdown, for example in breaking $SU(3)$ or in weak interaction theory. However, we have not been successful in doing this.

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[10]This matrix condition is invariant under the linear chiral transformations of the form of Eq. (8) and thus can be the basis for converting a linear representation in terms of the 18 π_i and σ_i into a nonlinear one in terms of the nine π_i alone. Such methods are reviewed in Ref. 1.

[11]That is the ordinary $SU(3)$ transformation generated on the quark fields by the matrices λ_a, $a=1, 2, \ldots, 8$; and the chiral transformation generated by $\lambda_a\gamma_5$, $a=0, 1, 2, \ldots, 8$.