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[1]All relevant prior investigations pertinent to double-spectral representations for scalar boxes, including definitions (2) and (3), as well as Eq. (7), are well exposited and documented in the monograph: D. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge Univ. Press, Cambridge, England, 1966).

[2]The only prior investigation of double-spectral representations for photon-photon scattering the authors are aware of is B. DeTollis, Nuovo Cimento 32, 757 (1964). In that work, an unparametrized form of one of the invariant amplitudes for photon-photon scattering is reduced to double-spectral form by direct application of the Cutkosky rules. The connection between Feynman-parametrized integrals and their double-spectral representations was not investigated. Not only can we reproduce DeTollis's result, but in consequence of our investigation can understand the source of the single-spectral terms in DeTollis's amplitude in terms of the gauge-invariant tensor he chose. However, we have not been able to verify the existence of some other set of tensors for this process that might allow of kinematic singularity-free unsubtracted representations for the amplitudes.

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Shadow Channels in Potential Theory*

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The effect of coupling a shadow channel to a real channel is studied in the framework of nonrelativistic potential-scattering theory. For simplicity, s-wave scattering with energy-independent δ-function potentials is considered. This model has the virtue of being exactly solvable in configuration space. The main features of the solution which are analogous to those expected in a relativistic field theory whose divergences have been removed using Sudarshan's shadow states, are as follows. The coupled problem reduces in the real channel to scattering by an effective energy-dependent pseudopotential. The physical scattering amplitude still satisfies elastic unitarity at all energies, but is piecewise analytic, with a point of nonanalyticity occurring at the threshold for the shadow channel.

I. INTRODUCTION

It is well known that the introduction of states with negative norms is one way of eliminating the divergence difficulties that occur in local relativistic quantum field theories.[1] To preserve the probability interpretation of such indefinite-metric theories, Sudarshan has introduced the idea of "shadow states."[2] Shadow states differ from ordinary states in that they "propagate" with half-advanced, half-retarded Green's functions. This ensures that they never become physical, which means that their introduction does not change the unitarity properties of the theory.[2,3] However, theories with shadow states are not expected to be globally analytic, but instead will be piecewise analytic with points of nonanalyticity at shadow thresholds.[2,3] Of course, such states have a profound effect on the dynamics of the theory, since it is just the additional forces due to the presence of the shadow states that lead to a convergent theory.

In this paper the effect of coupling a shadow channel to a real channel is investigated in the framework of nonrelativistic potential-scattering theory. For simplicity, s-wave scattering with real energy-independent δ-function potentials is considered. This model has the virtue that the coupled Schrödinger equations can be solved *exactly* in configuration space. It turns out that the two-channel problem reduces, in the real channel, to an effective one-channel problem with a real, energy-dependent (and in general nonlocal) pseudopotential. Furthermore, the physical scattering amplitude still satisfies elastic unitarity at all energies, but is piecewise analytic with a point of nonanalyticity at the threshold for the shadow channel. Consequently, this nonrelativistic model displays the main features of the field-theoretic problem: a modification of the forces in the physical channel accompanied by a change in the analyticity properties of the theory but no change in the unitarity properties.

The plan of this paper is as follows: In Sec. II the formal solution to the two-channel problem is written down in terms of the free-particle Green's functions in the different channels. Section III is devoted to the choice of Green's functions. An advanced Green's function is chosen (as usual) in the real channel. A half-advanced, half-retarded (i.e., principal-value) Green's function is necessary in a shadow channel to guarantee no outgoing flux of particles asymptotically. A novel feature of such a function is that it is piecewise analytic with a point of nonanalyticity at the threshold for "scattering" in the shadow channel. In Sec. IV the exact physical scattering amplitude for the coupled-channel problem with δ-function potentials is obtained. The properties of the solution (pseudopotential, elastic unitarity, piecewise analyticity) are discussed in Sec. V. Section VI contains a few concluding remarks tying this work together with work on shadow states in relativistic quantum field theory.

