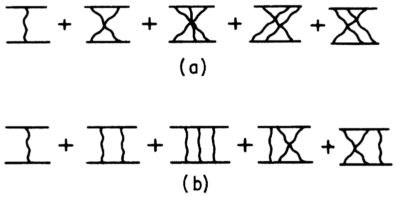

FIG. 5. (a) Graphs to sixth order which contribute to K_s. (b) Graphs to sixth order which contribute to K_u.

FIG. 6. The two eighth-order generalized ladder graphs which are *not* generated by the crossing-symmetric Bethe-Salpeter equations. It is easy to see that these graphs are two-body irreducible in neither the s nor u channels.

*Alexander von Humboldt-Stiftung Fellow. Present address: Institut für Physik, Johannes Gutenberg Universität, 6500 Mainz, Federal Republic of Germany.

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[3]If we are dealing with "scalar electrodynamics," where the "photons" are indeed massless, this last statement does not have very much meaning since the elastic and production thresholds coincide. However, if we are dealing with "massive scalar electrodynamics," they will not coincide, and elastic unitarity will indeed be satisfied between the elastic and production thresholds.

[4]R. J. Yaes, Ref. 2.

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Regge Behavior, Crossing Symmetry, and Unitarity in the Nakanishi Representation

M. O. Taha*
International Centre for Theoretical Physics, Miramare-Trieste, Italy
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The analyticity of the weight function ψ in the Nakanishi representation is discussed under certain assumptions and a simple inversion formula is proved. Two-particle unitarity is formulated as an integral equation for ψ. A crossing-symmetric asymptotic form for ψ corresponding to Regge behavior for the amplitude is obtained. The significance of this investigation for the construction of dual amplitudes is discussed.

I. INTRODUCTION

An integral representation for the scattering amplitude was proposed by Nakanishi[1] as an alternative to the Mandelstam representation. It was based on the general form of Feynman integrals for perturbation-theoretic graphs. On the basis of its construction it follows that this representation is valid to all orders in perturbation theory. Further, unlike the Mandelstam representation, the Nakanishi representation can be extended to N-point functions.

The unsubtracted form of the representation for the two-particle scattering amplitude $M(s,t,u)$ is

$$M(s,t,u)=A(s,t)+A(t,u)+A(u,s)\,, \tag{1.1}$$

$$A(s,t)=\int_0^1 \psi(xs+(1-x)t,x)\,dx\,, \tag{1.2}$$

$$\psi(z,x)=\frac{1}{(2\pi i)^2}\oint ds'\oint dt'\,\frac{A(s',t')}{[xs'+(1-x)t'-z]^2}\,. \tag{1.3}$$

The contours are large semicircles in the upper half-planes, where it is assumed that $A(s,t)$ is analytic. If $A(s,t)$ is bounded by $|xs+(1-x)t|^{-\delta}$, $\delta>0$, whenever $|xs+(1-x)t|>N$, for some N, where x is any nonnegative real number, then

$$\psi(z,x)=\frac{1}{1-x}\int_{4m^2}^{\infty}\frac{g(z'+i\epsilon,x)-g(z'-i\epsilon,x)}{z'-z}\,dz'\,, \tag{1.4}$$

where

$$g(z,x)=\frac{1}{(2\pi i)^2}\,\frac{\partial}{\partial z}\oint A\left(s,\frac{z-xs}{1-x}\right)ds\,, \tag{1.5}$$

with the same contour as above.

In the present work, however, *we shall also assume that the amplitude satisfies an unsubtracted Mandelstam representation,* which enables us to obtain variations on the Nakanishi representation and to give an explicit discussion on the analyticity of the function ψ. Our main object is to determine the high-energy Regge-pole form of the weight function ψ and the condition of two-particle unitarity on it. The primary motivation for this work is the hope that it might provide an approach to the construction of crossing-symmetric dual models with Regge behavior with desirable analytic features. In this respect we note the similarity of some such recent models[2] to the representation (1.2).

The arrangement of the material is as follows: In Sec. II, assuming that the amplitude satisfies an unsubtracted Mandelstam representation, we write ψ in terms of the double-spectral function ρ and discuss its domain of analyticity. In Sec. III, we write an integral formula for ψ in terms of the discontinuity $D_s(s,t)$ of $A(s,t)$ in s, under the assumptions that the amplitude satisfies a single-variable dispersion relation in s and that $D_s(s,t)$ vanishes as $t\to-\infty$. This formula for ψ is the main tool in our discussions of two-particle unitarity on ψ in Sec. IV and Regge asymptotic form in Sec. V. We are able to write the unitarity condition as an integral equation for ψ. The kernel H in this equation is given by an integral over a function of K, where K is the usual function in the two-particle unitarity integral for A. We comment in Appendix A on the calculation of H.

In Sec. V and Appendix B, we calculate the asymptotic form of ψ corresponding to an imposed form of Regge behavior for the amplitude. The limits $x\to+0$ and $1-x\to+0$ in ψ are, respectively, associated with s and t Regge behavior for A. A crossing-symmetric asymptotic form for ψ, simultaneously incorporating these limits, is also obtained.

