156 (1969).

[19]This is related to an interesting analog of the Sugawara model, pointed out in Ref. 21. In a free-quark field theory, one finds that

$$\sum_{a=1}^{8}[J_a^i(\vec{x},0),J_a^j(0)] = -i\times\tfrac{16}{3}\Theta^{ij}(0)\delta(\vec{x}) \text{ plus terms in } \delta^{ij},$$

where $\Theta^{\mu\nu}$ is the fermion stress-energy tensor, and the commutator is taken at equal time. In the Sugawara model, the coefficient $\frac{16}{3}$ is replaced by 3, and $\Theta^{\mu\nu}$ is the full stress-energy tensor. If the equal-τ commutator (4.27) is expanded in powers of $\frac{1}{2}x^3$, and summed over $SU(3)$ indices, the coefficient of $-\frac{1}{4}|\frac{1}{2}x^3|\partial_j\delta(\vec{x}_\perp)$ is $\frac{16}{3}\Theta^{j0}(0)$, plus a term involving the gradient of the axial-vector current. The relationship between the expansion in powers of $\frac{1}{2}x^3$ and equal-time commutators is discussed in Sec. V.

[20]Y. Frishman, Phys. Rev. Letters 25, 966 (1970); G. Altarelli, R. A. Brandt, and G. Preparata, Phys. Rev. Letters 26, 42 (1971).

[21]J. M. Cornwall, Phys. Rev. D 2, 578 (1970).

[22]L. Van Hove, Phys. Letters 24B, 183 (1967).

[23]R. Jackiw and G. Preparata, Phys. Rev. Letters 22, 975 (1969); S. L. Adler and Wu-Ki Tung, *ibid.* 22, 978 (1969).

[24]D. J. Gross and S. Treiman (unpublished report). We thank Dr. Gross for communicating his results to us.

PHYSICAL REVIEW D VOLUME 4, NUMBER 2 15 JULY 1971

Feynman Rules for the Yang-Mills Field: A Canonical Quantization Approach. I*

Rabindra Nath Mohapatra
Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, New York 11790
(Received 3 February 1971)

An attempt has been made to derive covariant Feynman rules for the massless Yang-Mills field, starting with canonical methods of quantization. In this paper we will summarize the techniques involved in such a program, along with a few preliminary results. Working in the radiation gauge ($\partial_i\vec{b}_i=0$), we find that there is an infinity of noncovariant vertices. We obtain a noncovariant set of rules to describe them to any order. Working with the suggested set of rules, we first prove that all tree diagrams can be described by a covariant set of Feynman rules. Secondly, to order g^2, we find that the one-loop diagram can also be made covariant. However, apart from the usual three-vector and four-vector vertices, the covariant loop contains an extra vertex of vector-scalar-scalar type and the scalar loop occurs with a weight factor of -2 with respect to the vector loop.

I. INTRODUCTION

In recent years, considerable attention has been given to the problem of obtaining covariant Feynman rules for the Yang-Mills field.[1] Because the Lagrangian for the massless Yang-Mills field obeys non-Abelian gauge symmetry, canonical methods of quantization are complicated due to the nonlinear nature of the constraints on the independent dynamical variables. Therefore, other less conventional methods were employed in deriving the Feynman rules for the field and accurate rules were suggested, first by Feynman,[2,2a] and later by Fadeev and Popov,[3] Mandelstam,[4] DeWitt,[5] and Fradkin and Tyutin.[6] Massless limits of massive gauge fields have also been studied[7] in this connection, but the resulting rules are found to violate both unitarity and Lorentz invariance and hence are incorrect. The method of canonical quantization, though complicated, is an unambiguous and more conventional procedure, and it serves to elucidate rather clearly the role of constraint equations in the derivation of the rules. The present paper is devoted to a study of this procedure. In this first of a series of papers, we will summarize the techniques involved in such a program, including a treatment of the constraint equations, and we will report a few preliminary results for tree and one-loop diagrams.

We will work in the radiation gauge [$\partial_i b_i^a(x)=0$, i.e., the field is transverse] and first isolate the independent dynamical variables, which will be postulated to satisfy the canonical commutation relations (CCR). In this gauge the interaction Hamiltonian is an infinite series in the coupling constant, each term of the series being noncovariant. Since we are working in a noncovariant gauge, the propagator is also noncovariant and contains the so-called normal-dependent terms.[8] First of all, we will suggest noncovariant rules for tree diagrams. However, we will prove them only up to order g^4 since all the essential elements in the proof are exhausted by fourth order. Going to higher order requires only a very complicated combinatorial analysis. We solve this problem in Appendix B. Using these noncovariant rules, we will show that in the case of tree diagrams to all orders in the coupling constant, the noncovariant terms drop out, giving rise to a covariant set of rules. We work only to order g^2 in the case of one-loop diagrams and demonstrate that in this case also, the noncovariant (or normal-dependent) terms cancel out. The resulting covariant rules for the loop are then found to have an extra vector-scalar-scalar vertex (of $\rho\pi\pi$ type), in addition to the usual triple- and quadruple-vector vertices. The extra vertex occurs only within the loop (the scalar loop) and has a weight factor of -2 relative to the vector loop. This result agrees with those obtained previously by others (see Refs. 2–6), using different methods. Since this procedure seems to work, we would like to extend it to higher orders with more loops in future publications.

In Sec. II, we write down the field equations, isolate the independent dynamical variables, and derive the interaction Hamiltonian in interaction representation. In Sec. III, we suggest the non-covariant rules for the tree diagrams and give a proof of the same to order g^4. Using the suggested rules in Sec. IV, we show that in the case of the tree diagrams, the noncovariant terms can be dropped. In Sec. V we treat the loop diagrams. In Appendix A we present a proof of the classical Hamilton's equations for a Yang-Mills field in the radiation gauge, which does not seem to exist in the literature. In Appendix B we prove the noncovariant set of rules suggested in Sec. III to any order in the case of tree diagrams, using combinatorial methods.

II. INTERACTION HAMILTONIAN IN RADIATION GAUGE

We will work with the SU(2) gauge group, although generalization to an arbitrary group is trivial. The gauge field $\vec{b}_\mu$ in that case is an isovector field and the Lagrangian for the b field only interacting with itself is the following[9]:

$$\mathcal{L} = -\tfrac{1}{4}\vec{f}_{\mu\nu}\cdot\vec{f}_{\mu\nu}, \tag{2.1}$$

where

$$\vec{f}_{\mu\nu} = \partial_\mu\vec{b}_\nu - \partial_\nu\vec{b}_\mu + g\vec{b}_\mu\times\vec{b}_\nu. \tag{2.2}$$

The corresponding gauge transformation for the field is

$$b_\mu^a \to b_\mu^a + \frac{1}{g}\nabla_\mu^{ab}\alpha^b(x), \tag{2.3}$$

where

$$\nabla_\mu^{ab} = \partial_\mu\delta_{ab} + g\epsilon_{acb}b_\mu{}^c \equiv (\partial_\mu + g\vec{b}_\mu\times)_{ab}. \tag{2.4}$$

The field equations are

$$\partial_\mu\vec{f}_{\mu\nu} + g\vec{b}_\mu\times\vec{f}_{\mu\nu} = 0 \tag{2.5}$$

and the canonical momenta are

$$\vec{\pi}_i = \frac{\partial\mathcal{L}}{i\partial(\partial_4\vec{b}_i)} = i\vec{f}_{4i}, \quad \vec{\pi}_4 = 0. \tag{2.6}$$

We will work in the radiation gauge; i.e., the fields satisfy the following condition:

$$\partial_i b_i^a(x) = 0, \quad \text{i.e., } \vec{b}_i \text{ is transverse.} \tag{2.7}$$

Notice that $\vec{\pi}_i$ has both longitudinal and transverse parts; we separate the two parts as follows:

$$\vec{\pi}_i = \vec{\pi}_i^t + \partial_i\vec{\phi}, \tag{2.8}$$

where $\partial_i\vec{\pi}_i^t = 0$ and $\vec{\phi}$ is a scalar under the rotation group. The field equations in (2.5) contain a dynamical part, giving the time evolution of the system and a constraint equation. The constraint equation is

$$\partial_i\vec{\pi}_i = -g\vec{b}_i\times\vec{\pi}_i. \tag{2.9}$$

Using Eq. (2.8), we can rewrite (2.9) as

$$\Delta\vec{\phi} + g\vec{b}_i\times\partial_i\vec{\phi} = -g\vec{b}_i\times\vec{\pi}_i^t, \tag{2.10}$$

where

$$\Delta = \sum_i \partial_i{}^2.$$

From Eq. (2.10), we see clearly that the constraint equation is nonlinear. However, we can write a formal solution for this. For that purpose, we define an operator

$$M \equiv \vec{b}_i\times\partial_i \tag{2.11}$$

and write $\vec{\chi} = -\vec{b}_i\times\vec{\pi}_i^t$. The formal solution for ϕ is

$$\vec{\phi} = \frac{g}{1+g\Delta^{-1}M}\Delta^{-1}\vec{\chi} \tag{2.12}$$

$$\equiv g\sum_{n=0}^{\infty}(-g\Delta^{-1}M)^n\Delta^{-1}\vec{\chi}, \tag{2.13}$$

where we have purposely dropped the space-time dependence of all the functions such as $\vec{\phi}$, $\vec{\chi}$. By now, we have isolated the independent dynamical variables $\vec{b}_i$ and $\vec{\pi}_i^t$; the longitudinal part of canonical momentum depends on $\vec{b}_i$ and $\vec{\pi}_i^t$ through Eq. (2.13).