II. TWO-CHANNEL PROBLEM – FORMAL SOLUTION

For simplicity, s-wave scattering is considered.[4] Assuming that channel 1 consists of two identical spinless particles of mass m_1 and channel 2 consists of two identical spinless particles of mass $m_2 > m_1$, the coupled Schrödinger equations are[5]

$$\left[\frac{d^2}{dr^2} + K_1{}^2 - U_1(r)\right] u_1(r) = U_{12}(r) u_2(r) \tag{1a}$$

and

$$\left[\frac{d^2}{dr^2} + K_2{}^2 - U_2(r)\right] u_2(r) = U_{21}(r) u_1(r). \tag{1b}$$

In Eqs. (1), $u_i(r) = r\psi_i(r)$, $U_i(r) = m_i V_i(r)$, $U_{ij}(r) = m_i V_{ij}(r)$, and $K_i{}^2 = m_i E_i$; E_i equals the total center-of-mass energy in channel i. The center-of-mass momenta in the two channels are related by

$$K_1{}^2 = K_2{}^2 + K_{\text{thresh}}{}^2, \tag{2a}$$

where

$$K_{\text{thresh}}{}^2 = 2m_1(m_2 - m_1) \tag{2b}$$

is the momentum squared in channel 1 at the inelastic threshold for channel 2 ($K_2 = 0$).

The formal solutions to Eqs. (1) are

$$u_1(r) = u_1^{\text{homo}}(r) + \int_0^\infty dr' G_1(r, r')[U_1(r')u_1(r') + U_{12}(r')u_2(r')] \tag{3a}$$

and

$$u_2(r) = u_2^{\text{homo}}(r) + \int_0^\infty dr' G_2(r, r')[U_2(r')u_2(r') + U_{21}(r')u_1(r')], \tag{3b}$$

where the homogeneous solutions are[5]

$$u_i^{\text{homo}}(r) = \frac{1}{K_i} \sin(K_i r), \tag{4}$$

and the Green's functions $G_i(r, r')$ are the solutions to

$$\left(\frac{d^2}{dr^2} + K_i{}^2\right) G_i(r, r') = \delta(r - r'), \tag{5}$$

which vanish when either r or r' is zero.

III. CHOICE OF GREEN'S FUNCTIONS

Writing the Green's function in a Fourier representation

$$G(r, r') = \int_{-\infty}^\infty dK' g(K') e^{iK'(r-r')}, \tag{6}$$

Eq. (5) requires that

$$g(K') = -\frac{1}{2\pi} \frac{1}{K'^2 - K^2}, \tag{7}$$

and so

$$G(r, r') = -\frac{1}{2\pi} \int_{-\infty}^\infty dK' \frac{e^{iK'(r-r')}}{K'^2 - K^2}. \tag{8}$$

To proceed further it must be decided how to treat the singularities which arise in the integrand of (8) when $K' = \pm K$ ($K \geq 0$). There are three alternatives: Displace the poles away from the real K' axis by adding $+i\epsilon$ or $-i\epsilon$ to K^2 ($\epsilon \to 0^+$), or leave the poles undisplaced and perform the integration using the principal-value prescription. The appropriate choice is dictated by the boundary conditions on the wave functions.

A. Real Channel

At energies above the threshold for scattering in a real channel there must be a nonzero flux of outgoing particles at large distances from the scattering center (if there is to be any scattering). In terms of the wave function, this translates into the statement that the inhomogeneous term must behave as an outgoing traveling wave asymptotically. This boundary condition can be satisfied by choosing the advanced Green's function

$$G_+(r, r') = -\frac{1}{2\pi} \int_{-\infty}^\infty dK' \frac{e^{iK'(r-r')}}{K'^2 - (K^2 + i\epsilon)}. \tag{9}$$

This integrates to

$$G_+(r, r') = -\frac{i}{2K} e^{iKr_>} e^{-iKr_<}, \tag{10}$$

where $r_< = \min(r, r')$ and $r_> = \max(r, r')$. To satisfy the boundary condition $u(0) = 0$, the Green's function must also satisfy

$$G_+(r, r') = 0, \quad r_< = 0. \tag{11}$$

The function (10), as it stands, does not satisfy this condition, but any solution to the homogeneous equation can be added to the inhomogeneous solution with impunity. Therefore, adding

$$h(r, r') = \frac{+i}{2K} e^{iKr_>} e^{iKr_<} \tag{12}$$

to (10) gives

$$G_+(r, r') = -\frac{1}{K}\sin(Kr_<)e^{iKr_>}, \tag{13}$$

which does vanish when $r_<$ is zero.

