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Self-Consistent Dispersion-Theoretic Electromagnetic Scattering Amplitudes

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A procedure for calculating self-consistent electromagnetic scattering amplitudes when there are bound states, applied in a previous paper to the electron-positron system, is extended to include spin-0–spin-0 and spin-0–spin-$\frac{1}{2}$ elastic scattering. The second-order amplitudes which result have the correct double-spectral functions and are cutoff-independent, analytic, and crossing-symmetric. The Regge trajectory functions which are required by self-consistency imply the usual Coulomb bound-state spectrum with reduced mass and recoil corrections, and the appropriate Regge asymptotic behavior. The correspondence between the amplitudes obtained by this procedure and by other means is discussed briefly.

I. INTRODUCTION

In a previous paper[1] we exhibited a method for constructing a self-consistent elastic electron-positron scattering amplitude which is Lorentz-covariant, analytic, cutoff-independent, and crossing-symmetric, which has the correct double-spectral functions through second order in the fine-structure constant, α, and which reduces to the usual Born term in lowest order. Moreover, the amplitude displays Regge asymptotic behavior and the positronium Regge poles. Self-consistency requires that the positronium poles appear at the correct position and with the proper residue, and that the amplitude possesses a well-defined Jacob-Wick expansion. The purpose of this paper is to extend the results of Ref. 1 to include the somewhat simpler processes described by spin-0–spin-0 and spin-0–spin-$\frac{1}{2}$ elastic scattering. Although these amplitudes have only a limited applicability to actual physical problems within the framework of pure quantum electrodynamics, they are, nevertheless, of theoretical interest since our procedure is considerably easier to follow in the absence of spin. In addition, with this simplification we obtain a convenient theoretical laboratory from which our procedure may be extended to higher orders more easily. Finally, the cases in which the spin of one or both particles is neglected are generally more familiar, which should facilitate comparison of our results with electromagnetic scattering amplitudes obtained by other means. Thus, in the following pages we will briefly review our procedure for the construction of self-consistent electromagnetic scattering amplitudes. We will then apply this technique to the calculation of the elastic scattering amplitude for two particles of spin 0. We will first consider the process in which the particles have unequal masses, and then specialize to the case of identical bosons. Finally, we will exhibit our results for the case in which one particle has spin $\frac{1}{2}$. We will also discuss briefly some relationships between our method and certain other approximations to the construction of electromagnetic scattering amplitudes when there are bound states.

II. THEORY

Because the perturbation expansion of the scattering amplitude does not converge when there are bound states,[2] the calculation of electromagnetic scattering amplitudes in which bound-state poles may appear occupies a rather dubious position. On the one hand, it is possible to write down an expression for the scattering *cross section*, based on perturbation theory, which so far has been adequate to account for the experimental situation.[3] However, the associated perturbation *amplitudes* are infrared divergent, which is unsatisfactory if one intends to employ them in another context. On the other hand, one can abandon a perturbation approach and, instead, attempt to construct the scattering solutions to the relativistic wave equations which have been developed. However, it has proved extremely difficult to obtain useful scattering solutions to the Bethe-Salpeter equation in physically interesting cases, and the Dirac and Klein-Gordon equations do not incorporate crossing or inelastic unitarity and, at best, are only accurate to lowest order. Accordingly, an alternative approach is desirable.

The basis for our calculation of electromagnetic scattering amplitudes can be found in a consideration of nonrelativistic potential theory. Accordingly, we wish to review here some aspects of the Coulomb amplitude obtained from the Schrödinger equation.

A. Coulomb Amplitude in Potential Theory

The Coulomb amplitude can be obtained in closed form from the solution of the Schrödinger equation in parabolic coordinates. For a particle of mass m and momentum p in an attractive potential, we find

$$A(E,z)=\frac{\alpha m}{2p^2}\,\frac{\Gamma(1-i\eta)}{\Gamma(1+i\eta)}\left(\frac{-t}{4p^2}\right)^{-1+i\eta}-\frac{1}{ip}\delta(1-z)\,, \tag{1}$$

where $\eta=\alpha m/p$, $t=-2p^2(1-z)$, $E=p^2/2m$, and $z=\cos\theta$ is the cosine of the scattering angle. $\delta(x)$ is the Dirac δ function. We see that $A(E,z)$ has Regge asymptotic behavior with only the leading trajectory contributing and, due to the Γ functions, the correct Coulomb bound-state poles. The partial-wave amplitudes obtained from (1) have the form

$$a_l(E)=\frac{1}{p}\,e^{i\delta_l}\sin\delta_l\,, \tag{2}$$

where $\delta_l=\arg\Gamma(l+1-i\eta)$, so that (1) satisfies elastic unitarity for all $E>0$.

If we now consider the perturbation series for the Coulomb amplitude, we can write

$$A(E,z)\simeq A_{(1)}(E,z)+A_{(2)}(E,z)+\cdots\,, \tag{3}$$

where $A_{(1)}(E,z)=-2m\alpha/t$ and

$$A_{(2)}(E,z)=\frac{(2m\alpha)^2}{4\pi p^2}\int_0^\infty\frac{dk}{k^2-p^2-i\epsilon}\,\frac{1}{\sqrt{K}}\,\ln\frac{\gamma^2-z+\sqrt{K}}{\gamma^2-z-\sqrt{K}}\,. \tag{4}$$

In Eq. (4), $K=(\gamma^2-z)^2-(\gamma^2-1)^2$, $\gamma=(k^2+p^2)/2kp$, and we have performed the angular integrations which appear in the second Born term. We note the following: (1) The first Born term is identical to the lowest-order term (in α) of the expansion[4] of the exact Coulomb amplitude. (2) The second Born term is infinite, as the integral (4) does not exist. However, if we assume that (4) has been suitably regularized[5] {it is similar in form to integrals which appear in the relativistic treatment [cf. Eq. (14)]}, then we find that, considered as a function of z, (4) has a branch cut along the positive real axis. The discontinuity across this cut for $z>1$ is given by

$$\mathrm{Im}_z A_{(2)}(E,z)=\frac{(2m\alpha)^2}{2p^2}\int_0^\infty\frac{dk}{k^2-p^2-i\epsilon}\,\frac{1}{\sqrt{K}}\,. \tag{5}$$