The Hamiltonian is

$$H = \int d^3x\,\mathcal{H}(x), \tag{2.14}$$

where the Hamiltonian density

$$\mathcal{H}(x) = \vec{\pi}_i\cdot\dot{\vec{b}}_i - \mathcal{L}(\vec{b}_i, \dot{\vec{b}}_i, \dots)\,. \tag{2.15}$$

Using constraint equations (2.9) and also performing partial integrations, we can write Eq. (2.15) as follows:

$$\mathcal{H}(x) = \tfrac{1}{2}\vec{\pi}_i\cdot\vec{\pi}_i + \tfrac{1}{4}\vec{f}_{ij}\cdot\vec{f}_{ij}, \tag{2.16}$$

where we have dropped all terms which are expressible as divergence of something, since there is an integration over space. We can separate the longitudinal and transverse parts of $\vec{\pi}_i$ and rewrite Eq. (2.16) as follows[10]:

$$\mathcal{H}(x)=\tfrac{1}{2}\vec{\pi}_i^t\cdot\vec{\pi}_i^t+\tfrac{1}{4}\vec{f}_{ij}\cdot\vec{f}_{ij}-\tfrac{1}{2}\vec{\phi}\cdot\Delta\vec{\phi} \tag{2.17}$$

because

$$\int\vec{\pi}_i^t\cdot\partial_i\vec{\phi}\,d^3x=0. \tag{2.18}$$

Notice that $\phi=O(g)$; therefore, in the absence of interaction $\phi\equiv 0$. At this stage, it might be interesting to check whether the classical Hamilton's equations are satisfied by $\vec{b}_i$ and $\vec{\pi}_i^t$. They indeed do satisfy Hamilton's equations, as we show in Appendix A.

Now we postulate the following canonical commutation relations among $\vec{b}_i$ and $\vec{\pi}_i^t$, following the guidelines of quantum electrodynamics[11]:

$$[b_i^a(\underline{x},t),\pi_j^{t,b}(\underline{y},t)]=i\delta_{ab}\delta_{ij}^t(\underline{x}-\underline{y}), \tag{2.19}$$

where

$$\delta_{ij}^t(\underline{x}-\underline{y})=(\delta_{ij}-\partial_i\partial_j/\Delta)\delta^3(x-y). \tag{2.20}$$

We will now go to the interaction picture, where canonical commutation relations remain unchanged. To avoid clumsiness, we denote the field variables in the interaction picture also by the same quantities $b_i(\underline{x_i},t)$ and $\pi_i^t(\underline{x_i},t)$.

The free Hamiltonian in this picture is

$$H_0=\int d^3x\,[\tfrac{1}{2}\vec{\pi}_i^t(x)\vec{\pi}_i^t(x)+\tfrac{1}{4}\vec{g}_{ij}\cdot\vec{g}_{ij}(x)], \tag{2.21}$$

where

$$\vec{g}_{ij}=\partial_i\vec{b}_j-\partial_j\vec{b}_i. \tag{2.22}$$

One can check using the CCR that

$$\vec{\pi}_i^t=\partial_0\vec{b}_i. \tag{2.23}$$

We are still in the radiation gauge and the $\vec{b}_i$'s satisfy the free-field equation and therefore have the following plane-wave decomposition:

$$b_\mu^a(x,t)=\sum_{\underline{k},\lambda=1,2}\frac{e_\mu(k,\lambda)}{\sqrt{2v}}\times[c_{\underline{k}\lambda}^a e^{i(\vec{k}\cdot\vec{x}-\omega t)}+c_{\underline{k}\lambda}^{a\dagger}e^{-i(\vec{k}\cdot\vec{x}-\omega t)}], \tag{2.24}$$

where

$$e_\mu(\underline{k},\lambda)=(e_i(\underline{k},\lambda),0)$$

with

$$k_i\cdot e_i(\underline{k},\lambda)=0, \tag{2.25}$$

$$\sum_{\lambda=1,2}e_i(\underline{k},\lambda)e_j(\underline{k},\lambda)=\delta_{ij}-\frac{k_ik_j}{\vec{k}^2}. \tag{2.26}$$

The fields are transverse, as is clear from Eq. (2.25). The interaction Hamiltonian is given by

$$H_{\text{int}}=\int d^3x\,\mathcal{H}_{\text{int}}(x)=\int d^3x\,[\mathcal{H}(x)-\mathcal{H}_0(x)],$$

where

$$\mathcal{H}_{\text{int}}(x)=\tfrac{1}{2}g\vec{b}_i\times\vec{b}_j\cdot\vec{g}_{ij}+\tfrac{1}{4}g^2\vec{b}_i\times\vec{b}_j\cdot\vec{b}_i\times\vec{b}_j-\tfrac{1}{2}\vec{\phi}\cdot\Delta\vec{\phi}. \tag{2.27}$$

If we use the expansion (2.13) for $\vec{\phi}$, we can see that $\mathcal{H}_{\text{int}}$ is an infinite series in powers of coupling constant and therefore has an infinite number of noncovariant vertices. Note that if we work in the same gauge in the case of quantum electrodynamics of charged particles, we have only one noncovariant vertex.[10,11]

III. NONCOVARIANT FEYNMAN RULES

To get the Feynman rules, we have to use the $\mathcal{H}_{\text{int}}(x)$ given in Eq. (2.27) in the Dyson expansion for the S matrix and do the Wick expansion for the T products. For the purpose, we have to know the vacuum expectation values, given below:

$$\langle 0|T(b_\mu^a(\underline{x_i},t)b_\nu^c(\underline{y_i},t'))|0\rangle=\frac{-i\delta_{ac}}{(2\pi)^4}\int\frac{e^{ik\cdot(x-y)}}{k^2-i\epsilon}d^4k\left[\delta_{\mu\nu}-\frac{k_\mu k_\nu-k\cdot\eta(k_\mu\eta_\nu+k_\nu\eta_\mu)+k^2\eta_\mu\eta_\nu}{k^2-(k\cdot\eta)^2}\right], \tag{3.1}$$

where $\eta_\mu=(0,0,0,1)\equiv\delta_{\mu 4}$.

This is obtained by the usual procedure using the plane-wave decomposition (2.24) and choosing the proper contour in the k_0 plane, and also using the fact that

$$\sum_{\lambda=1,2}\epsilon_\mu(k,\lambda)\epsilon_\nu(k,\lambda)=\delta_{\mu\nu}-\frac{k_\mu k_\nu-(k\cdot\eta)(k_\mu\eta_\nu+k_\nu\eta_\mu)+k^2\eta_\mu\eta_\nu}{k^2-(k\cdot\eta)^2}. \tag{3.2}$$

Also, we have

$$\langle 0|T(\partial_\alpha b_\mu^a(x)b_\nu^c(y))|\rangle=\frac{\delta_{ac}}{(2\pi)^4}\int\frac{e^{ik\cdot(x-y)}}{k^2-1\epsilon}d^4k\,k_\alpha\left[\delta_{\mu\nu}-\frac{k_\mu k_\nu-k\cdot\eta(k_\mu\eta_\nu+k_\nu\eta_\mu)+k^2\eta_\mu\eta_\nu}{k^2-(k\cdot\eta)^2}\right], \tag{3.3}$$

$$\langle 0|T(\partial_\alpha b^a_\mu(x)\partial_\beta b^c_\nu(y))|0\rangle = \frac{(-i)\delta_{ac}}{(2\pi)^4}\int \frac{e^{ik\cdot(x-y)}}{k^2-i\epsilon}\, d^4k\,(k_\alpha k_\beta - k^2\eta_\alpha\eta_\beta)\left[\delta_{\mu\nu} - \frac{k_\mu k_\nu - k\cdot\eta(k_\mu\eta_\nu+k_\nu\eta_\mu) - k^2\eta_\mu\eta_\nu}{k^2-(k\cdot\eta)^2}\right]. \tag{3.4}$$

It is easy to see that (3.1) denotes the propagator (see Fig. 1). Furthermore, since the polarization is a purely spacelike vector with fourth component zero, the first two terms in (2.27) can be effectively written as $\frac{1}{2}g\vec{b}_\mu\times\vec{b}_\nu\cdot\vec{g}_{\mu\nu}$ and $\frac{1}{4}g^2\vec{b}_\mu\times\vec{b}_\nu\cdot\vec{b}_\mu\times\vec{b}_\nu$, having the diagrammatic representation as in Fig. 1(b) and Fig. 1(c). Therefore we can also rewrite the interaction Hamiltonian as follows:

$$\mathcal{H}_I = \tfrac{1}{2}g\vec{b}_\mu\times\vec{b}_\nu\cdot\vec{g}_{\mu\nu} + \tfrac{1}{4}g^2\vec{b}_\mu\times\vec{b}_\nu\cdot\vec{b}_\mu\times\vec{b}_\nu + \mathcal{H}_{I'}. \tag{3.5}$$

The third term in Eq. (3.5) is however an infinite series, as noted earlier:

$$\mathcal{H}_{I'} = -\tfrac{1}{2}\int d^3x\,\vec{\phi}\cdot\Delta\vec{\phi} = -\tfrac{1}{2}g^2\int\sum_{m,n=0}^{\infty}(-g)^{m+n}(\Delta^{-1}M)^m\Delta^{-1}\vec{\chi}(x)\cdot(M\Delta^{-1})^n\vec{\chi}(x)d^3x. \tag{3.6}$$

