Equation (9) allows us to evaluate the $\pi\pi\sigma$ scattering amplitude in tree-graph approximation:

$$M(\sigma(k)\to\pi(q)+\pi(p))=M(k^2,p^2,q^2)=M(k^2,q^2,p^2)$$
$$=g[(d-1)(k^2-p^2-q^2)+d(\mu_\pi{}^2-k^2)]. \quad (10)$$

For on-shell pions we obtain Ellis's result[7]: $M(k^2,\mu_\pi{}^2,\mu_\pi{}^2)=g[(2-d)\mu_\pi{}^2-k^2]$. For off-shell pions, our amplitude differs from Ellis's because our assumption of PCAC relates our pion fields to his by the canonical transformation $e^{-dg\sigma}$. Our PCAC assumption also requires that our amplitude M satisfy

$$M(\mu_\pi{}^2,\mu_\pi{}^2,\mu_\pi{}^2)M(\mu_\pi{}^2,0,\mu_\pi{}^2)=0, \quad (11)$$

since the left-hand side of this equation is just the coupling of the σ pole in the $\pi\pi$ scattering amplitude at the Adler point.[8] Indeed, from Eq. (10) we see that

$$M(\mu_\pi{}^2,\mu_\pi{}^2,\mu_\pi{}^2)=g\mu_\pi{}^2(1-d)$$

and

$$M(\mu_\pi{}^2,\mu_\pi{}^2,0)=0.$$

If we demand that $M(k^2,p^2,q^2)$ be as smoothly varying as possible, i.e.,

$$\frac{\partial M}{\partial p^2}=\frac{\partial M}{\partial q^2}=g(1-d)=0,$$

then we find that $d=1$.[9] This value of d also produces a second-order zero in the coupling of the σ pole in the π-π amplitude at the Adler point.

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[1]Steven Weinberg, Phys. Rev. 166, 1568 (1968).

[2]Alex Hankey, Phys. Rev. D 3, 2543 (1971).

[3]Abdus Salam and J. Strathdee, Phys. Rev. 184, 1760 (1969).

[4]All commutators of fields with D and K^μ are evaluated at $x=0$.

[5]John Ellis, CERN Report No. CERN-TH-1245, 1970 (unpublished).

[6]C. G. Callan, Jr., S. Coleman, and R. Jackiw, Ann. Phys. (N. Y.) 59, 42 (1970).

[7]John Ellis, Nucl. Phys. B22, 478 (1970).

[8]Stephen L. Adler, Phys. Rev. 137, B1022 (1965).

[9]This value of d was obtained by Jackiw by a similar argument. See R. Jackiw, Phys. Rev. D 3, 1347 (1971).

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Analytic Properties of Three-Particle Amplitudes in the Scattering-Angle Variable*

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The elastic amplitude describing a scattering of a free particle on a two-particle bound state is here considered as a function of $\cos\theta$, where θ is the scattering angle. The two-body interactions were taken to be s-wave and separable. Our results are: (1) The on-shell amplitude is analytic inside a Lehmann ellipse for all real energies, except for a finite interval on the negative E axis, and (2) for negative energy, the off-shell amplitude is analytic in the whole $\cos\theta$ plane, apart from possible real left and right cuts, and in the Lehmann ellipse. Its asymptotic behavior in $\cos\theta$ is determined by the leading Regge trajectory.

I. INTRODUCTION

The study of multiparticle scattering amplitudes is essential not only for the calculation of cross sections, but also for the establishment of complete S-matrix theory and dispersion relations.

In nonrelativistic potential scattering, dispersion relations in the total energy for fixed directions of the individual momenta have been proved for three-particle amplitudes.[1]

Analytic properties in the scattering-angle variable for the three-particle amplitude were studied by Immirzi,[2] and by Hartle and Sugar.[3] Immirzi, using the invariance of the three-particle Green's function under rotation, has established a Lehmann ellipse in the $\cos\theta$ plane, in which the off-shell elastic amplitude describing a scattering of a particle off a two-particle bound state is analytic for all values of the total energy E. Hartle and Sugar have generalized Immirzi's result, and have also established the analyticity properties of the amplitude in the finite $\cos\theta$ plane, provided $E<0$. In both Refs. 2 and 3 the two-particle interactions were taken to be Yukawian.

In this paper we discuss analyticity in $\cos\theta$, for the three-particle amplitude for which the two-body interaction is separable and of s-wave type. There are two advantages in treating this kind of amplitude. Firstly, this amplitude satisfies an integral equation,[4] and hence the mathematical problem concentrates on the study of the kernel and the inhomogeneous term. Secondly, it is possible in this model to obtain some information about the asymptotic behavior of the amplitude in the $\cos\theta$ plane. This knowledge is important for the construction of the Mandelstam representation.