In Sec. VI we discuss the significance of our investigation for the construction of dual models. In particular, we point out that in the asymptotic form of ψ, besides crossing symmetry and Regge behavior, one has, already built in, a generalized notion of duality.

II. MANDELSTAM ANALYTICITY

Since it possesses a larger domain of validity, it is not surprising that the Nakanishi representation can be derived from the Mandelstam representation. Assuming an unsubtracted Mandelstam representation

$$A(s,t)=\frac{1}{\pi^2}\int\int\frac{\rho(s',t')}{(s'-s)(t'-t)}ds'dt', \tag{2.1}$$

one finds, on using the Feynman identity

$$\frac{1}{ab}=\int_0^1\frac{dx}{[xa+(1-x)b]^2},$$

that the function $\psi(z,x)$, defined by

$$\psi(z,x)=\frac{1}{\pi^2}\int\int\frac{\rho(s,t)ds\,dt}{[xs+(1-x)t-z]^2}, \tag{2.2}$$

satisfies

$$A(s,t)=\int_0^1\psi(xs+(1-x)t,x)\,dx. \tag{2.3}$$

To arrive at representation (2.3) we have assumed that the x and (s,t) integrations are interchangeable.

From these equations, or those of Sec. I, one immediately sees that crossing symmetry is maintained if and only if

$$\psi(z,x)=\psi(z,1-x). \tag{2.4}$$

From Eq. (2.2) one can also directly compute the domain of analyticity for $\psi(z,x)$. Suppose that $\rho(s,t)=0$, except when

$$H(s,t)\equiv(s-4m^2)(t-4m^2)-4m^4\geq 0.$$

$\psi(z,x)$ is then analytic in

$$\{(z,x):\ z<u(x)\},$$

where

$$u(x)=\min_{H=0}[xs+(1-x)t]. \tag{2.5}$$

A simple calculation then determines $u(x)$ to be

$$u(x)=4m^2+4m^2[x(1-x)]^{1/2}. \tag{2.6}$$

Alternatively, suppose that $\psi(z,x)$ is analytic in the above domain and that $A(s,t)$ is defined by Eq. (2.3). As x varies between 0 and 1, the one-parameter family of singularity curves of the integrand, namely

$$xs+(1-x)t=4m^2+4m^2[x(1-x)]^{1/2}, \tag{2.7}$$

generates an envelope which gives the singularity curve of $A(s,t)$. On computing the envelope to this family one in fact finds it, as expected, to be the curve $H(s,t)=0$, the correct Mandelstam boundary. We have therefore proved the result that the amplitude $A(s,t)$ defined by (2.3) has the correct Mandelstam boundary if and only if $\psi(z,x)$ is analytic in

$$\{(z,x):\ z-4m^2[x(1-x)]^{1/2}-4m^2<0\}. \tag{2.8}$$

One notes, further, that $\psi(xs+(1-x)t,x)$ is analytic in x, $x\in(0,1)$, for all (s,t) such that $s\leq 4m^2$, $t\leq 4m^2$. To see this, consider the function

$$F(x;\ s,t)=x(s-t)+t-4m^2-4m^2[x(1-x)]^{1/2},\qquad 0<x<1 \tag{2.9}$$

as a function of x at fixed s and t. This function has a single turning point, a minimum, at

$$x=\begin{cases}\frac{1}{2}-\Delta, & s-t\geq 0\\ \frac{1}{2}+\Delta, & s-t\leq 0\end{cases} \tag{2.10}$$

where

$$\Delta=\frac{1}{2}\left[1+\left(\frac{4m^2}{s-t}\right)^2\right]^{1/2}. \tag{2.11}$$

$\psi(xs+(1-x)t, x)$ is analytic in x, $x\in(0,1)$, for all (s, t) such that $F(x; s, t)<0$. For $s-t\geq 0$ the maximum value attainable by F is $s-4m^2$, at $x=1$, while the maximum value when $s-t\leq 0$ is $t-4m^2$, which is attained at $x=0$. It is thus seen that $\psi(xs+(1-x)t, x)$ is an analytic function of x, $0<x<1$, when $s\leq 4m^2$, $t\leq 4m^2$. The integral representation (2.3) is therefore well defined in the region $s\leq 4m^2$, $t\leq 4m^2$. Outside this region the amplitude $A(s, t)$ is obtained by analytic continuation.

