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# $T$-Matrix Perturbation Theory for the Faddeev Equations

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A $T$-matrix perturbation theory for the Faddeev equations is developed in two different ways giving slightly differing results for the first-order correction to the ground-state energy of the three-body system, and both expressions differ from the result recently obtained by Fuda. The application of the theory to the system of three identical spinless particles is discussed.

## I. INTRODUCTION

When solving the Faddeev equations,[1] a separable two-body $T$ matrix is often used as input, since this reduces the problem (after angular momentum decomposition) to one of solving one-dimensional (possibly coupled) integral equations instead of the more general two-dimensional equations. In many applications the potentials one wants to use are not separable, and consequently the corresponding two-body $T$ matrices will not be separable, and thus some separable approximation, $T_s$, to the actual $T$ matrix must be used. The difference between these two, $T_p = T - T_s$, the perturbing part of the $T$ matrix, may not be negligibly "small," and it is often useful to be able to calculate the first-order correction to the ground-state energy of the three-particle system.

Fuda[2] has written down such a perturbation expansion, but he uses a separable $T$ matrix that comes from a separable potential, while we take advantage of the considerable freedom in the particular breakup of the $T$ matrix into separable and nonseparable parts. Let $V=V_1+V_2$ with $V_1$ separable. Then

$$T=T_1+T_2+T_{12},$$

where $T_1$ is the solution of the Lippmann-Schwinger (LS) equation for $V_1$, $T_2$ is the solution for $V_2$, and $T_{12}$ contains both $V_1$ and $V_2$. Since $V_1$ is separable, so is $T_1$, and it is this that Fuda calls $T_s$. Thus, for Fuda, $T_p=T_2+T_{12}$. However, $T_1+T_{12}$ is also separable,[3-5] and thus it is possible to identify $T_2$ as $T_p$. The later breakup is advantageous, since $T_s$ includes some of the effects of $V_2$, and $T_p$ may be "small" even when $V_2$ is not. Yaes[5] has also developed a perturbation expression, but he emphasizes expansion in terms of the potential.

In Sec. II we derive two relations for the first-order correction to the energy, both of which differ from Fuda's, and in Sec. III we discuss how the formulas may be applied in order to find the first-order Coulomb correction to the ground-state energy for a system consisting of three identical, spinless, charged particles.

## II. THEORY

We first write the homogeneous Faddeev equations in the form

$$(E-H_0)|\psi\rangle = T(E)|\psi\rangle, \tag{2.1}$$

where

$$|\psi\rangle=\begin{pmatrix}|\psi^1\rangle\\|\psi^2\rangle\\|\psi^3\rangle\end{pmatrix} \tag{2.2}$$

and

$$T=\begin{pmatrix}0 & T_1 & T_1\\ T_2 & 0 & T_2\\ T_3 & T_3 & 0\end{pmatrix}, \tag{2.3}$$

with $T_i$ given by

$$T_i = V_i + V_i G_0 T_i. \tag{2.4}$$

In the above equation $V_i$ is the potential between particles $j$ and $k$ [throughout this paper $(ijk)=(123)$ or some cyclic permutation thereof], and $G_0$ is the free-particle Green's function given by

$$G_0(Z)=\frac{1}{Z-H_0}, \tag{2.5}$$

with $H_0$ the kinetic-energy operator. Thus $T_i$ is just the $T$ matrix, in three-body space, for particles $j$ and $k$. We note that Eq. (2.1) looks like a coupled-channel Schrödinger equation except that the "potential" $T(E)$ is energy-dependent.