Use of this Green's function gives an inhomogeneous term in the wave function which behaves like e^{iKr} at large r (traveling wave), guaranteeing a nonzero flux asymptotically for real positive momentum K. So it is the appropriate choice to describe a real channel.

B. Shadow Channel

In a shadow channel there must be no flux asymptotically (otherwise the state would be observable and therefore real). This boundary condition can be satisfied by choosing the half-advanced, half-retarded Green's function, which corresponds to evaluating (8) according to the principal-value prescription. Below the "scattering" threshold in the shadow channel, the poles in the integrand of (8) lie at $\pm i|K_s|$, and so the half-advanced, half-retarded Green's function is the same as the advanced (or retarded) Green's function:

$$G_s(r, r') = -\frac{1}{K_s}\sin(K_s r_<)e^{iK_s r_>}, \quad K_s^2 < 0. \tag{14a}$$

Above threshold the poles lie on the real K' axis, and performing the integration according to the principal-value prescription gives

$$G_s(r, r') = -\frac{1}{K_s}\sin(K_s r_<)\cos(K_s r_>), \quad K_s^2 \geq 0. \tag{14b}$$

For real, positive K_s the Green's function (14b) will lead to an inhomogeneous term in the wave function which behaves like $\cos(K_s r)$ at large r (standing wave), guaranteeing no outgoing flux in the shadow channel. Therefore, a principal-value Green's function turns out to be the appropriate choice to describe a shadow channel. It should be stressed that the principal-value Green's function is a bona fide Green's function and there is no reason not to use it once a motivation exists.

A point to be noted is that the shadow Green's function (14b) is *not* the analytic continuation of (14a). Consequently, this function is piecewise analytic with a point of nonanalyticity at $K_s = 0$. As will be seen below, this property of the shadow Green's function is reflected in the piecewise analyticity of the physical scattering amplitude in the coupled-channel problem.

IV. EXACTLY SOLVABLE MODEL

To study the effect of the coupling of a shadow channel (labeled 2) to a real channel (labeled 1), a special case in which Eqs. (3) are exactly solvable is considered: $V_1(r) = V_1\delta(r-a)$, $V_2(r) = 0$, and $V_{12}(r) = V_{21}(r) = V_{12}\delta(r-a)$, with V_1 and V_{12} real constants.[6,7] The coupled equations to be solved are[8]

$$u_1(r) = u_1^{\text{homo}}(r) + G_+(r, a)\,U_1 u_1(a) + G_+(r, a)\,U_{12} u_2(a) \tag{15a}$$

and

$$u_2(r) = G_s(r, a)\,U_{21} u_1(a). \tag{15b}$$

The solution is straightforward: Set $r = a$ in (15b) to obtain $u_2(a)$; substitute this into (15a); set $r = a$ in (15a) and solve for $u_1(a)$; and then substitute the results for $u_2(a)$ and $u_1(a)$ into (15a) and (15b). The exact solutions are

$$u_1(r) = u_1^{\text{homo}}(r) + \frac{[U_1 + U_{12}U_{21}G_s(a, a)]u_1^{\text{homo}}(a)G_+(r, a)}{1 - [U_1 + U_{12}U_{21}G_s(a, a)]G_+(a, a)} \tag{16a}$$

for the wave function in the real channel and

$$u_2(r) = \frac{U_{21}u_1^{\text{homo}}(a)G_s(r, a)}{1 - [U_1 + U_{12}U_{21}G_s(a, a)]G_+(a, a)} \tag{16b}$$

for the wave function in the shadow channel.

Note that the solution to the real-channel problem with no coupling to a second channel and potential $V_1(r) = V_1\delta(r-a)$ is

$$u_1^0(r) = u_1^{\text{homo}}(r) + \frac{U_1 u_1^{\text{homo}}(a)G_+(r, a)}{1 - U_1 G_+(a, a)}. \tag{17}$$

Comparison of Eqs. (16a) and (17) shows that the coupled-channel problem effectively reduces in the real channel to a one-channel scattering problem with the pseudopotential

$$U_{\text{pseudo}}(r) = [U_1 + U_{12}U_{21}G_s(a, a)]\delta(r-a), \tag{18}$$

which is *real* and energy-dependent.[9]