The function $\psi(z, x)$ may be presumed to obey the double dispersion relation

$$\psi(z, x)=\frac{1}{\pi^2}\int\int\frac{\phi(z', x')dz'dx'}{(z'-z)(x'-x)}, \tag{2.12}$$

where the integration extends over the complement to the domain (2.8), as well as the single-variable dispersion relation

$$\psi(z, x)=\frac{1}{2\pi i}\int_{4m^2}^{\infty}\frac{\xi(z', x)dz'}{z'-z}. \tag{2.13}$$

The discontinuity $\xi(z, x)$ of $\psi(z, x)$ in z, at fixed x, may be related to the double-spectral function ρ using Eq. (2.2). One obtains

$$\xi(z, x)=\frac{2i}{\pi}\frac{1}{(1-x)^2}\times\int\frac{\partial}{\partial t}[\rho(s, t)\theta(H(s, t))]\bigg|_{t=(z-xs)/(1-x)}ds, \tag{2.14}$$

where we have explicitly introduced the limit on the region of integration implied in Eq. (2.2). When the differentation in the integrand of Eq. (2.14) is effected, the δ term gives no contribution [since $\rho(s, t)$ vanishes on the boundary] while the θ term

$$\rho^{(1)}\left(s, \frac{z-xs}{1-x}\right)\theta\left(H\left(s, \frac{z-xs}{1-x}\right)\right),$$

$$\rho^{(1)}(x, y)\equiv(\partial\rho/\partial y)(x, y)$$

shows that $\xi(z, x)$ is nonzero for $z>4m^2+4m^2\times[x(1-x)]^{1/2}$ and is given there by the finite integral

$$\xi(z, x)=\frac{2i}{\pi(1-x)^2}\int_{s_-(z,x)}^{s_+(z,x)}\rho^{(1)}\left(s, \frac{z-xs}{1-x}\right)ds, \tag{2.15}$$

where

$$s_\pm(z, x)=\frac{1}{2x}\{z-4m^2\pm[(z-4m^2)^2-16m^4x(1-x)]^{1/2}\}. \tag{2.16}$$

III. THE INVERSION INTEGRAL

The inversion integral (1.3) of the Nakanishi representation is a complex double integral which, even in the real form (2.2) holding under conditions allowing a Mandelstam representation, is not particularly convenient for calculations involving quite simple functions. An alternative form giving more readily the function $\psi(z, x)$ when the amplitude $A(s, t)$ is known is therefore desirable. On the other hand, one observes that calculation of the discontinuity $D(s, t)$ in one of the energy variables is often a straightforward exercise, so that one may indeed assume that a single-integral representation for $\psi(z, x)$ in terms of $D(s, t)$ is almost as good as a direct inversion integral. We therefore proceed to prove, in the following, a formula giving $\psi(z, x)$ in terms of $D_s(s, t)$ for an amplitude $A(s, t)$ satisfying an unsubtracted dispersion relation in s,

$$A(s, t)=\frac{1}{2\pi i}\int_{4m^2}^{\infty}\frac{D_s(s', t)}{s'-s}ds', \tag{3.1}$$

with

$$\lim_{t\to-\infty}D_s(s, t)=0. \tag{3.2}$$

The formula to be proved is

$$\psi(z, x)=\frac{1}{2\pi i(1-x)^2}\int_{4m^2}^{\infty}ds\frac{\partial}{\partial t}D_s(s,t)\bigg|_{t=(z-xs)/(1-x)} \tag{3.3}$$

Since ψ is uniquely determined, given A, when (1.2) holds, it is sufficient to assume (3.3) and deduce (1.2) under conditions (3.1) and (3.2). From (3.3) we have, assuming that an interchange of the order of integrations is allowed,

$$I(s, t)\equiv\int_0^1\psi(xs+(1-x)t, x)dx$$
$$=\frac{1}{2\pi i}\int_{4m^2}^{\infty}ds'\int_0^1 dx\frac{D_s^{(1)}(s', t+(s-s')x/(1-x))}{(1-x)^2},$$

where $D_s^{(1)}(u, v)$ denotes $\partial D_s(u, v)/\partial v$. Introduce the variable $\lambda=t+(s-s')x/(1-x)$ in effecting the integral over x. Then $d\lambda/dx=(s-s')/(1-x)^2$, and for $s-s'<0$ (that is, $s<4m^2$), $(0, 1)\to(-\infty, t)$, giving

$$I(s, t)=\frac{1}{2\pi i}\int_{4m^2}^{\infty}\frac{ds'}{s'-s}\int_{-\infty}^{t}d\lambda\frac{\partial D_s(s', \lambda)}{\partial\lambda}$$

$$=\frac{1}{2\pi i}\int_{4m^2}^{\infty}\frac{D_s(s',t)}{s'-s}ds'=A(s,t).$$

This proves formula (3.3) under the stated conditions. In particular note that we have used condition (3.2).

It is clear that under similar conditions we also have the corresponding formula with $D_t(s,t)$:

$$\psi(z,x)=\frac{1}{2\pi i x^2}\int_{4m^2}^{\infty}dt\,\frac{\partial D_t(s,t)}{\partial s}\bigg|_{s=[z-(1-x)t]/x}. \tag{3.4}$$

For a crossing-symmetric amplitude, $D_t(s,t)=D_s(t,s)$, so that (3.3) and (3.4) again give $\psi(z,x)=\psi(z,1-x)$.