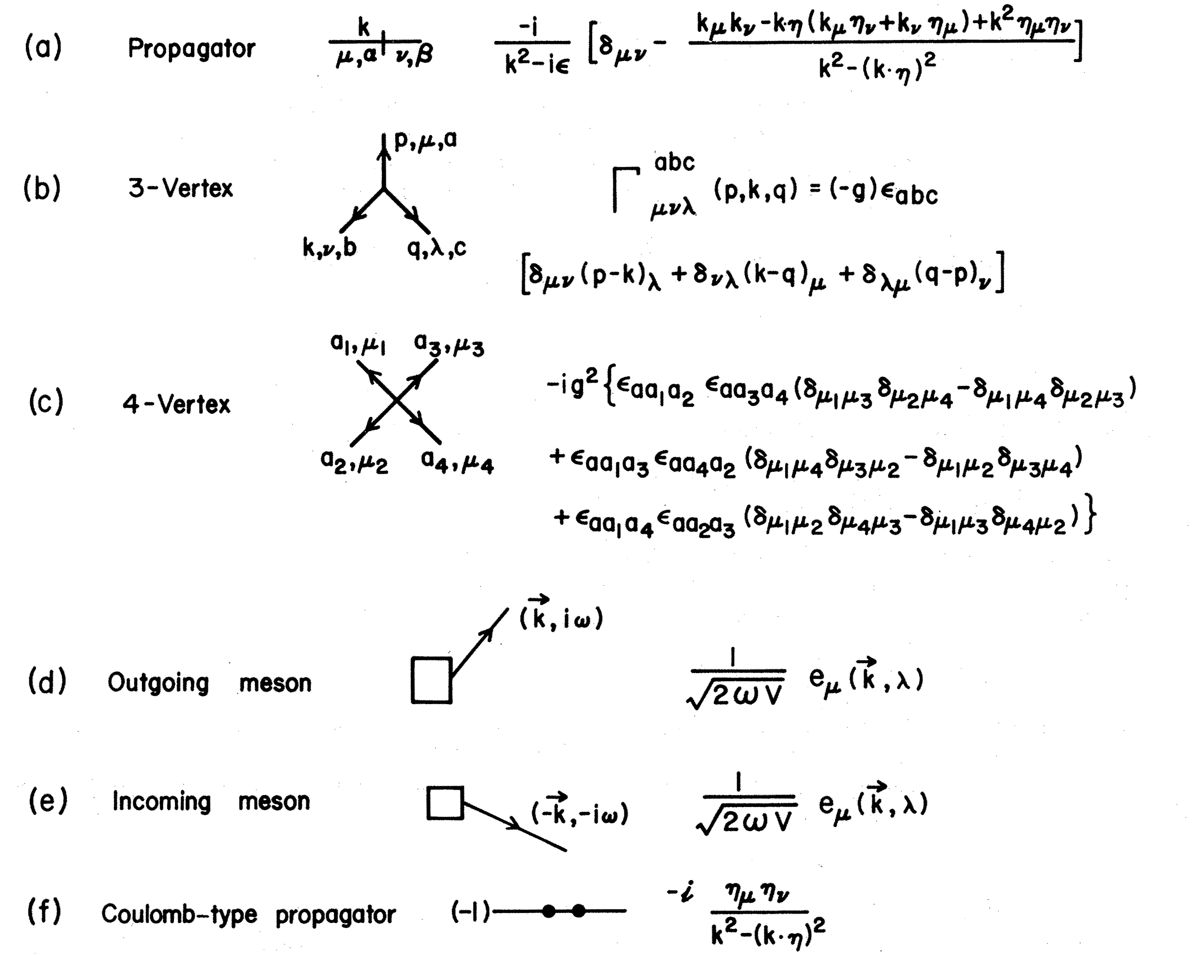

FIG. 1. (a)–(f) represent the correct Feynman rules for tree diagrams to all orders, with the understanding that (f) must be inserted in all combinations in place of propagator (a), to give rise to different diagrams to any particular order in g. [Please note that a dot stands for an η_μ and an unslashed and undotted line stands for the propagator $-i\delta_{ab}\delta_{\mu\nu}/(k^2-i\epsilon)$.] For the case of loops, we conjecture that the same Feynman rules remain, with the understanding that (1) proper symmetry numbers be multiplied to the loops involving all propagators of the type shown in Fig. 1(a) (see Ref. 8 for a definition of symmetry number), and (2) whenever there is a loop with all propagators in it being of the type shown in (f) and having only 3-vertices attached to it, a factor $\frac{1}{4}$ is to be multiplied to such a diagram. This conjecture is verified in case of the one-loop diagram to order g^2 and the general proof will be considered in a separate paper. In (e) and (f) the arrow over k denotes that it is a space vector, as opposed to the notation throughout the paper.

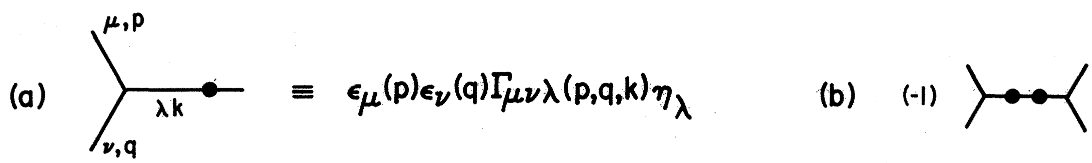

FIG. 2. All possible topological structures must be considered.

We suggest the following rules, which arise from Eq. (3.6).

For every propagator of the type Fig. 1(a), there should be a diagram with a propagator $-i\eta_\mu\eta_\nu/[k^2-(k\cdot\eta)^2]$ shown in Fig. 1(f) in the case of tree diagrams, which we call a Coulomb-type propagator. We prove this result to a few lower orders in the following paragraph. The general proof to all orders is given in Appendix B.

(a) To order g^2: Equation (3.6) becomes (since $b_0=0$)

$$\mathcal{H}_I(x)=-\tfrac{1}{2}g^2\Delta^{-1}(\vec{b}_\mu\times\partial_0\vec{b}_\mu)\cdot(\vec{b}_\nu\times\partial_0\vec{b}_\nu). \tag{3.7}$$

It is clear that in momentum space, Δ^{-1} becomes $1/[k^2-(k\cdot\eta)^2]$ and $\vec{b}_\mu\times\partial_0\vec{b}_\mu$ is equivalent to $\epsilon_\mu(p)\epsilon_\nu(q)\times\Gamma_{\mu\nu\lambda}(p,q,k)\eta_\lambda$ or the diagram shown in Fig. 2(a). Therefore, to order g^2, we have the diagram shown in Fig. 2(b), as suggested by Fig. 1. Note that the factor $\frac{1}{2}$ goes away in the actual calculation.

(b) To order g^3: One will have the following terms coming from $\mathcal{H}_{I'}(x)$:

$$-\tfrac{1}{4}g^3\int d^4x\,d^4y\,T(\vec{g}_{\mu\nu}(x)\cdot[\vec{b}_\mu(x)\times\vec{b}_\nu(x)]\Delta^{-1}(\vec{b}_\alpha\times\partial_0\vec{b}_\alpha)\cdot[\vec{b}_\beta\times\partial_0\vec{b}_\beta(y)])-ig^3\int d^4x\,T(\Delta^{-1}M\Delta^{-1}(\vec{b}_\mu\times\partial_0\vec{b}_\mu)\cdot(\vec{b}_\nu\times\partial_0\vec{b}_\nu)),$$

where

$$M=\vec{b}_i\times\partial_i\equiv\vec{b}_\mu\times\partial_\mu \tag{3.8}$$

since $b_0=0$, or $M=\frac{1}{2}\vec{b}_\mu\times\overleftrightarrow{\partial}_\mu$, which means

$$fMg=\tfrac{1}{2}(\vec{f}\cdot\vec{b}_\mu\times\partial_\mu\vec{g}-\partial_\mu\vec{f}\cdot\vec{b}_\mu\times\vec{g}). \tag{3.9}$$

Equation (3.9) follows because $\partial_\mu\vec{b}_\mu=\partial_i\vec{b}_i=0$. Now it is clear that the first term in Eq. (3.8) gives rise to a diagram of the type shown in Fig. 3(a). In the second term, the Δ^{-1}'s give the propagators $1/[k^2-(k\cdot\eta)^2]$, the expression $\vec{b}_\mu\times\partial_0\vec{b}_\mu$ gives $\epsilon_\mu(p)\epsilon_\nu(q)\Gamma_{\mu\nu\lambda}(p,q,k)\eta_\lambda$, and M in momentum space corresponds to

$$\epsilon_\nu(q)\eta_\mu\eta_\lambda\Gamma_{\mu\nu\lambda}(p,q,k), \tag{3.10}$$

shown in Fig. 3(b). As a result we get Fig. 3(c), thus substantiating our claim to order g^3.

(c) To order g^4: We will work out the rules to yet another order because a new type of term appears in this order which gives rise to the suggested rules. The contribution to the Dyson expansion to this order from Eq. (3.5) is the following:

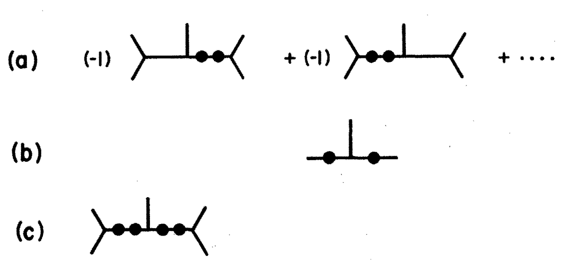

FIG. 3. Verification of the proposed Feynman rules in the case of tree diagrams.

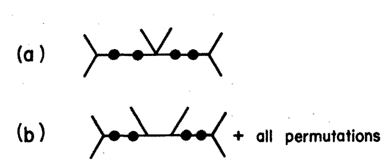

FIG. 4. Verification of the proposed Feynman rules in the case of tree diagrams.

$$\tfrac{3}{2}ig^4 \int d^4x\, T(\Delta^{-1}M\Delta^{-1}M\Delta^{-1}\vec{\chi}(x)\cdot\vec{\chi}(x)) - \tfrac{1}{2}g^4 \int d^4x\, d^4y\, T(\vec{g}_{\mu\nu}\cdot\vec{b}_\mu\times\vec{b}_\nu(x)\Delta^{-1}M\Delta^{-1}\vec{\chi}\cdot\vec{\chi}(y))$$
$$+\tfrac{1}{4}g^4 \int d^4x\, d^4y\, T(\vec{b}_\mu\times\vec{b}_\nu\cdot\vec{b}_\mu\times\vec{b}_\nu(x)\Delta^{-1}\vec{\chi}\cdot\vec{\chi}(x)) - \tfrac{1}{8}g^4 \int d^4x\, d^4y\, T(\Delta^{-1}\vec{\chi}\cdot\vec{\chi}(x)\Delta^{-1}\vec{\chi}\cdot\vec{\chi}(y)). \tag{3.11}$$