Our technique will be elementary; we will obtain a bound on the partial-wave amplitude for large l, where l is the angular momentum of the three-particle system. This estimation will be used to obtain the regions in which the partial-wave expansion and the Sommerfeld-Watson transform are absolutely convergent.

In obtaining the Lehmann ellipse, no continuation to complex l is needed, and the only tool used is the estimation of partial-wave amplitudes for large integer l. On the other hand, one should continue partial-wave amplitudes to complex l when trying to construct analytic properties in the whole $\cos\theta$ plane. This problem was first discussed by Aaron and Teplitz[5] when obtaining Regge trajectories for negative energy. Besides being faced here with the problem of unique continuation permitted by Carlson's theorem, there is a pathological $(-1)^{l+1}$ factor appearing in the Born term if exchange potentials are present.

It was shown by Drummond[6] and by Anselm *et al.*[7] that in the absence of special features one cannot define signatured two-body amplitudes that satisfy a unitary condition in which the intermediate states contain more than two particles. On the other hand, Aaron and Teplitz have shown that in the separable approach, this problem is solvable. They have defined signatured amplitudes, and have shown that they satisfy the same integral equations as the nonsignatured ones, but with signatured Born term and kernel. In obtaining the large-l behavior of partial-wave amplitudes, we will follow their technique closely.

Our results may be summarized as follows. (a) The on-shell amplitude is analytic in $\cos\theta$ inside a certain ellipse for all real E, except for a finite interval on the negative E axis. (b) The off-shell amplitude is analytic in the whole $\cos\theta$ plane – except for real cuts – and inside a certain ellipse, for all $E<0$. (c) For $E<0$, the asymptotic behavior of the amplitude in $\cos\theta$ is determined by the leading Regge trajectory.

Two results of this paper are therefore claimed beyond what is already contained in Ref. 3. Firstly, we obtain analytic properties for the on-shell amplitude for $E<0$, while there they are found for the off-shell one, and secondly, we get some knowledge about the asymptotic behavior in the $\cos\theta$ plane, although this is done only for large negative E.

In Sec. II we describe the model and write the integral equations for the total and partial-wave amplitudes, while the asymptotic form of the latter for large l is derived in Sec. III.

The analytic properties in $\cos\theta$ are derived in Sec. IV, and some conclusions are drawn in Sec. V. Two problems of mathematical technique are discussed in Appendices A and B.

II. DESCRIPTION OF THE MODEL

The model which we consider deals with three identical, nonrelativistic spinless particles with unit mass, interacting through a separable s-wave potential. Because of the interaction, a bound state (denoted by d, with binding energy α^2) of the two-particle subsystem may be produced. Our aim is to investigate the elastic amplitude describing the scattering of a free particle (denoted by n) on the d bound state, i.e., the process

$$n(\vec{k})+d(-\vec{k})\rightarrow n(\vec{k}')+d(-\vec{k}'), \tag{2.1}$$

where $\vec{k}$ ($\vec{k}'$) is the momentum of n before (after) the collision. This amplitude will be denoted by $T(\vec{k},\vec{k}',E)$, in which E is the c.m. energy of the three-particle system. The on-shell condition relates the magnitudes of $\vec{k}$ and $\vec{k}'$ to the c.m. energy through the relation (with $\hbar=1$)

$$k^2=k'^2=\tfrac{4}{3}(E+\alpha^2), \tag{2.2}$$

which shows that below threshold ($E<-\alpha^2$) the on-shell momenta are purely complex. For s-wave interaction, the physical amplitudes are functions of E and

$$\cos\theta=\hat{k}\cdot\hat{k}', \tag{2.3}$$

that is,

$$T(E,\cos\theta) \equiv T(\hat{k}[\tfrac{4}{3}(E+\alpha^2)]^{1/2},\hat{k}'[\tfrac{4}{3}(E+\alpha^2)]^{1/2},E+i0). \tag{2.4}$$

The separability of the interaction in momentum space implies the representation

$$v(\vec{q},\vec{q}')=\lambda g^*(\vec{q})g(\vec{q}'), \tag{2.5}$$

in which $\vec{q}$ ($\vec{q}'$) are relative momenta of the two-particle subsystem before (after) interacting, λ

is a strength parameter (determined by the requirement that the bound state be present at the binding energy α^2), and $g(\vec{q})$ is the d-state form factor. In the case of s-wave interaction only, $g(\vec{q})$ depends on q^2 alone and we have chosen it to be of Hulthén type:

$$g(\vec{q}) = N/(q^2+\beta^2), \tag{2.6}$$

where β is a constant giving the d state a composite structure, and N is determined by the normalization of the d wave function.