One observes that Eq. (3.3), which may be written in the form

$$\psi(z,x)=\frac{1}{2\pi i(1-x)^2}\int_{4m^2}^{\infty}D_s^{(1)}\left(s,\frac{z-xs}{1-x}\right)ds,$$

bears a close similarity to Eq. (2.15) for the discontinuity $\xi(z,x)$. In fact, Eq. (2.15) may be derived by taking the discontinuity in z of the above equation and using

$$\mathrm{Disc}D_s(s,t)|_t=-4\rho(s,t)\theta(H(s,t)).$$

Formula (3.3) is the main tool in our investigation of the asymptotic form of ψ corresponding to Regge behavior for $A(s,t)$. One may also use (3.3) to write the two-particle unitarity condition as an integral equation in ψ, which we proceed to do.

IV. TWO-PARTICLE UNITARITY FOR ψ

The s-channel unitarity condition on $A(s,t)$ may be written in the form

$$D_s(s,t)=\rho(s)\int_{a(s)}^{0}dt'\int_{a(s)}^{0}dt''\frac{A(s,t')A^*(s,t'')\theta(K)}{[K(s;t,t',t'')]^{1/2}}, \tag{4.1}$$

where

$$\rho(s)=\frac{i}{\pi^2}[s(s-4m^2)]^{-1/2},\quad a(s)=4m^2-s, \tag{4.2}$$

and

$$K(s;t,t',t'')=\frac{4tt't''}{s-4m^2}-(t^2+t'^2+t''^2)+2(tt'+tt''+t't''). \tag{4.3}$$

From Eq. (4.1) we have

$$\frac{\partial D_s(s,t)}{\partial t}=\int_{a(s)}^{0}dt'\int_{a(s)}^{0}dt''G(s;t;t',t'')A(s,t')A^*(s,t''), \tag{4.4}$$

where

$$G(s;t;t',t'')=-\rho K^{1/2}\frac{\partial K}{\partial t}\left[\frac{\theta(K)}{2K^2}+\delta'(K)\right]. \tag{4.5}$$

The dependence of ρ and K on the variables is as given in Eqs. (4.2) and (4.3). Substituting from Eqs. (1.2) and (3.3) into Eq. (4.4), we obtain the following equation for ψ:

$$\psi(z,x)=\frac{1}{2\pi i(1-x)^2}\int_{4m^2}^{\infty}ds\int_{a(s)}^{0}dt'\int_{a(s)}^{0}dt''\int_0^1 dx'\int_0^1 dx''$$
$$\times G\left(s;\frac{z-xs}{1-x};t',t''\right)\psi(sx'+(1-x')t',x')\psi^*(sx''+(1-x'')t'',x''). \tag{4.6}$$

This is the two-particle unitarity condition on ψ. To rewrite it in a different form, introduce the variables y' and y'' by

$$y'=\frac{t'}{a(s)},\quad y''=\frac{t''}{a(s)},$$

and two dummy δ integrations on the first variables of ψ and ψ^*. Equation (4.6) is then transformed into the following form:

$$\psi(z,x)=\frac{1}{2\pi i(1-x)^2}\int_{4m^2}^{\infty}\frac{ds}{(4m^2-s)^2}\int_0^1 dy'dy''dx'dx''\int_{-\infty}^{\infty}dz'dz''$$
$$\times\,\delta(z'-sx'-a(s)(1-x')y')\delta(z''-sx''-a(s)(1-x'')y'')\,G\left(s;\frac{z-xs}{1-x};a(s)y',a(s)y''\right)\psi(z',x')\psi^*(z'',x''). \tag{4.7}$$

It is now clear that in this equation we may explicitly integrate over all variables except the arguments of ψ and ψ^*. We thus finally obtain the two-particle unitarity equation for ψ in the form

$$\psi(z,x)=\int H(z;x;z',z'';x',x'')\psi(z',x')\psi^*(z'',x'')dx'dx''dz'dz'', \tag{4.8}$$

where the region of integration is the domain D in which $H\neq 0$, with $x',x''\in(0,1)$. The kinematical factor H is given by

$$H(z;x;z',z'';x',x'')=\frac{1}{2\pi i(1-x)^2}\int_{4m^2}^{\infty}\frac{ds}{(4m^2-s)^2}\int_0^1 dy'dy''\delta(z'-\cdots)\delta(z''-\cdots)G(s;\cdots), \tag{4.9}$$

where the dependence of G and the δ functions on their arguments is as given in Eq. (4.7).

When the function H has been calculated from (4.5) and (4.9), Eq. (4.8) is a direct integral equation for ψ. The actual calculation of H is a rather tedious job, on which we comment in Appendix A. The analytic properties of ψ imposed by Eq. (4.8) should, of course, be the same as already discussed in Sec. II, on the basis of the Mandelstam double-spectral function.