We did not write down the contribution of the first two terms in Eq. (3.5) since they would give rise to the usual diagrams shown in Figs. 1(a)–1(e). Note that in Eq. (3.11), the first term has a weight factor $\frac{3}{2}$, whereas to substantiate the rule postulated, we will have to have a weight factor of 2 there, as can be seen using Eqs. (3.9) and (3.10). Also, if our claim is correct, we ought to have the diagram shown in Fig. 4(a), which is obviously not manifested in Eq. (3.11). To see how one gets these two results, we have to look at the last term in Eq. (3.11). If we make a Wick expansion of the above term, we will first of all get all possible contractions of a pair. Among all these contractions, there is one term of the following type:

$$-\tfrac{1}{8}g^4\times 4 \int \epsilon_{a_1a_2a_3}\epsilon_{a_1a_4a_5}\epsilon_{a_6a_7a_8}\epsilon_{a_6a_9a_{10}} : \Delta^{-1}(b_\mu^{a_2}\partial_0 b_\mu^{a_3})(x)b_\nu^{a_4}(x)\Delta^{-1}(b_\lambda^{a_7}\partial_0 b_\lambda^{a_8})(y)b_\sigma^{a_9}(y) : \langle 0|T(\partial_0 b_\nu^{a_5}(x)\partial_0 b_\sigma^{a_{10}}(y))|0\rangle\, d^4x d^4y. \tag{3.12}$$

Using (3.4), one sees that

$$\langle 0|T(\partial_0 b_\nu^{a_5}(x)\partial_0 b_\sigma^{a_{10}}(y))|0\rangle = \delta_{a_5a_{10}} \frac{i\eta_\alpha\eta_\beta}{(2\pi)^4}\int \frac{e^{ik\cdot(x-y)}}{k^2-i\epsilon}\, d^4k\,(k_\alpha k_\beta - k^2\eta_\alpha\eta_\beta)\left[\delta_{\nu\sigma} - \frac{k_\nu k_\sigma - k\cdot\eta(k_\nu\eta_\sigma + k_\sigma\eta_\nu) + k^2\eta_\nu\eta_\sigma}{k^2-(k\cdot\eta)^2}\right]. \tag{3.13}$$

The $k_\alpha k_\beta$ term within (3.13) makes a contribution to the diagram in Fig. 4(b), whereas we have an extra term left out which, taken along with (3.12) after using the fact that $\eta\cdot b=0$, gives us the following extra term:

$$+i\frac{g^4}{2(2\pi)^4}\int d^4x d^4y d^4k\, e^{ik\cdot(x-y)}\left(\delta_{\nu\sigma} - \frac{k_\nu k_\sigma}{k^2-(k\cdot\eta)^2}\right) : \Delta^{-1}(\vec{b}_\mu\times\partial_0\vec{b}_\mu)\times\vec{b}_\nu(x)\cdot\Delta^{-1}(\vec{b}_\lambda\times\partial_0\vec{b}_\lambda)\times\vec{b}_\sigma(y) : \tag{3.14}$$

which is equal to

$$\tfrac{1}{2}ig^4\int d^4x : \Delta^{-1}(b_\mu\times\partial_0 b_\mu)\times b_\nu(x)\cdot\Delta^{-1}(\vec{b}_\lambda\times\partial_0\vec{b}_\lambda)\times\vec{b}_\nu(x) : +\tfrac{1}{2}ig^4\int d^4x : \Delta^{-1}M\Delta^{-1}M\Delta^{-1}\vec{\chi}(x)\cdot\vec{\chi}(x) : . \tag{3.15}$$

Thus, replacing the last term of Eq. (3.11) by Eq. (3.15), we see that we get the correct weight factor, substantiating our claim about Fig. 1(f), and also we get the diagram shown in Fig. 4(a). Therefore, we have proved that the rules suggested are indeed correct to order g^4. Here we would like to comment that we have checked the correctness of the rules through order g^8, but continuing to arbitrary order n becomes a difficult combinatorial problem, which we solve in Appendix B. Also, we would like to remark that there is no new type of term formed in higher orders.

IV. COVARIANTIZATION OF TREE DIAGRAMS TO ANY ORDER

The purpose of the present section is to show that all the noncovariant terms cancel each other in the case of tree diagrams. To show that, we will need the following theorem, which we prove at the end of this section.

Theorem 1. Let

$$T_{\mu_1\mu_2\cdots\mu_n}{}^{a_1a_2\cdots a_n}(k_1,k_2,\ldots,k_n)\epsilon_{\mu_1}(k_1)\epsilon_{\mu_2}(k_2)\cdots\epsilon_{\mu_n}(k_n)$$

stand for a set of *all* tree diagrams with n external particles, with polarization vectors $\epsilon_{\mu_1}(k_1),\ldots,$ etc., isotopic spins $a_1,\ldots,a_n$, and momenta $k_{1\mu_1},\ldots,k_{n\mu_n}$, constructed out of vertices $\Gamma_{\mu\nu\lambda}^{abc}$, $\Gamma_{\mu\nu\lambda\sigma}^{abcd}$ and propagators $-i\delta_{\mu\nu}\delta_{ab}/(k^2-i\epsilon)$ (to be called henceforth a covariant tree). If we replace any number of $\epsilon_{\mu_i}(k_i)$'s by the corresponding momenta, i.e., $k_{i\mu_i}$, then the result is zero, i.e.,

$$k_{1\mu_1}k_{2\mu_2}\cdots k_{l\mu_l}T_{\mu_1\mu_2\cdots\mu_l\mu_{l+1}\cdots\mu_n}{}^{a_1a_2\cdots a_l\cdots a_n}$$
$$\times\epsilon_{\mu_{l+1}}(k_{l+1})\cdots\epsilon_{\mu_n}(k_n)=0. \tag{4.1}$$

This result is diagrammatically shown in Fig. 5, where the crosses stand for momenta. Note that this general result is not true in the case of mas-

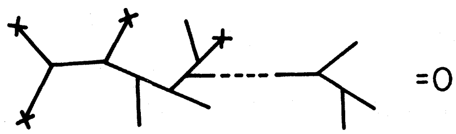

FIG. 5. A diagrammatic representation of Theorem 1. (All possible tree diagrams of this type must be considered.)

sive gauge fields even if all the external polarizations are taken to be transverse.[7] As a particular case of this result, we have the following identities:

$$p_\mu \Gamma^{abc}_{\mu\nu\lambda}(p, k, q)\epsilon_\nu(k)\epsilon_\lambda(q) = 0, \tag{4.2a}$$

$$p_\mu k_\nu \Gamma^{abc}_{\mu\nu\lambda}(p, k, q)\epsilon_\lambda(q) = 0, \tag{4.2b}$$

and

$$p_\mu k_\nu q_\lambda \Gamma^{abc}_{\mu\nu\lambda}(p, k, q) = 0. \tag{4.2c}$$

Proof. It is easy to see that

$$p_\mu \Gamma^{abc}_{\mu\nu\lambda}(p, k, q) = g\epsilon_{abc}[k^2\delta_{\nu\lambda} - \tfrac{1}{2}k_\nu(k-p)_\lambda] + g\epsilon_{acb}[q^2\delta_{\nu\lambda} - \tfrac{1}{2}q_\lambda(q-p)_\nu]. \tag{4.3}$$

This is diagrammatically shown in Fig. 6 (see Ref. 7). If k and q are on mass shell, since

$$k\cdot\epsilon(k) = 0 \quad \text{and} \quad k^2 = 0, \tag{4.4}$$

we have proved Eq. (4.2a). Multiplying Eq. (4.3) by k_ν on the right-hand side, we get

$$p_\mu k_\nu \Gamma^{abc}_{\mu\nu\lambda}(p, k, q) = -g\epsilon_{abc}\tfrac{1}{2}k^2 q_\lambda + g\epsilon_{acb}k_\nu[q^2\delta_{\nu\lambda} - \tfrac{1}{2}q_\lambda(q-p)_\nu]. \tag{4.5}$$

Therefore, when q is on the mass shell, the right-hand side vanishes and Eq. (4.2b) is proved. Equation (4.5) is also shown diagrammatically in Fig. 7, where the first term in Fig. 7 stands for the first term in Eq. (4.5). Equation (4.2c) is also easily proved to be correct.

As a first step towards proving the covariantization, we have to show that all diagrams having propagators of the type $(-i)\eta_\mu\eta_\nu/[k^2-(k\cdot\eta)^2]$ cancel. It is trivial to see to order g^2: They have opposite sign [see Figs. 1(a) and 1(f)] and they automatically cancel. The proof to any general order can be seen as follows: Consider a branch of a tree having n propagators and other branches going out of it as shown in Fig. 8, where the boxes stand for the rest of the tree diagrams attached to the branches.

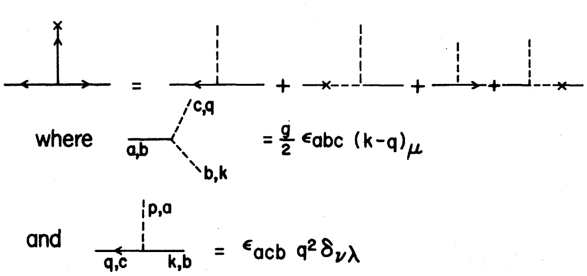

FIG. 6. Effect of multiplying a k_μ to a 3-vertex. (The cross stands for k_μ in all figures in the paper.)

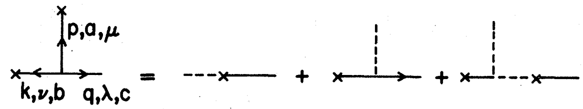

FIG. 7. Effect of multiplying crosses to two legs of a 3-vertex.