The model discussed here is obtained also as a special case of the Amado model,[8] in which all wave-function renormalization constants are zero.

In any case, $T(\vec{k},\vec{k}',E)$ is the solution of the integral equation

$$T(\vec{k},\vec{k}',E) = B(\vec{k},\vec{k}',E) + \int B(\vec{k},\vec{k}'',E)\tau(E-\tfrac{3}{4}k''^2)T(\vec{k}'',\vec{k}',E)d\vec{k}'', \tag{2.7}$$

where the Born term $B(\vec{k},\vec{k}',E)$ is given by

$$B(\vec{k},\vec{k}',E) = \frac{g^*(\vec{k}'-\frac{1}{2}\vec{k})g(\frac{1}{2}\vec{k}'-\vec{k})}{E-\frac{1}{2}[k^2+k'^2+(\vec{k}-\vec{k}')^2]} \tag{2.8}$$

and the propagator τ can be shown[4] to be given by

$$\tau(x) = \frac{-4\beta^2}{\pi N^2}\,\frac{(\beta+\alpha)^2(\beta+ix^{1/2})(\alpha+ix^{1/2})}{(x+\alpha^2)(ix^{1/2}+\alpha+2\beta)}\,. \tag{2.9}$$

It has a simple pole at $x=-\alpha^2$ (because of the presence of bound states at that point) and a cut from 0 to ∞ as a result of the possibility of the d breakup.

One now defines off- and on-shell partial-wave amplitudes by the decomposition

$$T(\vec{k},\vec{k}',E) = \sum_{l=0}^{\infty}(2l+1)T_l(k,k',E)P_l(\cos\theta), \tag{2.10a}$$

$$T(E,\cos\theta) = \sum_{l=0}^{\infty}(2l+1)T_l(E)P_l(\cos\theta), \tag{2.10b}$$

and, of course,

$$T_l(E) = T_l([\tfrac{4}{3}(E+\alpha^2)]^{1/2},[\tfrac{4}{3}(E+\alpha^2)]^{1/2},E+i0). \tag{2.11}$$

In the same way, one defines off- and on-shell partial-wave Born terms $B_l(k,k',E)$ and $B_l(E)$, respectively.

For a pure exchange potential, we have

$$B_l(k,k',E) = \frac{N^2}{(kk')^3}(-1)^{l+1}\left(\frac{Q_l(A)}{(B-A)(C-A)}+\frac{Q_l(B)}{(A-B)(C-B)}+\frac{Q_l(C)}{(A-C)(B-C)}\right), \tag{2.12}$$

where

$$A=\frac{k^2+k'^2-E}{kk'},\quad B=\frac{k^2+\frac{1}{4}k'^2+\beta^2}{kk'},\quad C=\frac{\frac{1}{4}k^2+k'^2+\beta^2}{kk'}. \tag{2.13}$$

B_l is not singular at $kk'=0$ or when $A=B$, $B=C$, or $C=A$.

If $k=k'$, as is the case for $B_l(E)$, then $B=C$ and Eq. (2.12) is replaced by

$$B_l(k,k,E) = \frac{N^2(-1)^{l+1}}{k^6(B-A)^2}\left(Q_l(A)-Q_l(B)+\frac{A-B}{B^2-1}\,l[BQ_l(B)-Q_{l-1}(B)]\right), \tag{2.14}$$

where a contiguous relation has been used in order to express the derivative of Q_l.

From $B_l(k,k',E)$, we define the signatured Born term

$$B_l^{\pm}(k,k',E) = \mp(-1)^{l+1}B_l(k,k',E), \tag{2.15}$$

which is free of the $(-1)^l$ factor. The signatured amplitude was shown in Ref. 5 to satisfy the integral equation

$$T_l^{\pm}(k,k',E) = B_l^{\pm}(k,k',E) + \int_0^{\infty}B_l^{\pm}(k,k'',E)\tau(E-\tfrac{3}{4}k''^2)T_l^{\pm}(k'',k',E)k''^2dk''. \tag{2.16}$$

Equation (2.16) for $E>0$ is solved in two steps.[9] First, the quantity $T_l^{\pm}(ke^{-i\varphi},k',E)$, in which $k'=[\frac{4}{3}(E+\alpha^2)]^{1/2}$, is found by the integral equation

$$T_l^{\pm}(ke^{-i\varphi},k',E) = B_l^{\pm}(ke^{-i\varphi},k',E) + \int_0^{\infty}B_l^{\pm}(ke^{-i\varphi},k''e^{-i\varphi},E)\tau(E-\tfrac{3}{4}k''^2e^{-2i\varphi})T_l^{\pm}(k''e^{-i\varphi},k',E)k''^2dk''e^{-3i\varphi}. \tag{2.17}$$