V. REGGE ASYMPTOTIC FORM

In this section we seek to determine the form of $\psi(z,x)$ corresponding to a Regge form for $A(s,t)$. Before we address ourselves to this matter, however, let us see what limit in $\psi(z,x)$ corresponds to the Regge limit. Making a change of variable in (1.2), we write it in the form

$$A(s,t)=\frac{1}{s}\int_0^s\psi(\lambda+(1-s^{-1}\lambda)t,\,s^{-1}\lambda(1-s^{-1}\lambda))\,d\lambda, \tag{5.1}$$

Thus as $|s|\to\infty$, at fixed t,

$$A(s,t)\sim\frac{1}{s}\int_0^s\psi(\lambda+t,\,s^{-1}\lambda)d\lambda, \tag{5.2}$$

provided that the limit in s is taken in such a way that the imposed path of integration (in general, a contour over the complex λ plane extending from the origin to the point at infinity) avoids the singularities of $\psi(z,x)$ as $x\to+0$. It should be noted that $s^{-1}\lambda$ remains real on the path of integration. From (5.2) we see that the Regge limit $|s|\to\infty$, at fixed t, is given by the asymptotic form of $\psi(z,x)$ as $x\to+0$.

Conversely, consider the limit $x\to+0$, at fixed z, for $\psi(z,x)$. One has

$$\psi(z,x)=\frac{1}{2\pi i(1-x)^2}\int_{4m^2}^{\infty}D_s^{(1)}\left(s,\frac{z-xs}{1-x}\right)ds$$
$$=\frac{1}{2\pi i(1-x)^2}\int_{4m^2x}^{\infty}D_s^{(1)}\left(x^{-1}\lambda,\frac{z-\lambda}{1-x}\right)d\lambda,$$

so that as $x\to+0$, at fixed z,

$$\psi(z,x)\sim\frac{1}{2\pi ix}\int_0^{\infty}D_s^{(1)}(x^{-1}\lambda,z-\lambda)d\lambda. \tag{5.3}$$

We thus see that the limit $x\to+0$ in $\psi(z,x)$ is given by the limit $s\to\infty$ in $D_s(s,t)$, i.e., by the Regge limit. The large-t behavior is similarly associated with the limit $1-x\to+0$.

We now seek to determine the asymptotic form of $\psi(z,x)$ corresponding to a Regge amplitude. Let us consider the amplitude given by

$$D_s(s,t)=f(t)s^{\alpha(t)}. \tag{5.4}$$

Then

$$D_s^{(1)}=s^{\alpha(t)}[\alpha'(t)f(t)\ln s+f'(t)],$$

and from Eq. (5.3) we obtain

$$\psi(z,x)\sim\frac{1}{2\pi ix}\int_0^{\infty}(x^{-1}\lambda)^{\alpha(z-\lambda)}$$
$$\times[\alpha'(z-\lambda)f(z-\lambda)\ln(x^{-1}\lambda)+f'(z-\lambda)]d\lambda,$$

i.e.,

$$\psi(z,x)\sim\frac{-\ln x}{x}\int_0^{\infty}F(z-\lambda)x^{-\alpha(z-\lambda)}\lambda^{\alpha(z-\lambda)}d\lambda, \tag{5.5}$$

where

$$F(z)=(1/2\pi i)\alpha'(z)f(z). \tag{5.6}$$

In Eq. (5.5) we must take $z\leq 4m^2$ so that the integrand remains real throughout the range of integration.

It is our purpose now to obtain an asymptotic form, as $x\to+0$, for the integral

$$I=\int_0^\infty F(z-\lambda)x^{-\alpha(z-\lambda)}\lambda^{\alpha(z-\lambda)}d\lambda\,. \tag{5.7}$$

In terms of the variable $\nu=-\ln x$, this integral becomes

$$I=\int_0^\infty F(z-\lambda)e^{h(\nu,\lambda,z)}d\lambda\,, \tag{5.8}$$

where

$$h(\nu,\lambda,z)=\alpha(z-\lambda)(\nu+\ln\lambda)\,. \tag{5.9}$$

In Appendix B we obtain the following asymptotic form for (5.8) as $\nu\to\infty$, using the generalized Laplace method[3]:

$$I\sim F(\alpha'\nu)^{-\alpha-1}e^{\nu\alpha}(1-\ln\nu/\nu)^{-\alpha-1}\Gamma(\alpha+1) \quad \text{as } \nu\to\infty\,, \tag{5.10}$$

where F, α, and α' denote $F(z)$, $\alpha(z)$, and $\alpha'(z)$. This asymptotic behavior holds, to order $1/\nu$, when $\alpha(z)>0$ and $\alpha'(z)>0$. Substituting in (5.5) we obtain

$$\psi(z,x)\sim G(z)x^{-\alpha(z)-1}(-\ln x)^{-\alpha(z)} \times\left[1+\frac{\ln(-\ln x)}{\ln x}\right]^{-\alpha(z)-1} \quad \text{as } x\to+0\,, \tag{5.11}$$

where

$$G(z)=(1/2\pi i)\Gamma(\alpha(z)+1)f(z)[\alpha'(z)]^{-\alpha(z)}\,. \tag{5.12}$$

Equation (5.11) thus gives the asymptotic form of ψ corresponding to an amplitude with asymptotic Regge behavior, as in (5.4). It should be noted that when the amplitude is calculated by taking ψ to be given by the right-hand side of (5.11), the resulting amplitude is not exactly the Regge amplitude with $D_s(s,t)$ given by (5.4), but an amplitude (in fact a much more involved function) with asymptotic behavior such that $D_s(s,t)\sim f(t)s^{\alpha(t)}$ when $s\to\infty$ at fixed t. We have explicitly checked and confirmed this asymptotic behavior, but shall not reproduce the calculation here. One also observes from Eq. (5.12) that the factor $\Gamma(\alpha(t)+1)^{-1}$, usually extracted from $f(t)$ by writing

$$f(t)=\frac{R(t)}{\Gamma(\alpha(t)+1)}\,,$$

is canceled in G, so that $G(z)$ is essentially given by the Regge residue $R(z)$.