First, let us break the propagators of the type shown in Fig. 1(a) in the following way:

$$\text{Fig. 1(a)} = \mathfrak{N}^{ab}_{\mu\nu}(k) + \frac{i\eta_\mu\eta_\nu\delta_{ab}}{k^2-(k\cdot\eta)^2}, \tag{4.6}$$

and let us call the first term a type-"N" and the second term a type-"A" propagator, and the one in Fig. 1(f), a type-"F" propagator. Writing all propagators in the form shown in Eq. (4.6) and multiplying the various terms, we will have diagrams with all type-A propagators and products of various numbers of N- and A-type propagators. Also, there will be diagrams with products of N- and F-type propagators and F- and A-type propagators. Our aim is to cancel all diagrams with A- and F-type propagators and be left with those with N-type propagators only. Let us first take the case when there is no N-type propagator present in the branch under consideration. Then there will only be diagrams with F- and A-type propagators. There will be one diagram with all A-type propagators. Then there will be $\binom{n}{1}$ terms with one F-type and $n-1$ A-type propagators with an extra relative minus sign with respect to the first diagram, and so on for two and more F-type propagators. On adding all these up, we obtain a diagram with all A-type propagators with the following weight factor:

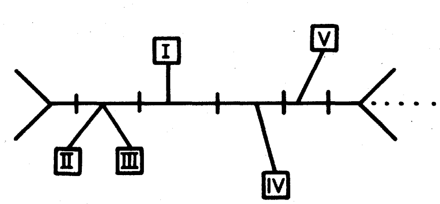

FIG. 8. A general noncovariant tree diagram to any order with propagators of type shown in Fig. 1(a).

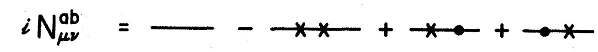

FIG. 9. Decomposition of an N-type propagator.

$$1-\binom{n}{1}+(-1)^2\binom{n}{2}+(-1)^3\binom{n}{3}+\cdots$$

$$=\sum_{r=0}^{n}(-1)^r\binom{n}{r}=(1-1)^n=0. \tag{4.7}$$

Therefore, we have proved that all such terms cancel. Now let us look at diagrams with one N-type and $n-1$ both A- and F-type propagators. The N-type propagator could be at various places. But for each position of the N-type propagator, there always exists a complete set of A- and F-type propagators which add up to give zero in the manner shown in Eq. (4.7). This can be generalized to any number of N-type propagators. Thus, in the particular branch considered in Fig. 8, we have only N-type propagators. Then, we can examine each of the branches coming out of this main branch and apply similar considerations. Therefore, after all this has been done, it is easy to see by inspection and a moment's thought that in all tree diagrams, we will have only N-type propagators and there are no F- or A-type terms at all in tree diagrams. Therefore, from this point onwards, our Feynman rules for tree diagrams will be those given by Fig. 1(b)–1(e) along with the propagator

$$\mathfrak{N}^{ab}_{\mu\nu}=\frac{-i\delta_{ab}}{k^2-i\epsilon}\left[\delta_{\mu\nu}-\frac{k_\mu k_\nu-k\cdot\eta(k_\mu\eta_\nu+k_\nu\eta_\mu)}{k^2-(k\cdot\eta)^2}\right]. \tag{4.8}$$

This is represented diagrammatically in Fig. 9.

In the next paragraph, we will show that the remaining noncovariant terms will drop out in the case of tree diagrams by virtue of Eq. (4.1).

To prove the above assertion, we first break up and multiply out all the terms in the various propagators. Then, there will be one term in which all the $\delta_{\mu\nu}$-type terms are multiplied and this is the term which we want at the end. Among the rest, we will have the following types of terms:

(a) Terms in which there is one or a number of k_μ's (denoted in Fig. 9 by a cross) or crosses attached to one leg or legs of a complete set of covariant trees defined in Theorem 1 with other legs being external (i.e., multiplied to polarization vectors). Such terms will drop out because of Eq. (4.1), when we consider a complete set.

(b) Of course, one might think that one could also have a dot attached to an external covariant tree. But, as shown in Fig. 9, every dot is accompanied by a cross. Therefore, the cross accompanying the dot will be attached to the leg of a covariant tree [where we call a 3-vertex a covariant tree also, since it satisfies (4.1)]. If all the other legs of this tree are external lines, then, of course, Eq. (4.1) will eliminate it. If not, then let us examine the other legs of the covariant tree to which it is attached. If all the legs have crosses planted on them, then we just have to take a complete set, and then by Theorem 1 the result vanishes. There is of course the possibility that, of the n legs of the tree, m ($m>0$) could be planted with crosses and $n-m$ could have dots on them. In that case, the dots have crosses accompanying them and the same considerations are now applied to this new cross and the covariant tree on whose leg the new cross is planted, until we reach the situation in which the cross is on the leg of a covariant tree, all the rest of whose legs are attached to polarizations, and this gives zero by Eq. (4.1). This then proves the assertion that all the noncovariant diagrams vanish. To illustrate the case (b), we draw a typical diagram to order g^6 in Fig. 10, where we see that there is a cross on the leg of an external covariant tree. Thus, in this section we have shown that with the set of noncovariant Feynman rules suggested in Sec. III, the tree diagrams can be described by a covariant set of rules as follows:

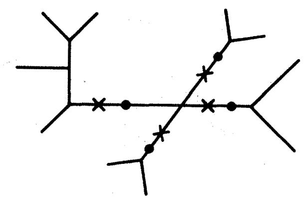

FIG. 10. A typical diagram to order g^8 after the decomposition of the N-type propagator has been made. Notice that, in this diagram, there is a cross sitting on one leg of an external covariant tree diagram. Therefore, when we combine all diagrams where the legs of the external covariant tree are permuted so as to give a complete set for it, this diagram will vanish by Theorem 1.

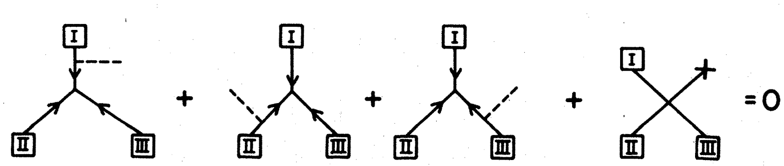

FIG. 11. Boxes stand for anything.

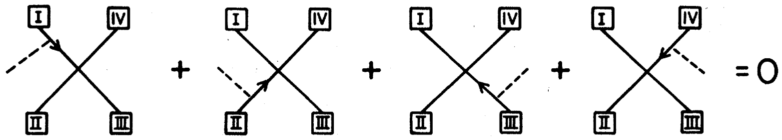

FIG. 12. Boxes stand for anything.

(1) the 3-vertex $\Gamma^{abc}_{\mu\nu\lambda}(b, k, g)$ as given in Fig. 1(b);
(2) the 4-vertex $\Gamma^{abcd}_{\mu\nu\lambda\sigma}$ as given by Fig. 1(c);
(3) the propagator $-i\delta_{\mu\nu}\delta_{ab}/(k^2 - i\epsilon)$;
(4) the external lines as given by Figs. 1(d) and 1(e).

Now we will prove Theorem 1 [Eq. (4.1)]. We will be using the identity in Fig. 6, for a cross sitting on a vertex. Before going further, let us recall that the identities given in Figs. 11 and 12 can be proved.[7]

Now take a covariant tree diagram with n external legs with an arbitrary number m of its legs crossed. Let us take one cross and use the identity shown in Fig. 6. There are two possibilities: (a) The leg adjacent to the one under consideration and attached to the same 3-vertex has a cross on it, and (b) it does not. In case (a), we can use the identity in Fig. 7 and get the result shown, Fig. 13. The second term in Fig. 13, by virtue of the identities in Figs. 11 and 12, will disappear. Therefore, what remains is a diagram with $n-1$ external legs, $m-1$ of which are crossed. Therefore, by induction we can go down until there remains only a crossed 3-vertex, which is zero by virtue of Eq. (4.2). Therefore, we have proved Theorem 1 in case (a).

In case (b), we use the identity in Fig. 6 and we get the result shown in Fig. 14. It is clear from Fig. 14 that the first and the third terms drop out because of the identities in Figs. 11 and 12. Therefore, we have a higher-order crossed covariant tree equal to lower-order crossed covariant trees, and a process of induction leads to the final result.

Another possibility is that a cross could be attached to a 4-vertex. In that case, one can see that when we take a complete set of trees, there will be corresponding diagrams so as to satisfy the identity of Fig. 11, thereby proving the theorem.

V. COVARIANTIZATION OF ONE-LOOP DIAGRAM TO ORDER g^2

The diagrams that arise in this simple case are shown in Fig. 15. A few words as to the origin of the various diagrams is in order. The origin of Fig. 15(a) is obvious; Fig. 15(b) arises because of the Wick contraction of the terms in

$$\tfrac{1}{2}ig^2\int d^4x\, T(\Delta^{-1}\vec{\chi}(x)\cdot\vec{\chi}(x)). \tag{5.1}$$

Figure 15(c) arises because of Wick expansion of the 4-vertex. The last two terms arise when in the Wick expansion of (5.1), we take the vacuum expectation value of the type $\langle 0|T(\partial_0 b^a(x)\partial_0 b^b_\nu(y))|0\rangle$. Now, let us write down the algebraic expression for each of them (omitting the integrations over k and q and the δ functions at the vertices):

Fig. 15(a):

$$-\frac{2g^2\delta_{ab}}{(2\pi)^4}[-\tfrac{1}{2}\Gamma_{\mu\mu_1\mu_2}(-p, k, q)\Gamma_{\nu_1\nu\nu_2}(-k, p, -q)P_{\mu_1\nu_1}(k)P_{\mu_2\nu_2}(q)]\epsilon_\mu(p)\epsilon_\nu(p), \tag{5.2}$$

where

$$P_{\mu_1\nu_1}(k) = \frac{1}{k^2 - i\epsilon}\left[\delta_{\mu_1\nu_1} - \frac{k_{\mu_1}k_{\nu_1} - k\cdot\eta(k_{\mu_1}\eta_{\nu_1} + k_{\nu_1}\eta_{\mu_1}) + k^2\eta_{\mu_1}\eta_{\nu_1}}{k^2 - (k\cdot\eta)^2}\right]; \tag{5.3}$$

Fig. 15(b):

$$-\frac{2g^2\delta_{ab}}{(2\pi)^4}\left[-\Gamma_{\mu\mu_1\mu_2}(-p, k, q)\Gamma_{\nu_1\nu\nu_2}(-k, p, -q)P_{\mu_1\nu_1}(k)\,\frac{\eta_{\mu_2}\eta_{\nu_2}}{q^2 - (q\cdot\eta)^2}\right]\epsilon_\mu\epsilon_\nu, \tag{5.4}$$

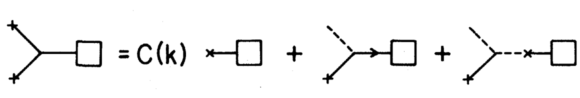

FIG. 13. Boxes stand for anything.