This integral (5) exists and has a branch cut for E real and >0. The discontinuity across this cut is just

$$\mathrm{Im}_E\,\mathrm{Im}_z A_{(2)}(E,z)\equiv\rho^{(2)}_{z,E}(E,z)=\pi\eta(2m\alpha/t)\,\theta(E)\,\theta(z-1)\,. \tag{6}$$

The expression above is identical to the second-order term in the expansion of the exact double-spectral function obtained from (1). Thus, although the perturbation series does not converge to the correct amplitude, the double-spectral function which can be extracted from the perturbation series does correspond to the exact double-spectral function, at least through second order in α. Since it can be shown that each term in the perturbation expansion for a superposition of Yukawa potentials maintains the correct cut structure,[6] it is plausible to assume that, even though the perturbation series may not converge, the double-spectral function obtained from the perturbation expansion may be correct to all orders. At any rate, the result (6) for the Coulomb amplitude anticipates our approach to the construction of relativistic electromagnetic scattering amplitudes when there are bound states.

B. Self-Consistent Electromagnetic Scattering Amplitudes

The results of Sec. II A for the Coulomb amplitude suggest that, while the perturbation series does not converge to the correct amplitude when there are bound states, the double-spectral functions which can be extracted from it do correspond to the exact double-spectral functions, at least through second order. We assume, here, that this result is also valid in the relativistic case. However, we also note that in S-matrix theory the double-spectral functions are supposed to be completely determined by unitarity (elastic plus inelastic) and crossing, which yields a singular, inhomogeneous, nonlinear integral equation for the elastic scattering amplitude or, equivalently, the elastic double-spectral functions. Our procedure then is as follows. Through second order, at least, the contribution of the inhomogeneous terms to the double-spectral functions (inelastic unitarity) can be calculated exactly by the usual procedure. We note that the amplitudes which appear in these terms have no bound states so that perturbation theory is applicable. In order to evaluate the elastic-unitarity contribution, we assume an initial trial form for the elastic amplitude which can be inserted into the unitarity integral to complete the specification of the double-spectral functions. (Note that the Born term cannot be used for this purpose as it would cause the integral to diverge.) The calculated double-spectral functions are then used as a basis for the construction of a second trail amplitude which can be reinserted into the unitarity integral to give a new estimate of the double-spectral functions. The process may be repeated until self-consistency is achieved and the calculated double-spectral functions remain unchanged, through terms of the desired order, upon iteration. For elastic scattering through second order, it is not necessary to iterate; our initial trial amplitude reproduces itself in the unitarity integral. This amplitude is obtained from the Born approximation by replacing the photon pole term by a function which has a form similar to that of the Coulomb amplitude (1). This choice can be motivated by the observation that the correct relativistic scattering amplitude must reduce to the Coulomb amplitude in the low-energy limit, if we are to believe nonrelativistic quantum mechanics at all. We will find that the amplitude obtained in this manner is analytic, crossing-symmetric, and cutoff-independent, that it has the correct second-order double-spectral functions and the usual Born term in lowest order. In addition, the amplitude displays Regge asymptotic behavior and the bound-state Regge poles. The position and residue of these poles are determined by self-consistency, so that the calculation is essentially a semibootstrap; the only poles whose position and residue are inserted *a priori* are those associated with the external particles. We feel that our procedure offers a simple, practical means to obtaining electromagnetic scattering amplitudes for processes which have bound states and for which the usual formulation of perturbation theory is inappropriate.

III. SPIN-0–SPIN-0 ELECTROMAGNETIC SCATTERING

We consider here the elastic scattering of two spinless particles which only interact electromagnetically, with kinematics as in Fig. 1. Particles 1 and 3 have mass m and particles 2 and 4 have mass μ. For spin-0 scattering there is a single Lorentz-invariant scalar amplitude $A(s,t,u)$, where $s=(k_1+k_2)^2$, $t=(k_1+k_3)^2$, and $u=(k_1+k_4)^2$ are the usual Mandelstam variables, and $s+t+u=2m^2+2\mu^2$. A complete set of unitarity diagrams for this amplitude, through second order in α, is given in Figs. 2 and 3. We see that, for nonidentical particles through second order, t-channel unitarity accounts for the inelastic contribution to the double-spectral functions and the u channel exhibits only the elastic component.[7] We can now consider the contribution of each diagram to the elastic scattering amplitude.

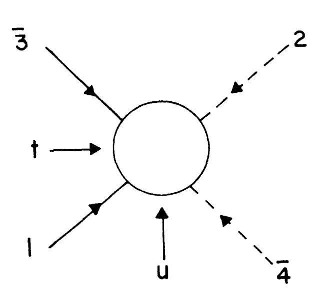

FIG. 1. Diagram of the scattering process. Unbroken line indicates mass m; dashed line, mass μ.

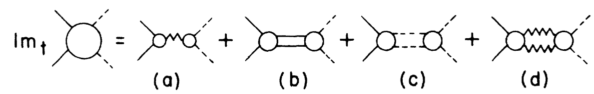

FIG. 2. t-channel unitarity through second order introduces the following contributions to the imaginary part of the scattering amplitude: (a) one-photon exchange, (b) two-boson exchange (mass m), (c) two-boson exchange (mass μ), and (d) two-photon exchange.

Diagram 2(a) represents the one-photon-exchange contribution, which is just the Born term. It can be written

$$A(s,t,u)_{\text{Born}}=\tfrac{1}{4}(s-u)(4\pi\alpha/t)\,, \tag{7}$$

where the factor $\frac{1}{4}(s-u)$ is due to the spin of the exchanged photon. We have found that the form of the bound-state Regge trajectory which appears in our final amplitude, while independent of the spins of the external particles, does depend on the vector character of the photon. Accordingly, we will include the photon spin at each stage of our calculation. The Born term, of course, does not contribute to the double-spectral functions. It is, however, necessary to the construction of our initial trial amplitude.