We now write

$$T(E)=T_0(E)+\lambda T_p(E), \tag{2.6}$$

where we assume that the solution of the Faddeev equations for the $T$ matrix $T_0$ is known; i.e., we know $|\psi_0\rangle$ and $E_0$, where

$$(E_0-H_0)|\psi_0\rangle=T_0(E_0)|\psi_0\rangle. \tag{2.7}$$

If we now expand $|\psi\rangle$, $E$, $T_0(E)$, and $T_p(E)$ in powers of $\lambda$ about their unperturbed values, we obtain

$$|\psi\rangle=|\psi_0\rangle+\lambda|\psi_1\rangle+\lambda^2|\psi_2\rangle+\cdots, \tag{2.8}$$

$$E=E_0+\lambda E_1+\lambda^2E_2+\cdots, \tag{2.9}$$

$$T_0(E)=T_0(E_0)+\lambda E_1\frac{\partial T_0(E)}{\partial E}\bigg|_{E=E_0}+\lambda^2\left(E_2\frac{\partial T_0(E)}{\partial E}\bigg|_{E=E_0}+\tfrac{1}{2}E_1{}^2\frac{\partial^2T_0(E)}{\partial E^2}\bigg|_{E=E_0}\right)+\cdots, \tag{2.10}$$

with an analogous expansion for $T_p(E)$. Using these expansions in Eq. (2.1) and comparing coefficients of different powers of $\lambda$, we obtain

$$[E_0-H_0-T_0(E_0)]|\psi_0\rangle=0, \tag{2.11}$$

$$[E_0-H_0-T_0(E_0)]|\psi_1\rangle+E_1\left(1-\frac{\partial T_0}{\partial E}\bigg|_{E=E_0}\right)|\psi_0\rangle-T_p(E_0)|\psi_0\rangle=0, \tag{2.12}$$

plus higher-order equations. We note that Eq. (2.11) is just the unperturbed Faddeev equation. After a little manipulation, Eq. (2.12) will give us $E_1$, the first-order correction to the energy. We now define the "conjugate wave function," $\langle\overline{\psi}|$, by

$$\langle\overline{\psi}|=\langle\psi|M=(\langle\psi_2|+\langle\psi_3|,\ \langle\psi_1|+\langle\psi_3|,\ \langle\psi_1|+\langle\psi_2|), \tag{2.13}$$

where

$$M=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}. \tag{2.14}$$

If we now multiply Eq. (2.12) on the left by $\langle\overline{\psi}_0|$, we obtain

$$E_1\left\langle\overline{\psi}_0\left|1-\frac{\partial T_0}{\partial E}\bigg|_{E=E_0}\right|\psi_0\right\rangle-\langle\overline{\psi}_0|T_p(E_0)|\psi_0\rangle=0, \tag{2.15}$$

since taking the Hermitian conjugate of Eq. (2.11) gives

$$\langle\overline{\psi}_0|[E_0-H_0-T_0(E_0)]=0. \tag{2.16}$$

Thus from Eq. (2.15) we have, for the first-order correction to the energy,

$$E_1=\langle\overline{\psi}_0|T_p(E_0)|\psi_0\rangle\Big/\left\langle\overline{\psi}_0\left|1-\frac{\partial T_0(E)}{\partial E}\bigg|_{E=E_0}\right|\psi_0\right\rangle. \tag{2.17}$$

If $[\partial T_0(E)/\partial E]_{E=E_0}$ is small compared to 1, we reproduce Fuda's result.

We can derive another equation for the first-order correction to the energy from a variational principle.[6] We define

$$J[\psi]=\langle\psi|L(E)|\psi\rangle, \tag{2.18}$$

where

$$L(E)=M[E-H_0-T(E)], \tag{2.19}$$

with

$$T(E)=T_0(E)+T_p(E).$$

Then $\delta J=0$ gives

$$L(E)|\psi\rangle=0 \quad\text{and}\quad \langle\psi|L(E)=0, \tag{2.20}$$

which are just the Faddeev equations.