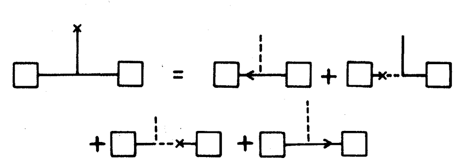

FIG. 14. Boxes stand for anything.

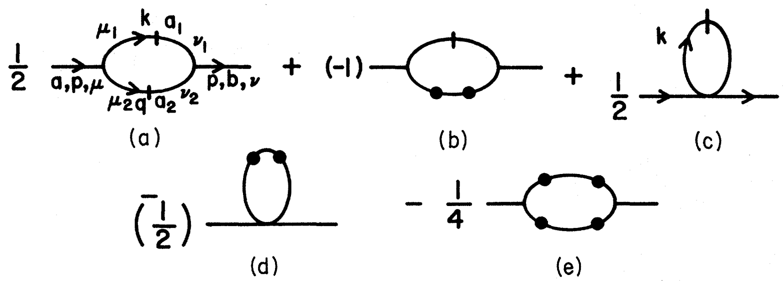

FIG. 15. Diagrams for the one-loop case to order g^2.

Fig. 15(c):

$$\frac{-2g^2\delta_{ab}}{(2\pi)^4}\left[\tfrac{1}{2}\left(2\delta_{\mu_1\nu_1}\delta_{\mu\nu}-\delta_{\mu_1\mu}\delta_{\nu_1\nu}-\delta_{\mu_1\nu}\delta_{\nu_1\mu}\right)\right]P_{\mu_1\nu_1}(q)\epsilon_\mu\epsilon_\nu, \tag{5.5}$$

Fig. 15(d):

$$\frac{-2g^2\delta_{ab}}{(2\pi)^4}\left[+\tfrac{1}{2}\left(2\delta_{\mu_1\nu_1}\delta_{\mu\nu}-\delta_{\mu_1\mu}\delta_{\nu_1\nu}-\delta_{\mu_1\nu}\delta_{\nu_1\mu}\right)\right]\frac{\eta_{\mu_1}\eta_{\nu_1}}{q^2-(q\cdot\eta)^2}\epsilon_\mu\epsilon_\nu, \tag{5.6}$$

Fig. 15(e):

$$\frac{-2g^2\delta_{ab}}{(2\pi)^4}\left\{-\tfrac{1}{4}\Gamma_{\mu\mu_1\mu_2}(-p,k,q)\Gamma_{\nu_1\nu\nu_2}(-k,p,-q)\frac{\eta_{\mu_1}\eta_{\nu_1}\eta_{\mu_2}\eta_{\nu_2}}{[k^2-(k\cdot\eta)^2][q^2-(q\cdot\eta)^2]}\right\}\epsilon_\mu(p)\epsilon_\nu(p). \tag{5.7}$$

We make a detailed diagrammatic expansion of Figs. 15(a), 15(b), and 15(c) using cross and dot for k_μ and η_μ, respectively (see Fig. 8), with momentum-dependent coefficients understood (Figs. 16–18). If one studies Fig. 16, one can easily see by using Eq. (4.2b) that parts (j), (l), (o), (p), (r), (s), and (t) of Fig. (16) vanish. Moreover, we have the following cancellations among figures, as can be easily checked:

$$16(e)+16(i)+17(a)=0, \tag{5.8}$$

$$16(m)+16(v)+17(c)=0, \tag{5.9}$$

$$16(q)+16(x)+17(d)=0, \tag{5.10}$$

$$16(d)+16(h)+18(b)=0, \tag{5.11}$$

$$18(e)+15(d)=0. \tag{5.12}$$

Also, using identities in Fig. 6, we can evaluate 16(b), 16(c), 16(f), 16(g), 16(k), and 16(n). For example,

$$16(k)+16(n)=-\frac{(k-q)_\mu(k-q)_\nu}{4[k^2-(k\cdot\eta)^2][q^2-(q\cdot\eta)^2]}-\frac{(k-q)_\mu(k-q)_\nu}{4k^2q^2}+\frac{(k-q)_\mu(k-q)_\nu}{2k^2[q^2-(q\cdot\eta)^2]}, \tag{5.13}$$

$$16(z)=-\frac{1}{2}\,\frac{(k-q)_\mu(k-q)_\nu}{[k^2-(k\cdot\eta)^2][q^2-(q\cdot\eta)^2]}, \tag{5.14}$$

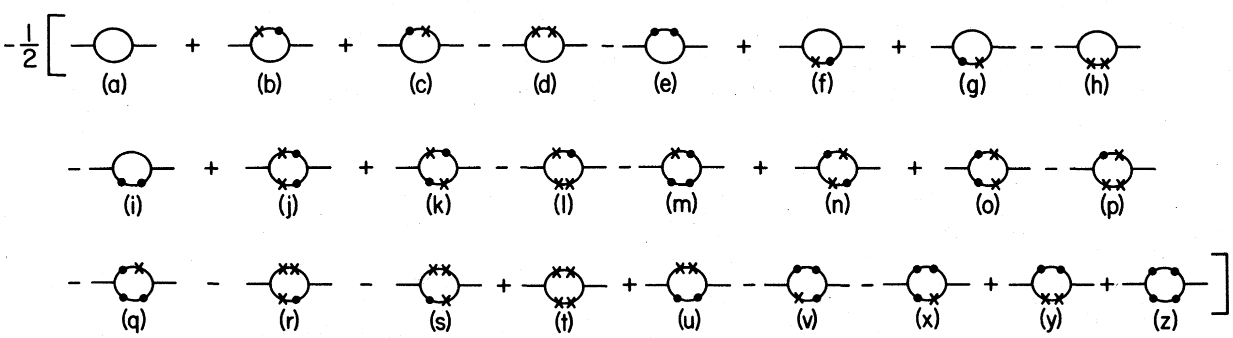

FIG. 16. Diagrammatic expansion of Fig. 15(a).

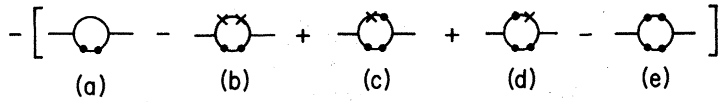

FIG. 17. Diagrammatic expansion of Fig. 15(b).

$$15(\mathrm{e}) = -\frac{1}{4}\frac{(k-q)_\mu (k-q)_\nu}{[k^2-(k\cdot\eta)^2][q^2-(q\cdot\eta)^2]}, \tag{5.15}$$

$$17(\mathrm{e}) = \frac{(k-q)_\mu (k-q)_\nu}{[k^2-(k\cdot\eta)^2][q^2-(q\cdot\eta)^2]}, \tag{5.16}$$

so that

$$17(\mathrm{e}) + 15(\mathrm{e}) + 16(\mathrm{z}) + \text{first term of Eq. (5.13)} = 0. \tag{5.17}$$

Moreover, using the identity in Fig. 6 again, we get

$$16(\mathrm{b}) + 16(\mathrm{f}) + 16(\mathrm{c}) + 16(\mathrm{g}) = \frac{2(k\cdot\eta)^2\delta_{\mu\nu}}{k^2[k^2-(k\cdot\eta)^2]} - \frac{(k-q)_\mu(k-q)_\nu}{2k^2[q^2-(q\cdot\eta)^2]} + \frac{(k-q)_\mu(k-q)_\nu}{2q^2k^2}. \tag{5.18}$$

Therefore, we see that

$$18(\mathrm{c}) + 18(\mathrm{d}) + \text{first term of (5.18)} = 0, \tag{5.19}$$

$$\text{second term of (5.18)} + \text{third term of (5.13)} = 0. \tag{5.20}$$

On taking into account all these equations, what is finally left is

$$\frac{-2g^2\delta_{ab}}{(2\pi)^4}\left[-\tfrac{1}{2}\Gamma_{\mu\mu_1\mu_2}(-p,k,q)\Gamma_{\nu_1\nu\nu_2}(-k,p,-q)\frac{\delta_{\mu_1\nu_1}}{k^2-i\epsilon}\,\frac{\delta_{\mu_2\nu_2}}{q^2-i\epsilon} + \frac{1}{4}\,\frac{(k-q)_\mu(k-q)_\nu}{(q^2-i\epsilon)(k^2-i\epsilon)}\right]\epsilon_\mu\epsilon_\nu. \tag{5.21}$$

First of all, it is easy to see that this expression is covariant. Furthermore the second term represents a scalar loop with vector-scalar-scalar coupling of the type $\frac{1}{2}g\epsilon_{abc}(k-q)_\mu$, where k and q represent the momenta of the scalar particles going out of the vertex. Also, since the scalar loop contains identical particles, we must have a symmetry number 2 in the denominator. Therefore, the scalar loop appears with an extra weight factor -2. All these are summarized in Eq. (5.22) and Fig. 19.

$$\frac{1}{(2\pi)^4}\left[\tfrac{1}{2}\Gamma^{aa_1a_2}_{\mu\mu_1\mu_2}(-p,k,q)\frac{-i\delta_{\mu_1\nu_1}}{k^2-i\epsilon}\,\frac{-i\delta_{\mu_2\nu_2}}{q^2-i\epsilon}\Gamma^{a_1ba_2}_{\nu_1\nu\nu_2}(-k,p,-q) - 2\times\tfrac{1}{2}g^2\epsilon_{aa_1a_2}\,\tfrac{1}{2}(k-q)_\mu\frac{(-i)}{k^2-i\epsilon}\,\frac{(-i)}{q^2-i\epsilon}\,\epsilon_{ba_1a_2}\tfrac{1}{2}(q-k)_\nu\right]$$
$$-\frac{2g^2\delta_{ab}}{(2\pi)^4}\,\frac{\delta_{\mu_1\nu_1}}{q^2-i\epsilon}\left[\tfrac{1}{2}(2\delta_{\mu_1\nu_1}\delta_{\mu\nu} - \delta_{\mu_1\mu}\delta_{\nu_1\nu} - \delta_{\mu_1\nu}\delta_{\nu_1\mu})\right]. \tag{5.22}$$

From these second-order calculations we can conclude that covariant Feynman rules for the one-loop case must contain a vector-scalar-scalar vertex apart from the usual 3-vertex and 4-vertex specified on Figs. 1(b) and 1(c) inside the loop. Moreover, the scalar loop must have a weight factor of -2. As has been said earlier, these results were also obtained from different considerations by various authors.[2-6] However, a second-order loop calculation is only an indication of nature of Feynman rules for the massless case and one must prove it to all orders in g, even for the one-loop case. This is presently under investigation.