The angle φ is chosen such that $B_l^{\pm}(ke^{-i\varphi},k',E)$ will be regular for $0<k<\infty$. It is not difficult to prove that Eq. (2.17) is a Fredholm equation for integer l. Second, the solution of Eq. (2.17) is used to obtain the physical amplitude by the quadrature

$$T_l^{\pm}(k,k',E) = B_l^{\pm}(k,k',E) + e^{-3i\varphi}\int_0^{\infty}B_l^{\pm}(k,k''e^{-i\varphi},E)\tau(E-\tfrac{3}{4}k''^2e^{-2i\varphi})T_l^{\pm}(k''e^{-i\varphi},k',E)k''^2dk'', \tag{2.18}$$

in which k also is put on-shell.

The solution of Eq. (2.16) for $E<0$ and real momenta is much simpler. In the range $-\alpha^2<E<0$ the only singularities are those of the propagator, and one can remove them either by the contour-deformation technique discussed above or by transforming the equation into a nonsingular one.[10] As for $E<-\alpha^2$, Eq. (2.16) is regular.

III. THE LARGE-l BEHAVIOR OF THE PARTIAL-WAVE AMPLITUDE

In this section we will determine the large-l behavior of $T_l^{\pm}(k,k',E)$. The discussion will be divided into two parts, according to the possible values of E, l, and the momenta. (The signature subscript will be dropped where no confusion may arise.)

A. $E>0$, l = Integer, and On-Shell Momenta

For $z\notin[-1,1]$ and integer l, one has[11]

$$Q_l(z)_{l\,\widetilde{\to}\,\infty}\,[z+(z^2-1)^{1/2}]^{-l-1/2}l^{-1/2}, \tag{3.1a}$$

$$Q_l'(z)\sim l\,Q_l(z). \tag{3.1b}$$

As long as we remain on the first sheet of the function $w=z+(z^2-1)^{1/2}$, we have $|w|>1$, as discussed in Appendix A. Consider now the Schmidt norm of the kernel:

$$M_l(k,k'',E)=B_l(ke^{-i\varphi},k''e^{-i\varphi},E)\times\tau(E-\tfrac{3}{4}k''^2e^{-2i\varphi})k''^2e^{-2i\varphi}, \tag{3.2}$$

$$I_l(E)\equiv\int_0^\infty\int_0^\infty|M_l(k,k'',E)|^2dkdk''. \tag{3.3}$$

Since Eq. (2.16) is a Fredholm equation, the integral is (absolutely) convergent, and we may find the limit of $I_l(E)$ for $l\to\infty$ by integrating the limit of the integrand. From Eq. (3.1) we get

$$\lim_{l\to\infty}I_l(E)=0. \tag{3.4}$$

This implies that the large-l behavior of $T_l(ke^{-i\varphi},k',E)$ is determined by that of $B_l(ke^{-i\varphi},k',E)$. When $T_l(ke^{-i\varphi},k',E)$ is put into Eq. (2.18) and the limit $l\to\infty$ is taken, the integral vanishes faster than $B_l(k,k',E)$. Consequently, $T_l(E)$ has the asymptotic behavior of $B_l(E)$, namely,

$$T_l(E)_{l\,\widetilde{\to}\,\infty}\,B_l(E),\qquad l=\text{integer},\quad E>0. \tag{3.5}$$

B. $E<0$, $\mathrm{Re}(l)\geq-\frac{1}{2}$, and Real Momenta

This case was discussed by Aaron and Teplitz in Ref. 5. Equation (2.16) remains Fredholm for all complex l such that $\mathrm{Re}(l)\geq-\frac{1}{2}$, and the kernel as well as the inhomogeneous term are analytic in l. Therefore, the solution $T_l(k,k',E)$ is analytic in l. As for the asymptotic behavior, they have proved[12]

$$T_l(k,k',E)_{l\,\widetilde{\to}\,\infty}\,B_l(k,k',E),\qquad \mathrm{Re}(l)\geq-\tfrac{1}{2},\quad E<0,\quad k,k'\ \text{real}. \tag{3.6}$$

Note that in the range $-\alpha^2<E<0$, the results are applied also for the on-shell amplitude as a special case.