An important property of variational principles is that they give second-order accuracy for some quantity when the trial function is good to first order. Let

$$|\psi\rangle=|\psi_0\rangle+|\delta\psi\rangle,\quad E=E_0+\delta E, \tag{2.21}$$

where

$$(E_0-H_0)|\psi_0\rangle=T_0(E_0)|\psi_0\rangle$$

and

$$(E-H_0)|\psi\rangle=[T_0(E)+T_p(E)]|\psi\rangle. \tag{2.22}$$

Then

$$\begin{aligned}J_0&=\langle\psi_0|L(E_0)|\psi_0\rangle\\&=(\langle\psi|-\langle\delta\psi|)L(E-\delta E)(|\psi\rangle-|\delta\psi\rangle)\\&=-\delta E\left\langle\psi\left|\frac{\partial L}{\partial E}\right|\psi\right\rangle+\text{(higher-order term)}\\&=-\delta E\left\langle\psi_0\left|\frac{\partial L}{\partial E}\bigg|_{E=E_0}\right|\psi_0\right\rangle+\text{(higher-order term)}.\end{aligned}$$

Thus, since

$$J_0=\langle\psi_0|L(E_0)|\psi_0\rangle=\langle\overline{\psi}_0|T_p(E_0)|\psi_0\rangle,$$

we have to second order

$$\delta E=\langle\overline{\psi}_0|T_p(E_0)|\psi_0\rangle\Big/\left\langle\overline{\psi}_0\left|1-\frac{\partial T(E)}{\partial E}\bigg|_{E=E_0}\right|\psi_0\right\rangle. \tag{2.23}$$

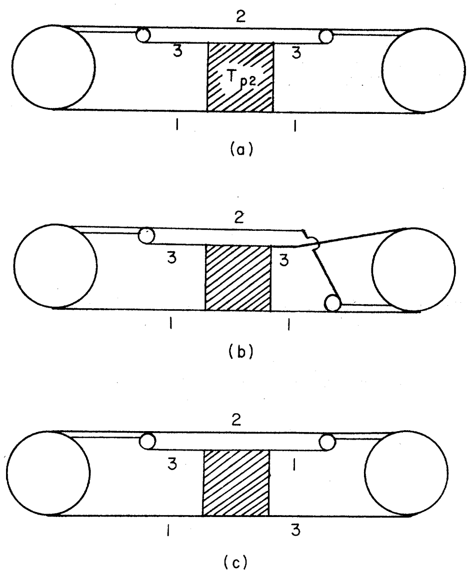


FIG. 1. (a) Graphical representation of $\langle\psi^1|T_{p2}|\psi^1\rangle$. (b), (c) Two different graphical representations of $\langle\psi^1|T_{p2}|\psi^3\rangle$.

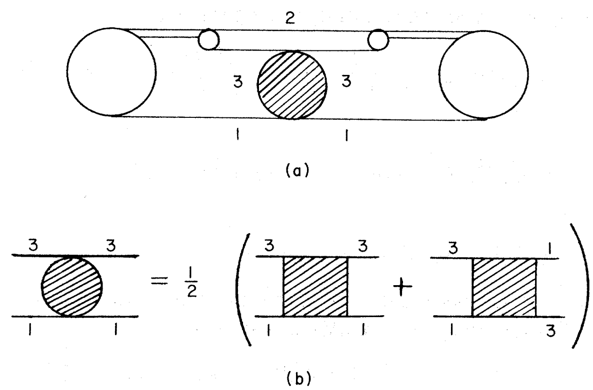


FIG. 2. (a) One half the sum of graphs (a) and (c) of Fig. 1. (b) Symmetrized form of the perturbing $T$ matrix.

We notice that (2.23) differs from (2.17) by the extra term $(\partial T_p/\partial E)_{E=E_0}$ in the denominator. The reason for this is that in the derivation of (2.17) it was assumed that $T_p$ was "small" compared to $T_0$, where no such condition is required in the derivation of (2.23). Thus, if $T_p$ is small compared to $T_0$, we expect $E_1$ given by Eq. (2.17) to be small compared to $E_0$. On the other hand, we expect that Eq. (2.23) will be useful when $\delta E$ is small compared to $E_0$, whether or not $T_p$ is small compared to $T_0$.