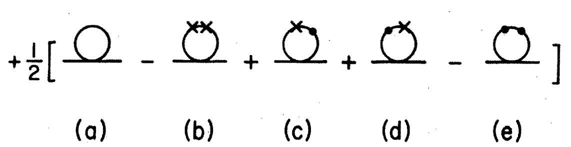

FIG. 18. Diagrammatic expansion of Fig. 15(c).

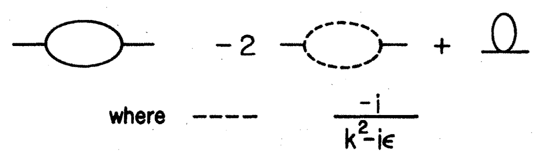

FIG. 19. Symmetry numbers are taken into account in this set of diagrams.

VI. CONCLUSION

In conclusion, we would like to say that in accordance with the claims made in the Introduction, we have outlined the techniques for obtaining covariant Feynman rules for the massless Yang-Mills field, using conventional canonical quantization methods. Using these techniques, we have shown that to all orders, tree diagrams can be described by a covariant set of rules. The one-loop diagram to order g^2 has also been shown to be covariant. These results must be generalized to all orders, with various number of loops, in order to have the complete solution to the problem. Here, we have worked in radiation gauge. There is another gauge,[12] known as the axial gauge, where the Hamiltonian is not an infinite series in coupling constants and may therefore be easier to deal with. This will be the subject of a forthcoming investigation by the author. If these covariant rules turn out to be the correct ones to all orders, then the theory is probably renormalizable, as opposed to the case of massive gauge fields.[13,14]

Added note. After writing this paper, it came to the attention of the author, through a paper by Schwinger,[15] that to the Hamiltonian density written in Eq. (2.17) one will have to add an extra term, in order that the radiation-gauge quantization described in Sec. II be consistent with Lorentz invariance. The extra term is

$$-\tfrac{1}{8}g^2\epsilon_{abc}\partial_k\mathfrak{D}_{ec}(\vec{x},\vec{x},t)\epsilon_{dbe}\partial_k\mathfrak{D}_{ea}(x,\vec{x},t), \qquad \text{where} \quad (\Delta+gM)\mathfrak{D}(\underline{x},\underline{x}')=\delta^3(x-x').$$

[Isospin indices have been suppressed in the second equation; M is defined in Eq. (2.11).] However, this term does not contribute to the tree diagrams (to all orders) and also does not contribute to the one-loop diagrams to order g^2. Therefore, the results of the paper remain unchanged.

ACKNOWLEDGMENTS

The author is grateful to Professor C. N. Yang for suggesting this investigation and providing encouragement and guidance throughout the course of this work. He also thanks Professor B. W. Lee, Professor B. McCoy, and Dr. R. F. Suitor for some discussions. Finally, he would like to thank his wife Manju for her patience during the time when the work was done.

APPENDIX A

Here, we prove the classical Hamilton's equations for the Yang-Mills field in radiation gauge. The Hamiltonian in this case is given by

$$H=\int \mathcal{H}d^3x=\int d^3x(\tfrac{1}{2}\vec{\pi}_i^t\cdot\vec{\pi}_i^t+\tfrac{1}{4}\vec{f}_{ij}\cdot\vec{f}_{ij}-\tfrac{1}{2}\vec{\phi}\cdot\Delta\vec{\phi}). \tag{A1}$$

We have to show that

$$\frac{\partial\vec{b}_i(x,t)}{\partial t}=\frac{\partial H}{\partial\vec{\pi}_i^t(x,t)} \tag{A2}$$

and

$$\frac{\partial\vec{\pi}_i^t(x,t)}{\partial t}=-\frac{\partial H}{\partial\vec{b}_i(x,t)}. \tag{A3}$$

Recall that

$$\vec{\phi}=g\frac{1}{1+g\Delta^{-1}M}\Delta^{-1}\vec{\chi}, \tag{A4}$$

where

$$\vec{\chi}=\vec{\pi}_i^t\times\vec{b}_i, \quad M=\vec{b}_i\times\partial_i. \tag{A5}$$

Also,

$$\partial_i b_i^a(x)=0 \quad \text{(gauge chosen)}. \tag{A6}$$

Note that

$$\int\vec{f}\cdot\Delta^{-1}\vec{h}d^3x=\int(\Delta^{-1}\vec{f})\cdot\vec{h}d^3x \tag{A7}$$

and

$$\int\vec{f}\cdot M\vec{h}d^3x=\int(M\vec{f})\cdot\vec{h}d^3x. \tag{A8}$$

Using these, we can write

$$-\tfrac{1}{2}\int d^3x\,\vec{\phi}\cdot\Delta\vec{\phi}=\tfrac{1}{2}\int\sum_{n=0}^{\infty}(n+1)(-g)^{n+3}(\Delta^{-1}M)^n\,\vec{\Delta\chi}\cdot\vec{\chi}(x)d^3x. \tag{A9}$$

Moreover, using Eq. (2.9), we can also solve for b_0 and get

$$\vec{b}_0=-\frac{g}{(1+g\Delta^{-1}M)^2}\Delta^{-1}\vec{\chi} \tag{A10}$$

$$=-g\sum_{n=0}^{\infty}(-g)^n(n+1)(\Delta^{-1}M)^n\Delta^{-1}\vec{\chi}. \tag{A11}$$

Now, differentiating (A1) with respect to $\vec{\pi}^t$ (remember that $\vec{b}_i$ and $\vec{\pi}_i^t$ are independent of each other) and adding a full divergence to the integral, we see that

$$\frac{\partial H}{\partial \vec{\pi}_i^t}=\frac{\partial \vec{b}_i}{\partial t}-g(\vec{b}_i\times\vec{b}_0)-\vec{b}_i\times\sum_{n=0}^{\infty}(-g)^{n+2}(n+1)(\Delta^{-1}M)^{n-1}\Delta\vec{\chi}. \tag{A12}$$

Using (A11), we get (A2). Q. E. D.

Let us now try to prove Eq. (A3). This one is more complicated because we are differentiating H with respect to $\vec{b}_i$ and there are a lot of terms to be differentiated. On differentiating H with respect to b_j^c we get

$$\frac{\partial H}{\partial b_i^c(x)}=\partial_j f_{ij}^c+g(\vec{b}_j\times\vec{f}_{ij})^c-\tfrac{1}{2}\int d^3y\frac{\partial}{\partial b_i^c(x)}(\vec{\phi}\cdot\Delta\vec{\phi}), \tag{A13}$$

$$-\frac{1}{2}\frac{\partial}{\partial b_i^c(x)}\int(\vec{\phi}\cdot\Delta\vec{\phi})d^3y=-\int d^3y\left(\frac{\partial\vec{\phi}}{\partial b_i^c}\right)\cdot\Delta\vec{\phi}. \tag{A14}$$

Let us then try to evaluate the right-hand side of (A14). For that purpose, notice that

$$\frac{\partial\vec{\phi}}{\partial b_i^c}=g\sum_{n=0}^{\infty}(-g)^n\frac{\partial}{\partial b_i^c(x)}(\Delta^{-1}M)^n\Delta^{-1}\vec{\chi}, \tag{A15}$$

$$\begin{aligned}\sum\int d^3y\,\frac{\partial}{\partial b_i^c(x)}(\Delta^{-1}M)^n\Delta^{-1}\vec{\chi}\cdot\Delta\vec{\phi}(-g)^n\\ =\sum_{n=0}^{\infty}(-g)^n d^3y\bigg\{\frac{\partial M}{\partial b_i^c(x)}[(\Delta^{-1}M)^{n-1}\Delta^{-1}\vec{\chi}\cdot\vec{\phi}+(\Delta^{-1}M)^{n-2}\Delta^{-1}\vec{\chi}\cdot(\Delta^{-1}M)\vec{\phi}+(\Delta^{-1}M)^{n-3}\Delta^{-1}\vec{\chi}\cdot(\Delta^{-1}M)^2\vec{\phi}\\ +\cdots+\Delta^{-1}\vec{\chi}\cdot(\Delta^{-1}M)^{n-1}\phi]+\frac{\partial\vec{\chi}}{\partial b_i^c(x)}(\Delta^{-1}M)^n\vec{\phi}\bigg\}.\end{aligned} \tag{A16}$$

It is easy to see that on summing up over n, one gets

$$\int d^3y\left[\partial_i\left(\frac{1}{1+g\Delta^{-1}M}\Delta^{-1}\vec{\chi}\right)\cdot\left(-g\frac{1}{1+g\Delta^{-1}M}\vec{\phi}\right)\right]-\int d^3y\left[\vec{\pi}_i^t\times\frac{1}{1+g\Delta^{-1}M}\vec{\phi}\delta^3(x-y)\right]. \tag{A17}$$

Using Eq. (A10), we get

$$\begin{aligned}\text{Eq. (A17)}&=[+g\epsilon_{acd}(\partial_i\phi^d)b_0^a+g\epsilon_{acd}\pi_i^{td}b_0^a]\\ &=g(\partial_i\vec{\phi}+\vec{\pi}_i^t)\times\vec{b}_0\\ &=g\vec{\pi}\times\vec{b}_0=g\vec{f}_{0i}\times\vec{b}_0.\end{aligned} \tag{A18}$$

Substituting (A18) in (A13), we get

$$\begin{aligned}\frac{\partial H}{\partial b_i^c(x)}&=(\partial_j\vec{f}_{ij}+g\vec{b}_j\times\vec{f}_{ij}+g\vec{b}_0\times\vec{f}_{0i})^c\\ &=-\partial_0\pi_i^c\quad\text{(using field equations).}\end{aligned} \tag{A19}$$

This is equal to the Eq. (A3) apart from an extra $\partial_0\pi_i^{\text{long}}$ in (A19). But this can always be subtracted out by adding a full-space divergence to the Hamiltonian density $\mathcal{H}(x)$ of the form

$$\vec{b}_i\cdot\partial_0\partial_i\vec{\phi}\equiv\partial_i(\vec{b}_i\cdot\partial_0\vec{\phi}). \tag{A20}$$

Therefore, we have proved both the classical Hamiltonian equations of motion.