Diagrams 2(b) and 2(c) only represent vacuum-polarization and vertex corrections to the Born term, they do not contribute to the double-spectral functions. These unitarity diagrams are essentially equivalent to the four Feynman diagrams of Fig. 4 plus renormalization. Although the Feynman integrals exhibit an infrared divergence, the unitariy diagrams 2(a) and 2(b) can be evaluated without the introduction of a cutoff if the photon pole terms are replaced by the appropriate generalization of the Coulomb amplitude (1). This substitution, which will be discussed in detail when we consider the elastic-unitarity diagram, Fig. 3, allows us to write the contribution of diagrams 2(b) and 2(c) to the imaginary part of the scattering amplitude as follows:

$$\mathrm{Im}_t A(s,t,u)=\tfrac{1}{4}(s-u)\,\mathrm{Im}_t\{(4\pi\alpha/t)[\Gamma(t)-1]\}\,, \tag{8}$$

where $\Gamma(t)=1+\gamma(t,m^2)+\gamma(t,\mu^2)$. $\gamma(t,M^2)$ satisfies a dispersion relation of the form

$$\gamma(t,M^2)=\frac{t}{\pi}\int_{4M^2}^{\infty}\frac{dx}{x(x-t)}\,\mathrm{Im}\gamma(x,M^2)\,, \tag{9}$$

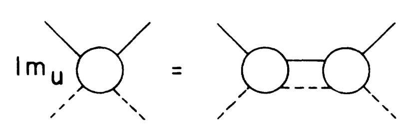

FIG. 3. u-channel elastic unitarity.

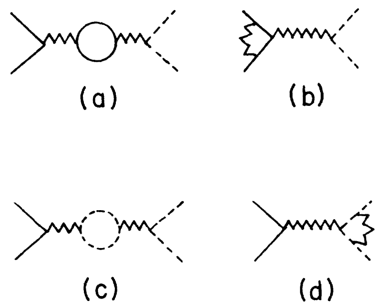

FIG. 4. Feynman diagrams which contribute to the spin-0 electromagnetic form factor. Diagrams (a) and (c) represent vacuum-polarization contributions; diagrams (b) and (d) are vertex corrections.

where

$$\mathrm{Im}\gamma(t,M^2)=(\alpha/q_M W_t)[\tfrac{1}{2}(2M^2-t)\psi(2)-\tfrac{2}{3}q_M{}^4/t]. \tag{10}$$

In Eq. (10), $q_M=\frac{1}{2}(t-4M^2)^{1/2}$, $W_t=2E_t=\sqrt{t}$, and $\psi(z)$ is the digamma function. Note that a subtraction is necessary in the definition of $\Gamma(t)$, since the electric charge remains a parameter in dispersion theory as well as in perturbation theory. The net result of the inclusion of diagrams 2(b) and 2(c) is that the Born term, Fig. 2(a), is multiplied by the spin-0 electromagnetic form factor, $\Gamma(t)$.

A. Double-Spectral Functions

Diagram 2(d) gives the first nonvanishing t-channel contributions to the double-spectral functions. It can be written in the form

$$\mathrm{Im}_t A(s,t,u)_{2\gamma}=\tfrac{1}{2}\rho_{2\gamma}\sum_{\text{spins}}\int d\Omega' R(K'')R^\dagger(K'), \tag{11}$$

where $\rho_{2\gamma}$ is the two-photon phase-space factor, the sum is over the polarizations of the intermediate-state photons, and $R(K)$ is the two-photon annihilation amplitude in the pole approximation. With kinematics as in Fig. 5,

$$\begin{aligned}R(K')=&M(s')(k_1\cdot\epsilon_5)(k_3\cdot\epsilon_6)\\&+M(u')(k_1\cdot\epsilon_6)(k_3\cdot\epsilon_5)+2\pi\alpha(\epsilon_5\cdot\epsilon_6),\end{aligned} \tag{12}$$

where

$$M(s')=4\pi\alpha/(s'-m^2)=4\pi\alpha/(-2k_1\cdot k_5),$$

$$M(u')=4\pi\alpha/(u'-m^2)=4\pi\alpha/(-2k_1\cdot k_6).$$

[$R(K'')$ can be obtained from $R(K')$ by the substitutions $1\leftrightarrow 4$ and $3\leftrightarrow 2$.] We note that the amplitude (12) is symmetric with respect to the exchange $5\leftrightarrow 6$ as required by Bose statistics, and is gauge-invariant since $k_5\cdot\epsilon_5=k_6\cdot\epsilon_6=0$. Using (12) we can explicitly perform the spin sums and angular integrations indicated in (11). We find that

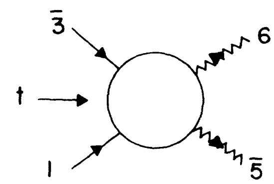

FIG. 5. Two-photon annihilation amplitude (t channel).

$$\begin{aligned}\mathrm{Im}_t A(s,t,u)_{2\gamma}&\\=\theta(t)[&\tfrac{1}{4}(u-m^2-\mu^2)^2 I_0(s,t,u)+\tfrac{1}{8}(t-2m^2)I_1(t)\\&+\tfrac{1}{8}(t-2\mu^2)I_2(t)+\tfrac{1}{4}I_3(t)]+(s\leftrightarrow u),\end{aligned} \tag{13}$$

where $\theta(z)$ is the unit Heaviside function and

$$\begin{aligned}I_0(s,t,u)&=\alpha^2\int\frac{d\Omega'}{(s'-m^2)(s''-\mu^2)}\\&=\frac{4\pi\alpha^2}{t(2q_uW_u)}\ln\frac{m^2+\mu^2-u+2q_uW_u}{m^2+\mu^2-u-2q_uW_u},\\I_1(t)&=\alpha^2\int\frac{d\Omega'}{s'-m^2}=\frac{-8\pi\alpha^2}{tv_m}Q_0\left(\frac{1}{v_m}\right),\\I_2(t)&=\alpha^2\int\frac{d\Omega'}{s''-\mu^2}=\frac{-8\pi\alpha^2}{tv_\mu}Q_0\left(\frac{1}{v_\mu}\right),\\I_3(t)&=\alpha^2\int d\Omega'=4\pi\alpha^2.\end{aligned} \tag{14}$$