The case in which $E<-\alpha^2$ and the momenta are on-shell (i.e., complex) is discussed in Appendix B, in which it is shown again that

$$T_l(E)_{l\,\widetilde{\to}\,\infty}\,B_l(E), \tag{3.7}$$

l = integer, $-\frac{4}{3}\alpha^2<E<-\alpha^2$, and $E<E_0=-3\beta^2-\alpha^2$.

For $B_l(k,k',E)$ we have, according to Eqs. (2.12), (2.14), and (3.1),

$$B_l(k,k',E)_{l\,\widetilde{\to}\,\infty}\,l^{-1/2}e^{-\gamma(l+1/2)},\quad k\neq k' \tag{3.8a}$$

$$B_l(k,k,E)_{l\,\widetilde{\to}\,\infty}\,l^{1/2}e^{-\gamma(l+1/2)}, \tag{3.8b}$$

where

$$\gamma=\min_{A,B,C}\ \mathrm{arc\,cosh}(A,B,C). \tag{3.9}$$

IV. ANALYTICITY IN THE $\cos\theta$ PLANE

We now arrive at the goal of our discussion, namely, the derivation of the analytic properties of $T(\vec{k},\vec{k}',E)$ in the variable $z=\cos\theta$. The technique we shall adopt here is identical to that used in the two-body potential scattering.[13]

Consider first Eq. (2.10b), for $E>0$. At large integer l one has

$$|P_l(\cos\theta)|<\frac{c\exp[(l+\frac{1}{2})\mathrm{Im}\theta]}{|\sin\theta|^{1/2}|l+\frac{1}{2}|^{1/2}}$$

and therefore, by (3.8b), the summand is bounded by

$$\frac{c\exp[(l+\frac{1}{2})(\mathrm{Im}\theta-\gamma)]}{|\sin\theta|^{1/2}}|2l+1|. \tag{4.1}$$

Thus, the partial-wave expansion for $T(\vec{k},\vec{k}',E)$ is absolutely convergent provided $\mathrm{Im}\theta<\gamma$, i.e., within an ellipse (in the $\cos\theta$ plane) whose semimajor axis is equal to $\cosh\gamma$.

From Eq. (2.13) we have for on-shell momenta

$$A=\frac{5}{4}+\frac{3\alpha^2}{4(E+\alpha^2)},\quad B=C=\frac{5}{4}+\frac{3\beta^2}{4(E+\alpha^2)}. \tag{4.2}$$

As long as E does not belong to the interval

$$\Delta\equiv[-3\beta^2-\alpha^2,-\tfrac{4}{3}\alpha^2] \tag{4.3}$$

[which exhibits the left cut of $T_l(E)$ in the E plane], one has $|A|,|B|>1$ and the semimajor axis of the

Lehmann ellipse is given by

$$a = \min(|A|, |B|). \tag{4.4}$$

Along the same line, we arrive at the conclusion that $T(\vec{k}, \vec{k}', E)$ for negative E and arbitrary real momenta is analytic in $\cos\theta$ inside a Lehmann ellipse with semimajor axis given by

$$a = \min(|A|, |B|, |C|). \tag{4.5}$$

For this $T(\vec{k}, \vec{k}', E)$ we will now find the analytic properties in the whole $\cos\theta$ plane. This is done by considering the integral

$$H = \frac{1}{i}\int_\Gamma \frac{T(\lambda, k, k', E)}{\cos\pi\lambda} P_{\lambda-1/2}(-\cos\theta)\lambda d\lambda, \tag{4.6}$$

where $T(\lambda, k, k', E) = T_{l+1/2}(k, k', E)$ and, as usual, the contour Γ encircles clockwise all physical points but not the Regge poles of $T(\lambda, k, k', E)$.

If H is convergent, the sum of the residues gives us $T(\vec{k}, \vec{k}', E)$. We want now to deform the contour Γ into the imaginary λ axis. The Regge poles, if they exist, will contribute real cuts in $\cos\theta$, whose branch points lie outside the Lehmann ellipse. It remains therefore to investigate the background integral.[14]

From the bound

$$\left|\frac{\lambda P_{\lambda-1/2}(-\cos\theta)}{\cos\pi\lambda}\right| \leq C|\sin\theta|^{-1/2}|\lambda|^{1/2} \times \exp[-|\mathrm{Re}\theta\,\mathrm{Im}\lambda| + \mathrm{Im}\theta\mathrm{Re}\lambda],$$

it follows from expressions (3.8) that the integrand in (4.6) is bounded by

$$C|\lambda|\exp[-|\mathrm{Re}\theta\,\mathrm{Im}\lambda| + (\mathrm{Im}\theta - \gamma)\mathrm{Re}\lambda].$$

It is therefore permissible to deform the contour of integration so that Γ becomes the λ imaginary axis, on which the integrand is bounded by

$$C|\lambda|e^{-|\mathrm{Re}\theta\mathrm{Im}\lambda|}.$$

Provided $\mathrm{Re}\theta \neq 0$, the integral for H is absolutely convergent and represents an analytic function of $\cos\theta$, with possible cuts on the ray $1 \leq \cos\theta < \infty$.