APPENDIX B

Here we will prove the noncovariant set of rules suggested in Sec. III, to order g^N in the case of tree diagrams. A look at Eq. (3.6) makes it immediately clear that we can rewrite it as follows:

$$\int d^4x\,T(\mathcal{H}_{I'}(x))=-\tfrac{1}{2}g^2\int d^4x\sum_{n=0}^{\infty}(-g)^n(n+1)T((\Delta^{-1}M)^{n-2}\Delta^{-1}\vec{\chi}(x)\cdot\vec{\chi}(x)). \tag{B1}$$

Note that the coefficient of the operators to order g^n is $\frac{1}{2}(n-1)$. However, Eq. (3.9) says that to substantiate the suggested noncovariant rules, the numerical coefficient of $(-g)^n$ should be 2^{n-3}. What we will prove in this appendix is that using the same type technique as in the fourth order, we can get back the correct coefficient.

Proof. To any general order g^N, one can easily convince himself that relevant terms are the following:

$$\tfrac{1}{2}(n-1)\int d^4x\, T((\Delta^{-1}M)^{n-2}\Delta^{-1}\vec{\chi}\cdot\vec{\chi}(x))$$
$$+\frac{1}{2\,!4}\sum_{r=1}^{n-3}(n-r-2)r\int d^4x\, d^4y T((\Delta^{-1}M)^{n-r-3}\Delta^{-1}\vec{\chi}(x)\cdot\vec{\chi}(x)(\Delta^{-1}M)^{r-1}\Delta^{-1}\vec{\chi}(y)\cdot\vec{\chi}(y))$$
$$+\frac{1}{3\,!2^3}\sum_{\substack{r_1,r_2,r_3\\ r_1+r_2+r_3=n-4}} r_1r_2r_3\int d^4x_1d^4x_2d^4x_3$$
$$\times T((\Delta^{-1}M)^{r_1}\Delta^{-1}\vec{\chi}(x_1)\cdot\vec{\chi}(x_1)(\Delta^{-1}M)^{r_2}\Delta^{-1}\vec{\chi}(x_2)\cdot\vec{\chi}(x_2)(\Delta^{-1}M)^{r_3}\Delta^{-1}\vec{\chi}(x_3)\cdot\vec{\chi}(x_3))+\cdots. \tag{B2}$$

To give the rules in the case of tree diagrams, only suitable terms in the Wick expansion of each T product need be taken. Further, we see from (3.14) that the vacuum expectation value $\langle 0|T(\partial_0 b_\nu^a(x)\partial_0 b_\sigma(y))|0\rangle$ gives two types of extra terms after integration in the k_0 plane, as shown in Eq. (3.14). In this appendix, we will treat only the $k_\sigma k_\nu/[k^2-(k\cdot\eta)^2]$ type terms of each order and remark that the treatment for the other terms is similar. Also notice that

$$-i\int d^4x\, d^4y\, e^{ik(x-y)}\frac{k_\sigma k_\nu}{k^2-(k\cdot\eta)^2}\vec{f}(x)\times\vec{b}_\nu(x)\cdot\vec{g}(y)\times\vec{b}_\sigma(y)=i\int d^4x\,\Delta^{-1}M\vec{f}\cdot M\vec{g}. \tag{B3}$$

Moreover, one can easily check the following if one ignores the $\delta_{\nu\sigma}$ term in Eq. (3.14) (for each contraction):

$$\int d^4x_1d^4x_2\cdots d^4x_m T((\Delta^{-1}M)^{r_1}\Delta^{-1}\vec{\chi}(x_1)\cdot\vec{\chi}(\Delta^{-1}M)^{r_2}\Delta^{-1}\vec{\chi}(x_2)\cdot\vec{\chi}(x_2)\cdots(\Delta^{-1}M)^{m}\Delta^{-1}\vec{\chi}(x_m)\cdot\vec{\chi}(x_m))$$
$$\approx 2^{m-1}m!\int d^4x\, T((\Delta^{-1}M)^{r_1+r_2+\cdots+r_m+m}\Delta^{-1}\vec{\chi}(x)\cdot\vec{\chi}(x))+\text{other terms coming from the Wick expansion.} \tag{B4}$$

I have written $\approx$ instead of an equality sign because I have kept only $k_\mu k_\nu$-type terms in each contraction of a pair of $\partial_0\vec{b}_\mu$'s. Moreover, the terms involving various normal orderings and contractions also have been ignored, since they do not contribute to the tree diagram of order $r_1+r_2+\cdots+m$. Using Eq. (B4), Eq. (B2) can be simplified and written as follows:

$$\text{Eq. (B2)}=\frac{1}{2}\Big((n-1)+\sum_r(n-r-2)r+\sum_{\substack{r,s,t\\ r+s+t=n-4}} rst+\cdots\Big)\int d^4x\, T((\Delta^{-1}M)^{n-2}\Delta^{-1}\vec{\chi}(x)\cdot\vec{\chi}(x)). \tag{B5}$$

Let us rewrite the coefficient in (B5) as follows:

$$I_n=\frac{1}{2}\Bigg((n-1)+\sum_{\substack{r_1,r_2\\ r_1+r_2=n-2}} r_1r_2+\sum_{\substack{r_1,r_2,r_3\\ r_1+r_2+r_3=n-4}} r_1r_2r_3+\cdots\Bigg). \tag{B6}$$

It is easy to see that the expression in (B6) is the coefficient of x^n in the following sum:

$$\frac{1}{2}\left(\frac{x^2}{(1-x)^2}+\frac{x^4}{(1-x)^4}+\frac{x^6}{(1-x)^6}+\cdots\right) \tag{B7}$$

or

$$I_n=\frac{1}{n!}\frac{d^n}{dx^n}f(x)\bigg|_{x=0}, \tag{B8}$$

where

$$f(x)=\tfrac{1}{2}\sum_{m=1}^{\infty}\frac{x^{2m}}{(1-x)^{2m}}. \tag{B9}$$

It is easy to sum up this series; one gets

$$f(x)=\frac{1}{2}\,\frac{x^2}{(1-x)^2}\,\frac{1}{1-x^2/(1-x)^2}=\frac{1}{2}\,\frac{x^2}{1-2x}. \tag{B10}$$

The coefficient of x^n in $f(x)$ is 2^{n-3}, as is required. It must be stressed that we have proved the general result only for one type of term, and we believe that the case when both $\delta_{\mu\nu}$- and $k_\mu k_\nu/[k^2-(k\cdot\eta)^2]$-type terms are mixed should be easy to obtain using similar methods. Complete derivation of the noncovariant rules to all orders with any number of loops is the subject of a forthcoming paper.[16]

---

*Work supported by the U. S. Atomic Energy Commission under Contract No. AT(30-1)-3668B.

[1]C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).

[2]R. P. Feynman, Acta Phys. Polon. 24, 697 (1963).

[2a]*Note added in proof.* The accurate Feynman rules to all orders were first suggested by DeWitt (Ref. 5) following the suggestions by Feynman (Ref. 2), who first pointed out the necessity of fictitious particles in the one-loop case.

[3]L. D. Faddeev and V. N. Popov, Phys. Letters 25B, 29 (1967).

[4]S. Mandelstam, Phys. Rev. 175, 1580 (1968).

[5]B. S. DeWitt, Phys. Rev. Letters 12, 742 (1964); Phys. Rev. 162, 1195 (1967).

[6]E. S. Fradkin and I. V. Tyutin, Phys. Letters 30B, 562 (1969).

[7]S. K. Wong, Phys. Rev. D 3, 945 (1971); H. Vandam and M. Veltman, Nucl. Phys. B7, 637 (1968); J. Reiff and M. Veltman, *ibid.* B13, 545 (1969).

[8]For a similar treatment of charged vector fields interacting with electromagnetic fields, see T. D. Lee and C. N. Yang, Phys. Rev. 128, 885 (1962).

[9]Our notation is as follows: An arrow on any quantity denotes that it is an isovector; $\mu,\nu,\alpha,\beta,\ldots$ stand for four Lorentz indices with the convention that $x=(x_1,x_2,x_3,ix_0)$; i, j, k, l will be used to denote only space indices. a, b, c, d will be used to denote isospin indices, when necessary.

[10]I. Khriplovich, Yadern. Fiz. 10, 409 (1969) [Soviet J. Nucl. Phys. 10, 235 (1970)].

[11]See J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[12]R. Arnowitt and S. I. Fickler, Phys. Rev. 127, 1821 (1962).

[13]D. G. Boulware, Ann. Phys. (N.Y.) 56, 140 (1970).

[14]Some other works related to the question of obtaining Feynman rules for Yang-Mills field are the following: C. M. Hindle, University of Cambridge report (unpublished); A. Maheshwari, Phys. Rev. D 2, 1436 (1970).

[15]J. Schwinger, Phys. Rev. 127, 344 (1962).

[16]Rabindra Nath Mohapatra, Phys. Rev. D (to be published).