In (14), $2q_uW_u=\{[u-(m+\mu)^2][u-(m-\mu)^2]\}^{1/2}$ and $v_M=q_M/E_t$. $Q_0(z)$ is a Legendre function of the second kind of degree zero. We see from (14) that only $I_0(s,t,u)$ has a branch cut in u. The cut in u of $I_0(s,t,u)$ may be taken along the real u axis and extends from $(m+\mu)^2$ to infinity. We find that the contribution of diagram 2(d) to the double-spectral functions can be written[8]

$$\begin{aligned}\rho_{tu}(s,t,u)_{2\gamma}&=\rho_{ts}(u,t,s)_{2\gamma}\\&=\tfrac{1}{4}(u-m^2-\mu^2)^2\,\frac{8\pi^2\alpha^2}{t(2q_uW_u)}\\&\quad\times\theta(t)\,\theta(u-(m+\mu)^2).\end{aligned} \tag{15}$$

The remaining parts of the second-order double-spectral functions are due to the u-channel elastic-unitarity diagram Fig. 3. We can write this term in the form

$$\mathrm{Im}_u A(s,t,u)=\tfrac{1}{2}\rho_2(u)\int d\Omega' A_0(s',t',u)A_0^*(s'',t'',u), \tag{16}$$

where $\rho_2(u)$ is the two-body phase-space factor for the intermediate state. With our normalization,

$$\rho_2(u)=[q_u/(2\pi)^2W_u]\,\theta(u-(m+\mu)^2). \tag{17}$$

If we attempt to determine the second-order contribution to the imaginary part of the elastic amplitude by inserting the Born term (7) into the right-hand side of Eq. (16), the result will be infrared divergent, as is well known from perturbation theory. In order to circumvent this difficulty, we will, instead, use an initial trial amplitude, $A_0(s,t,u)$, which can be obtained from the Born term by the substitution

$$\frac{4\pi\alpha}{t} \to f_0(s,t,u) = \left(\frac{-4\pi\alpha}{4q^2}\right)\frac{\Gamma(1-i\eta(u))}{\Gamma(1+i\eta(u))}\left(\frac{-t}{4q^2}\right)^{-1+i\eta(u)} + \left|\frac{\alpha}{2q_u W_u \eta(u)}\right| \frac{\delta(-t/4q_u{}^2)}{2\pi i \rho_2(u)}. \quad (18)$$

$f_0(s,t,u)$ has the same form[9] as the Coulomb amplitude (1) obtained from the Schrödinger equation, with the substitution of the correct relativistic two-body kinematical factors. The trajectory function $\eta(u)$ is to be determined by the requirement of self-consistency. This substitution (18) will admit a process of iteration since $f_0(s,t,u)$ has a well-defined partial-wave expansion, and we find that the amplitude which results can be made self-consistent. With the amplitude $A_0(s,t,u)$ defined above, the integral (16) can be evaluated explicitly. We obtain

$$\mathrm{Im}_u A(s,t,u) = \theta(u-(m+\mu)^2)\tfrac{1}{2}(u-m^2-\mu^2)\left(\frac{\alpha}{q_u W_u \eta(u)}\right)\mathrm{Im}_u[A_0(s,t,u)+\tfrac{1}{4}(u-m^2-\mu^2)f_1(s,t,u)+\tfrac{1}{4}f_2(s,t,u)]. \quad (19)$$

In Eq. (19), $A_0(s,t,u)$ is our original trial amplitude and

$$\begin{aligned} f_1(s,t,u) &= 4\pi\alpha(u-m^2-\mu^2)^{-1}|\Gamma(1-i\eta)|^2[P_{i\eta}(-z_u)-1], \\ f_2(s,t,u) &= 4\pi\alpha q_u{}^2(u-m^2-\mu^2)^{-1}|\Gamma(1-i\eta)|^2[i\eta P_{i\eta}(-z_u)], \end{aligned} \quad (20)$$

where $P_\nu(z)$ is a Legendre function of the first kind,

$$|\Gamma(1-i\eta)|^2 = \pi\eta(u)/\sinh\pi\eta(u),$$

and $z_u = 1 + t/2q_u{}^2$. We note the following: (1) If $\eta(u) = \alpha(u-m^2-\mu^2)/2q_u W_u$, our original trial amplitude is reproduced in Eq. (19). (2) For that value of $\eta(u)$, the remaining terms in (19) are of order α^2, and (3) Eq. (19) does not exhibit any spurious poles. The elastic-unitarity contribution to the double-spectral functions can be obtained relatively simply from Eq. (19) and the result will be independent of the actual form of $\eta(u)$ through terms of second order in α. We find

$$\begin{aligned} \rho_{ut}(s,t,u)_{\text{elastic}} &= \rho_{tu}(s,t,u)_{2\gamma}, \\ \rho_{us}(s,t,u)_{\text{elastic}} &= 0. \end{aligned} \quad (21)$$

The complete second-order double-spectral functions associated with the unitarity diagrams Figs. 2(d) and 3 are given by the sum of (15) and (21), so that

$$\rho_{tu}(s,t,u) = 2\rho_{ts}(u,t,s) = 2\rho_{tu}(s,t,u)_{2\gamma}, \quad (22)$$

where $\rho_{tu}(s,t,u)_{2\gamma}$ is defined by (15).