Defining the physical scattering amplitude T_1 by

$$u_1(r) \underset{r\to\infty}{\sim} u_1^{\text{homo}}(r) - T_1 e^{iK_1 r}, \tag{19}$$

it follows from Eq. (17) that

$$T_1^0 = -\frac{U_1[(1/K_1)\sin(K_1 a)]^2}{1 - U_1 G_+(a, a)} \tag{20}$$

for scattering with no coupled shadow channel, and from Eq. (16a) that

$$T_1 = -\frac{[U_1 + U_{12}U_{21}G_s(a, a)][(1/K_1)\sin(K_1 a)]^2}{1-[U_1 + U_{12}U_{21}G_s(a, a)]G_+(a, a)} \quad (21)$$

for scattering with a coupled shadow channel.

V. DISCUSSION

One obvious effect of the presence of the shadow channel is a change in the dynamics. With no coupled shadow channel the scattering in the real channel is determined by the direct potential $V_1(r)$. Coupling in a shadow channel introduces new forces, as evidenced by the fact that the effective potential felt in the real channel becomes the pseudopotential (18).

From Eq. (20) it explicitly follows that the scattering amplitude for the uncoupled one-channel problem satisfies elastic unitarity at all energies:

$$\mathrm{Im}(T_1^0) = K_1 |T_1^0|^2, \quad K_1 \geq 0. \quad (22)$$

Likewise, from Eq. (21) it follows that

$$\mathrm{Im}(T_1) = K_1 |T_1|^2, \quad K_1 \geq 0, \quad (23)$$

and consequently the physical scattering amplitude also satisfies elastic unitarity at all energies if the channel coupling to the real channel is a shadow channel.[10]

It also happens that the physical scattering amplitude is piecewise analytic with a point of nonanalyticity occurring at the shadow threshold. This follows explicitly from (21) and from the fact that the shadow Green's function is piecewise analytic with a point of nonanalyticity at $K_s = 0$, as was pointed out above. This property of the physical scattering amplitude would be reflected in a cusplike behavior of the scattering cross section at the energy corresponding to the shadow threshold. However, similar threshold anomalies can arise in the coupling of real channels, and so are in no way unique to theories with piecewise-analytic scattering amplitudes.[11]

VI. CONCLUSION

In a fairly simple (but exactly solvable) potential model, the effect of the introduction of a shadow channel has been elucidated. Basically, the dynamics of the theory are modified but the unitarity properties are unchanged, while global analyticity no longer holds.[12] Similar features are expected in a relativistic field theory with shadow states. In a theory of this latter type, however, things are complicated by the fact that there can be many-particle shadow states, and it is the entire many-particle state (not some particular particle) that has principal-value propagation.[13] Nevertheless, in every order of perturbation, unitarity is guaranteed and nonanalyticity observed. Consequently, even though only the lowest-order terms in a perturbation solution are usually explicitly calculable, the role played by the shadow states in such theories should not be mysterious.[14] If the role of the shadow states has appeared so to the reader, it is hoped that this work has thrown some new light on the matter.

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[1]P. A. M. Dirac, Proc. Roy. Soc. (London) A180, 1 (1942); W. Pauli and F. Villars, Rev. Mod. Phys. 21, 434 (1949); E. C. G. Sudarshan, in *Fundamental Problems in Elementary Particle Physics: Proceedings of the Fourteenth Conference of the Solvay Institute of Physics* (Interscience, London, 1968), p. 97; T. D. Lee and G. C. Wick, Nucl. Phys. B9, 209 (1969).

[2]E. C. G. Sudarshan, Proc. Indian Acad. Sci. 69A, 133 (1969); in *Coral Gables Conference on Fundamental Interactions at High Energy II*, edited by A. Perlmutter *et al.* (Gordon and Breach, New York, 1970), Vol. II, pp. 80–93; Fields and Quanta (to be published).

[3]A. M. Gleeson, R. J. Moore, H. Rechenberg, and E. C. G. Sudarshan, Phys. Rev. D 4, 2242 (1971).

[4]This choice is arbitrary. The analysis could, in principle, be carried out in any partial wave, and consequently the conclusions drawn are valid for the complete scattering amplitude.

[5]R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966), Chap. 11. Local potentials are assumed and natural units are used.