Had we considered Regge behavior as $t\to\infty$ at fixed s, given by

$$D_t(s,t)\sim f(s)t^{\alpha(s)}\,,$$

we would have obtained, for $\psi(z,x)$ as $(1-x)\to+0$,

$$\psi(z,x)\sim G(z)(1-x)^{-\alpha(z)-1}[-\ln(1-x)]^{-\alpha(z)} \times\left[1+\frac{\ln(-\ln(1-x))}{\ln(1-x)}\right]^{-\alpha(z)-1}\,, \tag{5.13}$$

which is obtained from (5.11) by imposing crossing symmetry on ψ.

A crossing-symmetric asymptotic form for $\psi(z,x)$ may in fact be written immediately from an expression for ψ near $x=0$. For, suppose that $\psi(z,x)=\psi(z,1-x)$ and that

$$\psi(z,x)\sim\psi_1(z,x) \quad \text{as } x\to+0\,. \tag{5.14}$$

Define $\psi_2(z,x)$ by

$$\psi(z,x)=\psi_1(z,x(1-x))\psi_2(z,x)\,.$$

Then $\psi_2(z,x)\sim 1$ as $x\to+0$ and $\psi_2(z,x)=\psi_2(z,1-x)$. Thus $\psi_2(z,x)\sim 1$ also as $1-x\to+0$. Thus

$$\psi(z,x)\sim\psi_1(z,x(1-x)) \quad \text{as } x\to+0 \text{ or } 1-x\to+0\,. \tag{5.15}$$

From (5.11) we may therefore write the crossing-symmetric form

$$\psi(z,x)\sim G(z)[x(1-x)]^{-\alpha(z)-1}[-\ln(x(1-x))]^{-\alpha(z)} \times\left\{1+\frac{\ln[-\ln(x(1-x))]}{\ln(x(1-x))}\right\}^{-\alpha(z)-1}\,, \tag{5.16}$$

as $x\to+0$ or $(1-x)\to+0$. This, of course, includes both (5.11) and (5.13). When the asymptotic form in (5.16) is substituted for ψ in (1.2), one obtains a crossing-symmetric amplitude with Regge behavior. The resulting amplitude will also possess some desirable analytic properties, such as correct threshold cuts and nonvanishing Mandelstam double-spectral functions. Its exact analytic structure, however, requires detailed investigation which we have not undertaken. In Sec. VI we comment on the significance for the construction of dual models.

VI. DUALITY

The relevance of our investigation to the construction of dual models is indeed rather obvious: starting from a Regge amplitude and following our procedure in Sec. V, one ends up with a crossing-symmetric amplitude with Regge behavior with nonvanishing double-spectral functions in which *a generalized notion of duality is already incorporated.* As was pointed out by Cohen-Tannoudji *et al.*,[2] it is essentially the association of s and t with x and $1-x$, respectively (called by these authors "s-x duality") that produces many correct analytic and unitary features in the amplitude. In fact one may further observe that this property of the integrand enables the amplitude to maintain a dual structure in the usual sense. For, when a certain behavior in s is associated with one end of the region of integration, then the corresponding behavior in t is associated with the other end. In particular, both the high-energy behavior in s and the resonance poles in t arise from the end point $x=0$.

Thus, under conditions favorable for the expansion of the integrand, the usual Regge-pole resonance duality is maintained.

Dual amplitudes based on the Nakanishi representation are, however, quite different from the type of amplitude discussed by Cohen-Tannoudji *et al.*,[2] which is of the form

$$A(s,t)=\int_0^1 x^{-\alpha(t(1-x))-1}(1-x)^{-\alpha(sx)-1}f(sx)f(t(1-x))dx, \tag{6.1}$$

so that the integrand is not a function of $xs+(1-x)t$ as in the Nakanishi representation. This amplitude, however, possesses many desirable features, including Mandelstam analyticity, and appears to be a positive improvement on the Veneziano amplitude. It also, of course, presents several difficulties. It remains to be seen whether the use of models of the type implied by (5.16) leads to further improvements. The dual amplitude which we have recently[4] suggested on the basis of the Nakanishi representation neglects the logarithmic factors in (5.16). It now appears that these logarithmic factors are rather important and a dual model taking them into account is under investigation.