B. Self-Consistent Scattering Amplitude

With the information presented above, it is possible to construct an amplitude which has the double-spectral-function terms indicated in Eq. (22) and which also has the appropriate vacuum-polarization and vertex corrections. Employing (19) and (21), with (8), we find

$$A(s,t,u) = 2F(s,t,u) + F(u,t,s), \quad (23)$$

where

$$F(s,t,u) = F_0(s,t,u) + \tfrac{1}{4}(u-m^2-\mu^2)f_1(s,t,u) + \tfrac{1}{4}f_2(s,t,u) \quad (24)$$

and

$$F_0(s,t,u) = \tfrac{1}{4}(s-u)\Gamma(t)f_0(s,t,u). \quad (25)$$

Thus, $F_0(s,t,u)$ is just our original trial amplitude multiplied by the form factor $\Gamma(t)$. $A(s,t,u)$ defined by (23) is cutoff-independent. It can be shown explicitly (cf. Ref. 1) that the δ-function terms in (23) reduce to the appropriate matrix elements, for each channel, of $-iI$, where I is the identity operator, so that the S matrix corresponding to the transition amplitude (23) is analytic. By construction (23) exhibits the proper double-spectral function terms. If we set

$$\eta(u) = \alpha(u-m^2-\mu^2)/2q_u W_u, \quad (26)$$

then (23) will be self-consistent. This is proved by observing that, for this value of $\eta(u)$, $A_0(s,t,u)$ reproduces itself in the elastic-unitarity integral (16). Since the additional terms in (23) are all of order α^2, their introduction into the unitarity integral can have no effect on the second-order double-spectral functions. Thus, the reintroduction of (23) into the elastic-unitarity integral will produce no additional second-order contributions to the double-spectral functions. It can also be verified that (23) has a well-defined partial-wave expansion so that it will admit a process of iteration. Moreover, $A(s,t,u)$ displays Regge asymptotic behavior and, with the trajectory function (26), the correct Coulomb bound-state Regge poles, including reduced mass and recoil effects.[10] In addition, (23) reduces to the usual Born term in lowest order. We also find that, since $(u-m^2-\mu^2)/2W_u$ is equal to the reduced mass at threshold, (23) reproduces the nonrelativistic Coulomb amplitude at low energy. Finally, we point out that in addition to the unitarity cuts which we have examined, $A(s,t,u)$ exhibits a left-hand cut which is required by self-consistency. This cut is due to the factor $W_u=\sqrt{u}$ which appears in the trajectory function and which originates in the relativistic two-body phase-space factor (17). For this reason, it is probably an inescapable feature of relativistic scattering. We note that both the Klein-Gordon and Dirac Coulomb scattering amplitudes have a cut structure of this type. We conclude that (23) represents a satisfactory electromagnetic scattering amplitude for two, spinless, nonidentical particles which is accurate through terms of second order in α.

C. Identical Particles

If we consider the elastic scattering of two identical bosons of mass m, then our remarks of Secs. III A and III B must be modified slightly. In particular, the statement of unitarity must be represented in the t channel by the diagrams of Fig. 6. Moreover, the u-channel unitarity diagrams are identical to those of the t channel, with the substitution $3 \leftrightarrow 4$, so that the total amplitude will be symmetric under exchange of identical particles as required by Bose statistics. We note that diagram 6(c) will incorporate vertex corrections in addition to elastic unitarity. Diagram 6(d) has been included to maintain a correspondence between the scattering amplitudes for identical particles and for the case $m \neq \mu$. In second order, this term only contributes a vacuum-polarization correction to the form factor. We find that the amplitude for the scattering of identical bosons can be obtained from the amplitude (23) by setting $m=\mu$ and symmetrizing with respect to the exchange $t \leftrightarrow u$, provided we replace the form factor $\Gamma(t)$ which appears in (25) by $\Gamma'(t)$, where

$$\Gamma'(t)=1+\frac{t}{\pi}\int_0^\infty \frac{dx}{x(x-t)}\mathrm{Im}\Gamma'(x) \qquad (27)$$

and

$$\mathrm{Im}\Gamma'(t)=\theta(t-4m^2)(\alpha/q_m W_t)[(2m^2-t)\psi(2)-\tfrac{2}{3}q_m{}^4/t] + \theta(t-4\mu^2)(\alpha/q_\mu W_t)(-\tfrac{2}{3}q_\mu{}^4/t)\,. \qquad (28)$$

We see that $\Gamma'(t)$ differs from $\Gamma(t)$ only in the vertex corrections, as might be expected.

We note finally that the scattering amplitudes for identical particles and for the case $m\neq\mu$ are related by means of unitarity so that in order to obtain the higher-order corrections we must consider the two processes simultaneously. The problem is rendered somewhat less intractable by the observation that the amplitudes which appear in the unitarity integrals need only be determined to order $(n-1)$ to obtain the nth-order correction to the amplitude under consideration. Thus, provided no divergences are encountered, our procedure may be continued by iteration to higher orders.

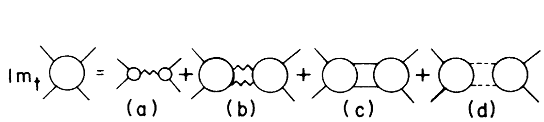

FIG. 6. t-channel unitarity through second order for identical particles. Diagram (a) is the one-photon-exchange contribution to the imaginary part of the amplitude. (b) Two-photon exchange. (c) Elastic unitarity. (d) Two-boson exchange (mass μ).

IV. SPIN-0–SPIN-$\frac{1}{2}$ ELECTROMAGNETIC SCATTERING

We should now like to indicate the results of our procedure when one particle has spin $\frac{1}{2}$. If we consider an elastic scattering process, with kinematics as in Fig. 1, in which particles 1 and 3 have mass m, spin $\frac{1}{2}$, particles 2 and 4 have mass μ, spin 0, then the scattering amplitude in the t channel, assuming parity conservation etc., can be written in the form[11]

$$T(k_3,k_4;k_1,k_2)=\bar{v}(k_3)[A(s,t,u)+\not{Q}B(s,t,u)]\,u(k_1)\,, \qquad (29)$$

where $Q=k_4-k_2$. $A(s,t,u)$ and $B(s,t,u)$ are Lorentz-invariant scalar amplitudes and the spinor normalization is that of Bjorken and Drell.[12] We remark that while we have written (29) in terms of the usual 4-component Dirac spinors, the actual calculation employed the 2-component formalism[13] in order to reduce the spinor algebra to manageable proportions. The results were then translated into the more familiar 4-component form.