## III. APPLICATION

In this section we will develop Eq. (2.17) or Eq. (2.23) for the case of three identical spinless charged particles. For this special case we have

$$\langle\bar{\psi}|T|\psi\rangle = 6(\langle\psi^1|T_2|\psi^1\rangle + \langle\psi^1|T_2|\psi^3\rangle) \tag{3.1}$$

and

$$\langle\psi|\psi\rangle = 6\langle\psi^1|\psi^3\rangle. \tag{3.2}$$

Then Eq. (2.23) becomes

$$\delta E = \frac{\langle\psi^1|T_{p2}|\psi^1\rangle + \langle\psi^1|T_{p2}|\psi^3\rangle}{\langle\psi^1|\psi^3\rangle - \langle\psi^1|(\partial T_2/\partial E)_{E=E_0}|\psi^1\rangle - \langle\psi^1|(\partial T_2/\partial E)_{E=E_0}|\psi^3\rangle}. \tag{3.3}$$

We give in Fig. 1 a graphical representation of the two terms in the numerator of Eq. (3.3). From the second graphical representation of $\langle\psi^1|T_{p2}|\psi^3\rangle$, we see that

$$\tfrac{1}{2}(\langle\psi^1|T_{p2}|\psi^1\rangle + \langle\psi^1|T_{p2}|\psi^3\rangle)$$

can be represented as in Fig. 2. We can thus write Eq. (3.3) as

$$\delta E = \frac{\langle\psi^1|T_{p2}|\psi^1\rangle}{\frac{1}{2}\langle\psi^1|\psi^3\rangle - \langle\psi^1|[\partial T_2(E)/\partial E]_{E=E_0}|\psi^1\rangle}, \tag{3.4}$$

where we must now use the symmetrized $T$ matrices for $T_p$ and $T_0$.

For three identical spinless particles we have

$$\langle\vec{p}_1\vec{p}_2\vec{p}_3|T_1|\vec{p}_1\vec{p}_2\vec{p}_3\rangle = (2\pi)^3\delta(\vec{K}-\vec{K}')(2\pi)^3\delta(\vec{q}_i-\vec{q}_i')t_i(\vec{k}_i,\vec{k}_i', E - k^2/12\mu - \tfrac{3}{4}q_i^2/2\mu), \tag{3.5}$$

where $t_i(\vec{k},\vec{k}',E)$ is the two-body $T$ matrix for particles $j$ and $k$ (i.e., the solution to the two-body Lippmann-Schwinger equation), $E$ is the two-body internal-energy variable, $\mu = m/2$ ($m$ = mass of each particle), and as usual

$$\vec{K} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3, \quad \vec{k}_i = \tfrac{1}{2}(\vec{p}_j - \vec{p}_k), \quad \vec{q}_i = \tfrac{1}{3}(\vec{p}_j + \vec{p}_k - 2\vec{p}_i). \tag{3.6}$$

We also note that the $\psi^i$'s all have the same functional form $\psi$, so that the Faddeev equations reduce to a single homogeneous integral equation,

$$\psi(\vec{k},\vec{q})=\frac{2}{E-k^2/2\mu-\frac{3}{4}q^2/2\mu}\int\frac{d^3q'}{(2\pi)^3}\,t_s(\vec{k},\vec{q}\,'+\tfrac{1}{2}\vec{q},E-\tfrac{3}{4}q^2/2\mu)\psi(\vec{q}+\tfrac{1}{2}\vec{q}\,',\vec{q}\,'),\tag{3.7}$$

where

$$\psi(\vec{p}_1,\vec{p}_2,\vec{p}_3)=\psi(\vec{k}_1,\vec{q}_1)+\psi(\vec{k}_2,\vec{q}_2)+\psi(\vec{k}_3,\vec{q}_3)\tag{3.8}$$

and

$$t_s(\vec{k},\vec{q})=\tfrac{1}{2}[t(\vec{k},\vec{q})+t(\vec{k},-\vec{q})]\tag{3.9}$$

is just the symmetrized two-particle $T$ matrix (we will suppress the subscript $s$ in what follows).