Since we have already established analyticity inside the Lehmann ellipse, the possible cuts actually lie on the ray

$$\gamma \leq \cos\theta < \infty. \tag{4.7}$$

The nonsignatured amplitude is obtained from the signatured one by the relation

$$T(\cos\theta) = \tfrac{1}{2}[T^+(\cos\theta) + T^+(-\cos\theta) + T^-(\cos\theta) - T^-(-\cos\theta)], \tag{4.8}$$

and may therefore have additional cuts at

$$-\infty < \cos\theta \leq -\gamma. \tag{4.9}$$

This cut is a direct consequence of the exchange nature of the potential in Eq. (2.12).

The above derivation predicts the cuts in $\cos\theta$, but tells nothing about the location of the branch points. However, these points may be found by brute force, on iterating Eq. (2.7). We do not enter into the detailed calculations, since a similar technique has been used by Hartle and Sugar.[3] It can be shown that each term in the iteration series contributes a branch point whose distance from the physical region is larger than that of the previous term. The first singularity is therefore the nearest pole of $B(\vec{k}, \vec{k}', E)$.

V. CONCLUSIONS

For the on-shell amplitude, we have established a Lehmann ellipse in the $\cos\theta$ plane in which it is analytic. The foci of the ellipse are at ± 1, and the semimajor axis is given by

$$a = \min(|A|, |B|),$$

where A and B are given in Eqs. (4.2). There is no Lehmann ellipse for $E \in \Delta$, where Δ is given in (4.3).

The off-shell amplitude $T(\vec{k}, \vec{k}', E)$ for $E < 0$ is an analytic function of $\cos\theta$ in the whole $\cos\theta$ plane – except for real positive and negative cuts – and inside the Lehmann ellipse.

An obvious advantage of our method over the others is the possibility of establishing the asymptotic behavior in $\cos\theta$ once the leading Regge trajectory is known. For example, it was shown by Aaron and Teplitz (who used a method suggested by Tiktopoulos[15]) that the integration contour Γ in Eq. (4.6) may be deformed to be the line $\mathrm{Re}\lambda = -n$ for any positive integer n. Since the leading Regge trajectory approaches -3 as $E \to -\infty$, this implies a $(\cos\theta)^{-3}$ behavior for large $\cos\theta$ and large negative E. The difference between our results and those of Refs. 2 and 3 is due to the special feature of the separable approach, in which the scattering off a bound state is treated very much as a two-body problem would be. The number $\frac{5}{4}$ appearing in the equation for semimajor axis results from the inequality of the n and the d masses. However, the main difficulty, which appears both here and in Ref. 3, is the impossibility of deriving analyticity in $\cos\theta$ in the whole $\cos\theta$ plane for $E > 0$. This is disappointing, since otherwise one could prove a Mandelstam representation for the three-body amplitude. The intractability of three-body problems above the breakup threshold is clearly model-independent.

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APPENDIX A: THE FUNCTION $w=z+(z^2-1)^{1/2}$

In two-body potential scattering, one encounters the function $w=z+(z^2-1)^{1/2}$, in which z is real and greater than 1. For example, in the case of the Yukawa potential, z is equal to $1+\mu^2/2k^2$. In order to investigate w for complex z, it is convenient to explore the properties of the inverse function (the so-called Zukovski's function[16])

$$z=\tfrac{1}{2}(w+1/w). \tag{A1}$$

Setting $w=re^{i\theta}z=u+iv$, one has

$$\frac{u^2}{[\frac{1}{2}(r+1/r)]^2}+\frac{v^2}{[\frac{1}{2}(r-1/r)]^2}=1. \tag{A2}$$

The circle $r=c$ is therefore mapped on an ellipse with semimajor axis $\frac{1}{2}(r+r^{-1})$ and with foci at ± 1. In particular, the circle $|w|=1$ is mapped on the segment $[-1,1]$.

It is easy to see that the unit circle $|w|<1$ is mapped on the whole z plane less $[-1,1]$, and since $z(w)=z(1/w)$, the domain $|w|>1$ is mapped in the same manner. As for the function w, it has two sheets connected by the interval $[-1,1]$. The first sheet is mapped on the domain $|w|>1$, while the second one is mapped on $|w|<1$. In our calculations z always lies on the first sheet, and hence we have $|w|>1$.