We wish to determine the amplitude (29) for the process in which the particles only interact electromagnetically. Through second order in α, we find that the amplitude can be determined self-consistently and analytically by means of the procedure discussed in Secs. II and III, and that no essential theoretical complications are admitted by the introduction of spin. Accordingly, we will only indicate the major results and proceed with a minimum of discussion.

The unitarity diagrams of Figs. 2 and 3 again represent a complete set through second order, with the understanding that particles 1 and 3 now have spin $\frac{1}{2}$. The Born term, diagram 2(a), can be written

$$T(k_f;k_i)_{\rm Born}=(-4\pi\alpha/t)\bar{v}(k_3)[\Gamma_1(0)\not{Q}-(1/2m)\Gamma_2(0)(s-u)]u(k_1)\,, \tag{30}$$

where $\Gamma_1(t)=F_1(t)+\Gamma_2(t)=F_1(t)+\kappa F_2(t)$, and $F_1(t)$ and $F_2(t)$ are the usual charge and magnetic-moment form factors, respectively, with $F_1(0)=F_2(0)=1$. $\kappa=\frac{1}{2}(g-2)$ is the anomalous part of the magnetic moment. For the case in which there are only electromagnetic interactions, $\kappa=\alpha/2\pi$ in lowest order. As before, the only second-order effects of diagrams 2(b) and 2(c) are vacuum-polarization and vertex corrections to the Born approximation. If we include these diagrams, we find that, in Eq. (30), $\Gamma_1(0)$ is replaced[14] by $\Gamma(t)=\Gamma_1(t)+\gamma(t,\mu^2)$ [cf. Eqs. (9) and (10)] and $\Gamma_2(0)$ by $\Gamma_2(t)$. The properties of $\Gamma_1(t)$ and $\Gamma_2(t)$ or, equivalently, $F_1(t)$ and $F_2(t)$ are discussed in many places.[15] We do note, however, that both $F_1(t)$ and $F_2(t)$ are automatically cutoff-independent in our calculation by virtue of the substitution (18) for the photon pole terms.

A. Double-Spectral Functions

The two-photon-intermediate-state contribution to t-channel unitarity [diagram 2(d)] can be evaluated explicitly using the usual pole approximations to the two-photon annihilation amplitudes. We find that the contribution to the imaginary part of each of the invariant amplitudes can be written

$$\mathrm{Im}_t A(s,t,u)_{2\gamma}=\left(\theta(t)\sum_{i=0}^{3}q_i^A(s,t,u)I_i(s,t,u)\right)+(u\leftrightarrow s)$$

and

$$\mathrm{Im}_t B(s,t,u)_{2\gamma}=\left(\theta(t)\sum_{i=0}^{2}q_i^B(s,t,u)I_i(s,t,u)\right)-(u\leftrightarrow s)\,, \tag{31}$$

where

$$\sum_{i=0}^{3}q_i^A(s,t,u)I_i(s,t,u)=2m(u-m^2)\Delta^{-1}[t(u+\mu^2-m^2)I_0(s,t,u)+(u-s)I_1(t)+(4\mu^2-t)I_2(t)]-(m/4q_m{}^2)[2m^2I_1(t)+I_3(t)]$$

and

$$\begin{aligned}\sum_{i=0}^{2}q_i^B(s,t,u)I_i(s,t,u)=&\tfrac{1}{2}(u-m^2-\mu^2)I_0(s,t,u)\\&+\Delta^{-1}\{t(u+m^2-\mu^2)(u-m^2)I_0(s,t,u)+[(u+m^2-\mu^2)(u-s)-\mu^2(4m^2-t)]I_1(t)\\&+[(u+m^2-\mu^2)(4\mu^2-t)-\mu^2(u-s)]I_2(t)\}\,.\end{aligned} \tag{32}$$

In Eq. (32), the integrals $I_i(s,t,u)$ are given by (14) and $\Delta=4[su-(m^2-\mu^2)^2]$. From (31) and (32), we find the following contributions to the double-spectral functions:

$$\rho_{tu}^A(s,t,u)_{2\gamma}=\rho_{ts}^A(u,t,s)_{2\gamma}=q_0^A(s,t,u)\frac{8\pi^2\alpha^2}{t(2q_uW_u)}(t)\,\theta(u-(m+\mu)^2)\,,$$

$$\rho_{tu}^B(s,t,u)_{2\gamma}=-\rho_{ts}^B(u,t,s)_{2\gamma}=q_0^B(s,t,u)\frac{8\pi^2\alpha^2}{t(2q_uW_u)}\theta(t)\,\theta(u-(m+\mu)^2)\,. \tag{33}$$

In order to evaluate the u-channel unitarity contribution (Fig. 3) to the double-spectral functions in the case of spin-0–spin-$\frac{1}{2}$ elastic scattering, we employ an initial trial amplitude, $T_0(k_f;k_i)$, which is derived from the Born approximation (30) by means of the following substitution:

$$-\frac{4\pi\alpha}{t}\to f_0(s,t,u)=\left(\frac{4\pi\alpha}{4q_u{}^2}\right)\frac{\Gamma(1-i\eta(u))}{\Gamma(1+i\eta(u))}\left(\frac{-t}{4q_u{}^2}\right)^{-1+i\eta(u)}+\left|\frac{\alpha}{2q_uW_u\eta(u)}\right|\frac{\delta(-t/4q_u{}^2)}{2\pi i\rho_2(u)}\,. \tag{34}$$