Variations on the asymptotic form in (5.16) may be obtained by using different forms for the Regge input, for example in terms of Legendre functions. Some analytic features of the resulting models would in this case be different, although crossing symmetry and Regge behavior are satisfied. Such differences may therefore be usefully exploited. One may also determine the asymptotic form for ψ corresponding to a Regge cut and construct amplitudes with cut-cut or pole-cut structure. Furthermore, it is possible to incorporate certain bounds and to satisfy given fixed-angle asymptotic behaviors without much difficulty, since the corresponding restrictions on ψ are easily obtained. Crossing symmetry, if lost, appears to be immediately recoverable at any stage.

It thus appears that a great advantage of prospective dual models based on the Nakanishi representation would be their direct link with a general theory which, in principle, can provide a source for further improvements. We finally observe that the Nakanishi representation has an N-point generalization which could, perhaps, be used for the construction of N-point dual amplitudes. Such a construction may, alternatively, be sought as a generalization of 4-point dual amplitudes obtained from the 4-point Nakanishi representation.

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APPENDIX A: REMARKS ON THE CALCULATION OF $H(z;x;z',z'';x',x'')$

To maintain explicit symmetry in z', z'' and x', x'', we use the δ functions in (4.9) to effect the integrations over y' and y''. This gives

$$H(z;x;z',z'';x',x'')=\frac{1}{2\pi i(1-x)^2}\int_{4m^2}^{\infty}\frac{ds}{(s-4m^2)^2}G\left(s;\frac{z-xs}{1-x};\frac{z'-x's}{1-x'},\frac{z''-x''s}{1-x''}\right)\Lambda\left(\frac{z'-x's}{(4m^2-s)(1-x')},\ \frac{z''-x''s}{(4m^2-s)(1-x'')}\right), \tag{A1}$$

where

$$\Lambda(x,y)=\theta(x)\theta(1-x)\theta(y)\theta(1-y). \tag{A2}$$

Use of these θ functions breaks up the integral into several pieces. Consider, for example, the first two θ functions in (A2), which state the inequalities

$$0>z'-4m^2x'-x'(s-4m^2)>-(1-x')(s-4m^2).$$

These may be written in the form

$$\frac{z'}{x'}<s<4m^2+\frac{z'-4m^2x'}{2x'-1}\quad\text{when}\quad x'>\tfrac{1}{2},$$

$$s>\frac{z'}{x'},\ 4m^2+\frac{4m^2x'-z'}{1-2x'}\quad\text{when}\quad x'<\tfrac{1}{2}.$$

Simple algebraic manipulations then show that the integral over s breaks into three parts:

$$\lambda_1'\int_{a_1'}^{b_1'} ds + \lambda_2'\int_{a_2'}^{b_2'} ds + \lambda_3'\int_{a_3'}^{b_3'} ds ,$$

where

$$\begin{aligned} &\lambda_1' = \theta(z' - 4m^2x')\,\theta(x' - \tfrac{1}{2}), \quad a_1' = \frac{z'}{x'}, \quad b_1' = \frac{z' - 4m^2(1-x')}{2x'-1};\\ &\lambda_2' = \theta(z' - 4m^2x')\,\theta(\tfrac{1}{2} - x'), \quad a_2' = \frac{z'}{x'}, \quad b_2' = \infty; \\ &\lambda_3' = \theta(4m^2x' - z')\,\theta(\tfrac{1}{2} - x'), \quad a_3' = \frac{4m^2(1-x') - z'}{1-2x'}, \quad b_3' = \infty . \end{aligned} \tag{A3}$$

Thus on using all the θ functions in (A2), one obtains the following decomposition:

$$H = \sum_{i,j=1}^{3} \gamma_{ij} I_{ij} , \tag{A4}$$

where

$$I_{ij} = \frac{1}{2\pi i(1-x)^2}\int_{\alpha_{ij}}^{\beta_{ij}} \frac{ds}{(s-4m^2)^2}\, G\left(s; \frac{z-xs}{1-x}; \frac{z'-x's}{1-x'}, \frac{z''-x''s}{1-x''}\right), \tag{A5}$$

$$\gamma_{ij} = \lambda_i'\lambda_j'' , \tag{A6}$$

$$\alpha_{ij} = \max\{a_i', a_j''\}, \qquad \beta_{ij} = \min\{b_i', b_j''\}. \tag{A7}$$

The double primes in (A6) and (A7) indicate that z' and x' in (A3) are to be replaced by z'' and x'' to obtain the corresponding functions. It should be noted that the form of the limits in (A7) shows that a contribution of the form $\gamma_{ij}I_{ij}$ again decomposes into a number (at most four) of integrals multiplied by θ functions.

We now briefly consider the general form of the integrand in (A5). Since in Eq. (4.5) the δ' term may be trivially integrated, let us consider only the θ term. This contributes to $G(s;\ t;\ t', t'')/(s-4m^2)^2$ the term

$$\frac{-i}{\pi^2} s^{-1/2}(s-4m^2)^{-5/2}\left(\frac{2t't''}{s-4m^2} + t' + t'' - 2\right)\left(\frac{4tt't''}{s-4m^2} - (t^2 + t'^2 + t''^2) + 2(tt' + tt'' + t't'')\right)^{-3/2},$$

so that its contribution to the integrand in (A5) takes the form

$$\frac{P_2(s)\,\theta(P_4(s))}{(s-4m^2)^2 P_3(s)[P_4(s)]^{1/2}}, \tag{A8}$$

where $P_n(s)$ is a polynomial of order n in s. The integral (A5) may therefore be expressed in terms of elliptic integrals of the first, second, and third kinds.[5] Thus the complete calculation of H is seen to be straightforward, though rather tedious and lengthy.