APPENDIX B: THE ON-SHELL AMPLITUDE BELOW THRESHOLD

In this Appendix we will prove that at large (integral) l, the behavior of $T_l(E)$ for $-\frac{4}{3}\alpha^2<E<-\alpha^2$ and $E<-3\beta^2-\alpha^2$ is determined by that of $B_l(E)$.

It turns out that the region $-\frac{4}{3}\alpha^2<E<-\alpha^2$ exhibits no problem, since one can solve Eq. (2.16) for real positive k, and then integrate again to get the amplitude for on-shell k. The proof of the asymptotic behavior is the same as the one carried out in Sec. III for $E>0$. Note, however, that one cannot use this argument for complex l since the argument A defined in Eq. (4.2) is negative, and one needs the formula

$$Q_l(-z)=-e^{\pm il\pi}Q_l(z),$$

which holds only for integer l. The situation for

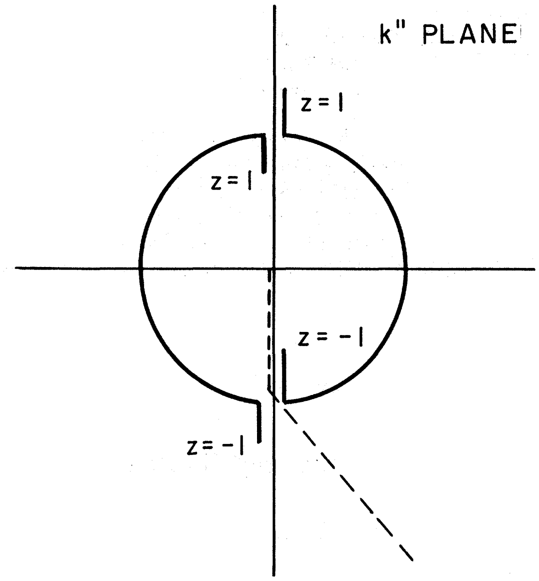

FIG. 1. The cut of the half-on-shell Born term $B_l(k,k'',E)$ in the k'' plane for $E<-3\beta^2-\alpha^2$ (solid line), and the integration contour of Eq. (2.17) which avoids this cut (dashed line).

$E<-3\beta^2-\alpha^2$ is more complicated. For example, for

$$z=\frac{k^2+\frac{4}{3}\alpha^2+\frac{1}{3}E}{k[\frac{4}{3}(E+\alpha^2)]^{1/2}} \tag{B1}$$

consider the function $Q_l(z)$ appearing in the half-on-shell Born term. In the k plane, $Q_l(z)$ has two cuts, given by

$$k=(1/\sqrt{3})\{(E+\alpha^2)^{1/2}z\pm[E(z-1)+\alpha^2(z-4)]^{1/2}\}, \tag{B2}$$

where z is in the range $[-1,1]$. These two cuts are shown in Fig. 1.

It is therefore necessary to replace the contour of integration in Eq. (2.17) with a different contour, which is also shown in Fig. 1, and avoid these cuts. Observe that unlike the ray $ke^{-i\varphi}$, this path is dependent on the value of E; but the problem is solved and one again arrives at the conclusion that

$$T_l(E)\underset{l\to\infty}{\simeq}B_l(E),\quad E<-3\beta^2-\alpha^2,\quad l=\text{integer}. \tag{B3}$$

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[1] M. H. Rubin, R. L. Sugar, and G. Tiktopoulos, Phys. Rev. 146, 1130 (1966); 159, 1348 (1967); 162, 1555 (1967).

[2] G. Immirzi, Nuovo Cimento 34, 1361 (1964).

[3] J. B. Hartle and R. L. Sugar, Phys. Rev. 169, 1104 (1968).

[4] A. N. Mitra, Nucl. Phys. 32, 529 (1962); C. A. Lovelace, Phys. Rev. 135, B1225 (1964).

[5] R. Aaron and Vigdor L. Teplitz, Phys. Rev. 167, 1284 (1967).

[6] I. Drummond, Phys. Rev. 140, B1368 (1965).

[7] A. Anselm *et al.*, Ann. Phys. (N. Y.) 37, 227 (1966).

[8] R. D. Amado, Phys. Rev. 132, 485 (1963).

[9] J. H. Hetherington and L. H. Schick, Phys. Rev. 135,

B935 (1965).

[10]Y. Avishai, Phys. Rev. D 3, 3232 (1971).

[11]Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 2, p. 162.