We note that $f_0(s,t,u)$ in (34) differs from the corresponding function in the spinless case by the relative sign of the δ-function part. This change will maintain the proper phase relationships so that our final expression can be identified with the transition amplitude, and is necessitated by the spinor factors which appear in (30). If the trial amplitude which we thus obtain is inserted into the elastic-unitarity integral, we find the following contribution to the imaginary part of each invariant amplitude in the u channel:

$$\mathrm{Im}_u A(s,t,u) = \theta(u-(m+\mu)^2)\tfrac{1}{2}(u-m^2-\mu^2)\frac{\alpha}{q_u W_u \eta(u)} \times \mathrm{Im}_u[A_0(s,t,u) - p_1^A(s,t,u)f_1(s,t,u)]$$

and (35)

$$\mathrm{Im}_u B(s,t,u) = \theta(u-(m+\mu)^2)\tfrac{1}{2}(u-m^2-\mu^2)\frac{\alpha}{q_u W_u \eta(u)} \times \mathrm{Im}_u[B_0(s,t,u) - p_1^B(s,t,u)f_1(s,t,u)]\,,$$

where

$$p_1^A(s,t,u) = 4m(u-m^2)(u+\mu^2-m^2)/\Delta\,,$$
$$p_1^B(s,t,u) = 2(u-m^2)(u+m^2-\mu^2)/\Delta\,. \tag{36}$$

$f_0(s,t,u)$ is defined in Eq. (20) and Δ is given below Eq. (32). $A_0(s,t,u)$ and $B_0(s,t,u)$ are our original trial amplitudes. As before, we note that they will reproduce themselves in the unitarity integral if $\eta(u)=\alpha(u-m^2-\mu^2)/2q_uW_u$. Apparently, this result for the trajectory function is independent of the spin of the external particles. We also find that Eq. (35) does not have any spurious poles. Given the imaginary part of the scalar amplitudes (35), it is relatively simple to extract the second-order u-channel contributions to the double-spectral functions. We find

$$\rho_{ut}^{A,B}(s,t,u)_{\text{elastic}} = \rho_{tu}^{A,B}(s,t,u)_{2\gamma}\,,$$
$$\rho_{us}^{A,B}(s,t,u)_{\text{elastic}} = 0\,. \tag{37}$$

The complete second-order double-spectral functions for spin-0–spin-$\frac{1}{2}$ scattering are given by the sum of (33) and (37), so that

$$\rho_{tu}^{A}(s,t,u) = 2\rho_{ts}^{A}(u,t,s) = 2\rho_{tu}^{A}(s,t,u)_{2\gamma}\,,$$
$$\rho_{tu}^{B}(s,t,u) = -2\rho_{ts}^{B}(u,t,s) = 2\rho_{tu}^{B}(s,t,u)_{2\gamma}\,, \tag{38}$$

where $\rho_{tu}^{A,B}(s,t,u)_{2\gamma}$ is defined by (33).

B. Self-Consistent Scattering Amplitude

With the results of Secs. III and IV A we can construct a second-order spin-0–spin-$\frac{1}{2}$ elastic scattering amplitude which has the double-spectral-function terms of Eq. (38) and also the appropriate vertex corrections due to the electromagnetic form factors. We find

$$A(s,t,u) = 2F^A(s,t,u) + F^A(u,t,s)\,,$$
$$B(s,t,u) = 2F^B(s,t,u) - F^B(u,t,s)\,, \tag{39}$$

where

$$F^{A,B}(s,t,u) = F_0^{A,B}(s,t,u) - p_1^{A,B}(s,t,u)f_1(s,t,u)\,. \tag{40}$$

As in the spin-0 case, $F_0^{A,B}(s,t,u)$ is just the original trial amplitude with the form factor added. Explicitly,

$$F_0^A(s,t,u) = (-1/2m)\Gamma_2(t)(s-u)f_0(s,t,u)\,,$$
$$F_0^B(s,t,u) = \Gamma(t)f_0(s,t,u)\,, \tag{41}$$

where $f_0(s,t,u)$ is given by Eq. (34). The form of the trajectory function for which the amplitude defined by (39) is self-consistent is again given by (26), so that (39) will also exhibit the correct bound states and Regge asymptotic behavior. Finally, we note that our second-order spin-0–spin-$\frac{1}{2}$ electromagnetic scattering amplitude has a well-defined Jacob-Wick expansion so that it can provide a suitable basis for further iteration.

V. DISCUSSION

By iterating unitarity using a trial amplitude which is similar in form to that obtained from the solution of the Schrödinger equation with a Coulomb potential and by imposing the requirement of self-consistency on the double-spectral functions which result, we have obtained analytic second-order electromagnetic scattering amplitudes which exhibit most of the features expected of a "good" scattering amplitude. Moreover, our procedure allows the calculation of electromagnetic scattering amplitudes which have bound-state poles, and for which the usual formulation of perturbation theory is inappropriate. It is, then, of some interest to compare our results with certain of the approximations which have previously been developed.