APPENDIX B: ASYMPTOTIC FORM OF I AS $\nu \to \infty$

$$I = \int_0^\infty F(z-\lambda)e^{h(\nu,\lambda,z)}\,d\lambda , \tag{B1}$$

$$h(\nu,\lambda,z) = \alpha(z-\lambda)(\nu + \ln\lambda). \tag{B2}$$

To apply the generalized Laplace method,[3] consider the equation $\partial h/\partial\lambda = 0$. This gives

$$\beta(z-\lambda) = \lambda(\nu + \ln\lambda), \tag{B3}$$

where

$$\beta(z) = \alpha(z)/\alpha'(z). \tag{B4}$$

From (B3) we see that if, in the neighborhood of a turning point, $\lambda\nu$ tends to a finite value as $\nu \to \infty$, then $\lambda\nu \to \beta(z)$. On the other hand, when, in the neighborhood of a turning point, $\lambda\nu \to \infty$ as $\nu \to \infty$, then $\beta(z-\lambda)$ must tend to infinity near that point. It is thus sufficient to assume that $\beta(z)$ remains finite for all $z < 4m^2$ to ensure that (B3) is satisfied only in the neighborhood of $\lambda = 0$. Further, to ensure that this neighborhood is in the region of integration, we must restrict ourselves to values of z in (B1) such that $\beta(z) > 0$. We then observe that

$$\partial^2 h/\partial\lambda^2 \sim -\alpha(z)/\lambda^2$$

as $\lambda \to +0$, with $\nu \sim \beta(z)/\lambda$. Thus for z such that $\alpha(z)>0$, $\alpha'(z)>0$, the dominant contribution to I comes from the neighborhood of $\lambda = 0$.

Define u by $\lambda = \beta u/\nu$. Then $u \sim 1$ is dominant and

$$I = \frac{\beta(z)}{\nu}\int_0^\infty F\left(z - \frac{\beta u}{\nu}\right)\exp\left[\alpha\left(z-\frac{\beta u}{\nu}\right)\left(\nu + \ln\frac{\beta u}{\nu}\right)\right]du\,. \tag{B5}$$

Using the mean-value theorem,

$$\alpha\left(z-\frac{\beta u}{\nu}\right) = \alpha(z) - \frac{\beta u}{\nu}\alpha'(z) + \frac{\beta^2 u^2}{2\nu^2}\alpha''\left(z - \theta\frac{\beta u}{\nu}\right), \quad 0<\theta<1\,.$$

Therefore,

$$\alpha\left(z-\frac{\beta u}{\nu}\right)\left(\nu+\ln\frac{\beta u}{\nu}\right) = (\nu - \ln\nu)\left[\alpha(z) - \frac{\beta u}{\nu}\alpha'(z)\right] + \alpha(z)\ln\beta u + g(\nu,u,z)\,, \tag{B6}$$

where

$$g(\nu,u,z) = \frac{\beta^2 u^2}{\nu^2}\left(\nu + \ln\frac{\beta u}{\nu}\right)\alpha''\left(z-\theta\frac{\beta u}{\nu}\right) - \frac{\beta u}{\nu}\alpha'(z)\ln\beta u\,. \tag{B7}$$

We now note that

$$F(z-\beta u/\nu)\exp g(\nu,u,z) = F(z)[1+O(1/\nu)] \quad \text{as} \quad \nu\to\infty\,.$$

Thus

$$I \sim \frac{\beta(z)}{\nu}F(z)\int_0^\infty \exp\left\{(\nu-\ln\nu)\left[\alpha(z) - \frac{\beta u}{\nu}\alpha'(z)\right] + \alpha(z)\ln\beta u\right\}du = F(z)\beta^{\alpha+1}\nu^{-\alpha-1}e^{\nu\alpha}\int_0^\infty e^{\gamma u}u^\alpha du\,, \tag{B8}$$

where

$$\gamma = \alpha(z)\left(\frac{\ln\nu}{\nu} - 1\right). \tag{B9}$$

Since $\alpha(z)>0$, and, for large ν, $\gamma<0$, the integral in (B8) is well defined and we finally obtain

$$I \sim F(\alpha'\nu)^{-\alpha-1}e^{\nu\alpha}\left(1-\frac{\ln\nu}{\nu}\right)^{-\alpha-1}\Gamma(\alpha+1) \quad \text{as} \quad \nu\to\infty\,, \tag{B10}$$

where F, α, and α' denote $F(z)$, $\alpha(z)$, and $\alpha'(z)$. This result holds to order $1/\nu$ for all $z<4m^2$ such that $\alpha(z)>0$, $\alpha'(z)>0$.

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*On leave of absence from the Department of Physics, University of Khartoum, Sudan.

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