[6]If $V_1(r)$ and $V_{12}(r)$ are chosen to have different ranges, an exact solution also exists, but the algebra is tedious and the resulting expressions messy. No new essential features come in, and so only the case of equal ranges will be discussed here.

[7]Exact solutions of Eqs. (3) also exist for square-well potentials or separable potentials. For the coupling of two real channels, see A. N. Kamal and H. J. Kreuzer, Phys. Rev. D 2, 2033 (1970).

[8]The homogeneous term in the shadow-channel wave function has been set equal to zero to ensure only "outgoing" waves in the shadow channel.

[9]If the potentials $V_1(r)$ and $V_{12}(r)$ had been chosen to have

different ranges, the pseudopotential would be real, energy-dependent, and nonlocal.

[10]This differs from the coupling of two real channels, in which case elastic unitarity holds only up to the inelastic threshold. Cf. H. Feshbach, Ann. Phys. (N.Y.) 5, 357 (1958).

[11]See, for example, R. G. Newton, Ref. 5, Chap. 17.

[12]Implications of the breakdown of global analyticity on high-energy phenomenology are currently under investigation (private communication from M. G. Gundzik).

[13]E. C. G. Sudarshan, Fields and Quanta (to be published).

[14]Some progress has been made in solvable Lee-type models with shadow states. Cf. C. A. Nelson and E. C. G. Sudarshan, University of Texas at Austin Report No. CPT-94, 1971 (unpublished).

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Broken Coupling Constants, Smoothness, and Scalar Mesons in the Gell-Mann–Oakes–Renner Model

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Considering SU(3)-octet broken scalar-pseudoscalar meson (*PPS*) couplings as well as "smoothness" in the chiral model of Gell-Mann, Oakes, and Renner (GMOR), we obtain the spectrum of scalar mesons in terms of the known masses of the speudoscalar mesons only. The agreement of the predicted spectrum with experiment is reasonably good. The broken pieces of the coupling constants are determined as well and are found to be comparable to their corresponding symmetric pieces. The effects of the broken couplings in the numerical results, such as decay widths, have also been examined. We have demonstrated that the usual assumption of the Goldstone bosons required by exact chiral symmetry emerges automatically from the smoothness relations. A few other interesting aspects of our work are finally discussed.

I. INTRODUCTION

During recent years various studies based on theories of chiral symmetry[1-11] and broken scale invariance[12-14] have indicated convincingly that scalar mesons have indeed a significant role in determining the coupling constants and masses of the hadrons. Working in different formalisms, many authors have predicted the masses and decay rates of scalar particles; however, none of these predictions can be tested decisively because of the lack of precise experimental information. In a broken chiral $SU(3)\otimes SU(3)$ model, Glashow[3] first obtained the scalar-meson spectrum in terms of the known masses of pseudoscalar mesons assuming that the mass term of the Lagrangian has the simple $(3,\bar{3})\oplus(\bar{3},3)$ transformation under the group $SU(3)\otimes SU(3)$ and is dominated by scalar-meson tadpoles. His predicted spectrum is not realistic in the sense that it includes a scalar particle with complex mass and the scalar kaon (so-called κ meson) with a very low mass ($\approx$770 MeV) which is not admitted by present experimental evidence.[15]

In subsequent works, several authors[5-7] have shown that a large mass of the κ meson is favored in the GMOR model of chiral symmetry.[1] In this model, one obtains a simple algebraic relation between the κ-meson mass (m_κ) and the coupling constants f_K and f_π, which are experimentally determined from the rates of $K\to l\nu$ and $\pi\to l\nu$ decays, respectively. For a reasonable value of the ratio $f_K/f_\pi \simeq 1.2$, m_κ is found to be approximately 1200 MeV,[6,7] in close agreement with experiment.

On the other hand, the complete spectrum of the scalar mesons has been obtained by Pande[8] who worked in the same model (GMOR), utilizing the constraints of various subsymmetries contained in chiral $SU(3)\otimes SU(3)$ symmetry.[9] He also considered the effect of SU(3) breaking through conventional mixing phenomena and predicted a small mixing angle ($\theta_S \approx 11°$) for scalar particles. This work is, however, open to criticism as it incorporates a very drastic assumption that masses of