Using the above relations, we have

$$\langle\psi^1|T_2|\psi^1\rangle=\int d^3k_2d^3q_2d^3k_2'\psi(-\tfrac{1}{2}\vec{k}_2-\tfrac{3}{4}\vec{q}_2,\vec{k}_2-\tfrac{1}{2}\vec{q}_2)t_2(\vec{k}_2,\vec{k}_2',E_0-\tfrac{3}{4}q_2{}^2/2\mu)\psi(-\tfrac{1}{2}\vec{k}_2-\tfrac{3}{4}\vec{q}_2,\vec{k}_2'-\tfrac{1}{2}\vec{q}_2).\tag{3.10}$$

If we let

$$\vec{x}=\tfrac{1}{2}\vec{q}_2,\quad \vec{y}=\vec{k}_2-\vec{k}_2',\quad \vec{z}=\vec{k}_2+\vec{k}_2',\tag{3.11}$$

and use the fact that

$$\psi(\vec{k},\vec{q})=\psi(-\vec{k},\vec{q}),\tag{3.12}$$

we get

$$\langle\psi^1|T_2|\psi^1\rangle=\int d^3x\,d^3y\,d^3z\,\psi(\tfrac{3}{2}\vec{x}+\tfrac{1}{4}\vec{y}+\tfrac{1}{4}\vec{z},-\vec{x}+\tfrac{1}{2}\vec{y}+\tfrac{1}{2}\vec{z})\times t_p(\tfrac{1}{2}\vec{y}+\tfrac{1}{2}\vec{z},-\tfrac{1}{2}\vec{y}+\tfrac{1}{2}\vec{z},E_0-\tfrac{3}{2}x^2/\mu)\psi(\tfrac{3}{2}\vec{x}-\tfrac{1}{4}\vec{y}-\tfrac{1}{4}\vec{z},-\vec{x}-\tfrac{1}{2}\vec{y}+\tfrac{1}{2}\vec{z}),\tag{3.13}$$

with an analogous expansion for $\langle\psi^1|(\partial T_2/\partial E)_{E=E_0}|\psi^1\rangle$. Similarly we have

$$\langle\psi^1|\psi^3\rangle=(2\pi)^3\int d^3y\,d^3z\,\psi(\vec{z}+\tfrac{1}{2}\vec{y},\vec{y})\psi(\vec{y}+\tfrac{1}{2}\vec{z},\vec{z}).\tag{3.14}$$

The nine-dimensional integral can be reduced to six dimensions by noting that the integrand depends only on $x$, $y$, $z$, $\vec{x}\cdot\vec{y}$, $\vec{x}\cdot\vec{z}$, and $\vec{y}\cdot\vec{z}$. Similarly the six-dimensional integral can be reduced to three dimensions as the integrand depends only on $y$, $z$, and $\vec{y}\cdot\vec{z}$. Note that we are not restricting the perturbing part of the $T$ matrix to act in only the $s$ wave, and thus the integral will necessarily be more cumbersome than Fuda's.

We now let $t_p=t_c$, the Coulomb $T$ matrix. For $t_c$ and $\partial t_c/\partial E$ we could use an integral representation of the Coulomb $T$ matrix[7] given by

$$t_c(\vec{k}_1,\vec{k}_2,E)=4\pi V_0[1+I(x)]/(\vec{k}_2-\vec{k}_1)^2,\tag{3.15}$$

where

$$I(x)=-4a\int_0^1\frac{t^a dt}{(1+t)^2-x(1-t)^2},\tag{3.16}$$

$$a=\frac{\mu V_0}{(-2\mu E)^{1/2}},\quad x=1-\frac{(k_2{}^2-2\mu E)(k_1{}^2-2\mu E)}{-2\mu E(\vec{k}_2-\vec{k}_1)^2}.$$

For identical particles of charge $q$, $V_0=q^2$, and the Coulomb $T$ matrix has no poles as a function of $E$. There are, however, singularities at $\vec{k}_1=\vec{k}_2$ and at $E=k_1{}^2/2\mu$ and $E=k_2{}^2/2\mu$. The latter singularities do not occur in Eq. (3.13), since $E=E_0-\frac{3}{2}x^2/\mu$ is always below threshold and the singularities at $\vec{k}_1=\vec{k}_2$ are smoothed over in the integration.[8]

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