[12]For complex l, $I_l(E)$ approaches a constant as $|l|\to\infty$. Hence, one needs the assumption that Eq. (2.18) does not have an eigenvalue $\lambda=1$ for $|l|\to\infty$.

[13]V. De Alfaro and T. Regge, *Potential Scattering* (North-Holland, Amsterdam, 1965), Chap. 9.

[14]One needs, of course, to assume that the sum of the Regge poles is convergent. For sufficiently large and negative E, there are no Regge poles to the right of $\mathrm{Re}\,\lambda=0$, since it was noticed by S. Mandelstam [Lawrence Radiation Laboratory Report No. UCRL-17250 (unpublished)] and proved also by Aaron and Teplitz (Ref. 5) that the leading Regge trajectory approaches -3 asymptotically as $E\to-\infty$.

[15]G. Tiktopoulos, Phys. Rev. 133, 1231 (1964).

[16]Z. Nehari, *Conformal Mapping* (McGraw-Hill, New York, 1952), p. 269.

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Allowed Domains in Theories of Broken Chiral Symmetry*

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We consider theories in which the explicit breaking of the chiral symmetry $SU_3\times SU_3$ of the energy density Θ_{00} is of the form $\epsilon_0 u_0+\epsilon_8 u_8$, where ϵ_0, ϵ_8 are real constants and u_0 and u_8 are scalar densities in the representation $(3,\overline{3})+(\overline{3},3)$. We propose an extension of this picture, in which the $SU_3\times SU_3$-invariant part of Θ_{00} has the form $\overline{\Theta}_{00}+\epsilon_9 u_9$, where $\overline{\Theta}_{00}$ is $U_3\times U_3$ invariant and the $SU_3\times SU_3$-invariant scalar u_9 breaks $U_3\times U_3$ in a specific way. Specifically, it is assumed that the ninth axial charge F_0^5 transforms u_9 into a pseudoscalar v_9 according to $i[F_0^5,u_9]=\kappa v_9$, where $i[F_0^5,v_9]=-\kappa u_9$. (The parameter κ labels the representation.) Given this group structure, the analysis of Okubo and Mathur may be extended to find allowed domains for the symmetry-breaking parameters ϵ_0, ϵ_8, ϵ_9 and the vacuum expectation values $\langle u_0\rangle$, $\langle u_8\rangle$, and $\langle u_9\rangle$. The positivity conditions now restrict $\langle u_9\rangle$ so that the allowed domains occupy certain volumes in a three-dimensional space.

I. INTRODUCTION

Many authors have attempted to understand how the approximate hadron symmetry $SU_3\times SU_3$ is broken. In addition to the conventional problem of assigning correct group properties to operators which explicitly violate the symmetry, there is the problem of describing the "spontaneous breakdown" of the symmetry. Apparently the dynamics underlying the low-mass spectrum is such that even when the explicit symmetry-breaking terms are removed, the solutions do not belong to representations of the chiral group, but rather to some subgroup, usually supposed to be SU_3. In this picture the vacuum is not chiral invariant even in the symmetry limit and the associated massless pseudoscalar octet is the principal manifestation of the symmetry.

Perhaps the most promising model[1-3] (rather, class of models) ascribes the explicit symmetry breaking to scalar components (u_0 and u_8) of the representation $(3,\overline{3})+(\overline{3},3)$. Introducing coupling parameters ϵ_0 and ϵ_8, we may write the energy density $\Theta_{00}(x)$ in the form

$$\Theta_{00}(x)=\hat{\Theta}_{00}(x)+\epsilon_0 u_0+\epsilon_8 u_8, \tag{1.1}$$

where $\hat{\Theta}_{00}(x)$ is invariant under $SU_3\times SU_3$. The representation $(3,\overline{3})+(\overline{3},3)$ is usefully described by the nonet of Hermitian scalar densities u_i and pseudoscalar densities v_i ($i=0,1,\ldots,8$) which obey the (equal-time) commutation relations

$$\begin{aligned}
[F_i, u_j]&=if_{ijk}u_k,\\
[F_i, v_j]&=if_{ijk}v_k,\\
[F_i^5, u_j]&=-id_{ijk}v_k,\\
[F_i^5, v_j]&=id_{ijk}v_k.
\end{aligned} \tag{1.2}$$

In order to compare the solutions of (1.1) with their counterparts in the symmetric limit, it is necessary to have some understanding of the analytic behavior of the solutions as a function of the parameters ϵ_0 and ϵ_8. In particular, we wish to be able to turn on the symmetry-breaking terms adiabatically without drastic alteration of the nature of the solution in the symmetric limit. As emphasized by Dashen,[4] this really means that we have to study the solution as we *turn off* the symmetry-breaking