In order to obtain electromagnetic scattering amplitudes for processes in which bound states may appear, the basic approach has been to attempt to sum certain subclasses of Feynman diagrams to obtain a convenient analytic form.[16] Perhaps the simplest example of this procedure is found in the relativistic wave equations. It has long been known, for example, that both the Klein-Gordon and Dirac equations with a Coulomb potential are equivalent to an approximate sum of a particular subclass of Feynman diagrams.[17] The scattering amplitudes derived from these equations do have the correct bound-state poles and Regge asymptotic behavior. However, this gain is somewhat offset by the fact that crossing symmetry is not maintained. If we compare the Klein-Gordon or Dirac Coulomb amplitude with that part of our amplitude [Eq. (23) or (39)] which has the appropriate double-spectral function (there is only *one* double-spectral function in potential theory) and neglect the vacuum-polarization and vertex corrections and the additional factors associated with the photon spin, we find that there is agreement only in the leading term. This is partly due to the fact that the Coulomb potential alone does not *exactly* represent the sum of the Feynman diagrams considered. The remaining discrepancy may be ascribed to the fact that the scattering amplitudes obtained from the relativistic wave equations satisfy elastic unitarity for all values of the energy (in one channel), whereas the actual physical amplitudes do not satisfy elastic unitarity for any value of E. This is, of course, due to the possibility of intermediate-state photons which cause elastic and inelastic threshold to overlap. Similar remarks can also be applied to the Bethe-Salpeter equation. Although the diagrams which are considered may be summed exactly and the equation has the additional virtue of preserving the correct two-particle kinematics, the result is considerably more difficult to solve. In fact, we are aware of no satisfactory scattering solutions to the Bethe-Salpeter equation when the exchanged particle is a massless vector boson, so that direct comparison with our amplitude is difficult except in the correspondence limit. However, even if these solutions were available, there would remain the difficulties with unitarity and crossing symmetry, and, while a modification of the Bethe-Salpeter equation has been proposed[18] which includes crossing symmetry it has so far proved intractable.[19] Finally, we can consider the relativistic generalizations of the eikonal approximation. If we compare the leading term of our amplitude (23) with the corresponding expression obtained in an eikonal approximation,[20] we find that the asymptotic behavior is essentially the same in each case. At first sight this result may be surprising. Our amplitude is based on the form of the Coulomb amplitude, which should be a good approximation at low energy, while the eikonal form is essentially a high-energy result. However, we note that in the eikonal expansion there is considerable cancellation between subsidiary terms, so that the asymptotic behavior of the amplitude is dominated by the leading Regge trajectory, and in the Coulomb case the leading Regge pole accounts for the complete amplitude.

We have, thus, exhibited a relatively simple procedure for dealing with scattering processes in which bound states may appear. The amplitude which results is analytic, cutoff-independent, and crossing-symmetric, reduces to the nonrelativistic Coulomb amplitude in the low-energy limit and has the correct double-spectral-function terms through second order in α. Moreover, the amplitude displays Regge asymptotic behavior and the bound-state Regge poles. Self-consistency requires that the poles appear at the correct position and with the proper residue and that the amplitude possesses a well-defined partial-wave expansion. In particular, the spin-0–spin-0 and spin-0–spin-$\frac{1}{2}$ amplitudes presented above offer a convenient theoretical basis from which further progress can be made. It should be possible to extend this work to higher order in α and, at least, determine the trajectory functions which result from the requirement of self-consistency. (Note that effectively only the double-spectral functions need be calculated for this purpose.) These can be compared with the results of the perturbation calculations of the bound-state energy spectrum to give a further check on our assumptions. At any rate, we feel that our procedure affords a convenient and accurate means to the calculation of electromagnetic scattering amplitudes when there are bound states, and which may also prove useful in the determination of the higher-order corrections to the bound-state energies.

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Nonlinear Hadron Couplings from Divergence Conditions. III. SU(3) Multiplets

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The Lagrangian density for a system of the pseudoscalar-meson nonet, the vector-meson nonet, and the spin-$\frac{1}{2}$-baryon octet is obtained by the SU(3) extension of the nonlinear Lagrangian density for pions, ρ mesons, and nucleons. The nonpolynomial Lagrangian density for the pseudoscalar mesons is derived by a general procedure, which is applicable to all models and includes a rigorous treatment of the mass term. The effect of the SU(3)-symmetry breaking is taken into account by introducing the mass matrix, and the broken divergence conditions for the vector and axial-vector currents are given in an explicit form.

I. INTRODUCTION

The nonlinear couplings of pions, ρ mesons, and nucleons were derived in two earlier papers[1,2] by imposing suitable divergence conditions on the source functions in the π- and ρ-field equations. We shall now conclude our investigation by extending the results to the pseudoscalar-meson nonet P, the vector-meson nonet V_μ, and the spin-$\frac{1}{2}$-baryon octet B.

The extensive literature on the SU(3) extension of nonlinear Lagrangian densities has been reviewed, for instance, by Weinberg[3] and by Gasiorowicz and Geffen,[4] where the main complication arises from the nonpolynomial nature of the pion Lagrangian density. For, in the SU(3) extension the role of the 2×2 matrix

$$\tau_i\pi_i\,,$$

with $i=1, 2, 3$, must be replaced by the 3×3 matrix

$$\lambda_i P_i\,,$$

with $i=0, 1, 2, \cdots, 8$, where the λ_i are Gell-Mann's SU(3) matrices.[5] This transition is not entirely straightforward owing to the fact that while $(\tau_i\pi_i)^2$ is a multiple of the unit matrix, $(\lambda_i P_i)^2$ does not possess this simple property. Because of the mathematical difficulties, the SU(3) extension of the nonlinear pion Lagrangian density has so far been given only for specific models, and the treatment of the pseudoscalar-meson mass term is especially inadequate in the existing literature. We shall, however, describe a general scheme that will enable us to carry out the SU(3) extensions of all pion models.

As we shall see, the divergence conditions that apply to the ρ-π-N system can be maintained for the V-P-B system as long as the SU(3) symmetry is preserved, but they are no longer valid when this symmetry is broken. We shall also investigate the effect of the SU(3)-symmetry breaking on the divergence conditions, and for this purpose we shall follow the symmetry-breaking mechanism of an earlier paper,[6] which is not only remarkably simple but also gives the symmetry-breaking terms explicitly rather than merely specifying their transformation properties.

We shall generally follow the same notation as in Refs. 1 and 2 with appropriate extensions for the SU(3) multiplets. The pseudoscalar-meson nonet, the vector-meson nonet, and the baryon octet will be denoted either by the usual 3×3 matrices P, V_μ, and B or by the nine-component vectors P_i, $V_{\mu,i}$, and B_i, the relationship between the matrix and component forms being given by

$$P=\lambda_i P_i/\sqrt{2}\,,\quad V_\mu=\lambda_i V_{\mu,i}/\sqrt{2}\,,\quad B=\lambda_i B_i/\sqrt{2}\,, \tag{1.1}$$