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Canonical Light-Cone Commutators*

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We study the canonical structure of current commutators on the light cone, as embodied in Kogut and Soper's formulation of quantum electrodynamics in the infinite-momentum frame. These commutators incorporate dynamical information far beyond that contained in conventional equal-time commutation relations. An analog of the Bjorken-Johnson-Low limit is developed for calculating matrix elements of light-cone commutators from conventional time-ordered products. Some apparently new sum rules for electroproduction are obtained. They relate Fourier transforms of structure functions to bilocal products of operators. We suggest possible directions for constructing phenomenological light-cone commutators in hadronic physics. These commutators can be used to describe entire Regge trajectories, and might find application to electroproduction and neutrino-induced production experiments.

I. INTRODUCTION

It has been known for some time that the knowledge of current commutators on the light cone would provide useful theoretical and experimental information for particle physicists. For example, it has been shown[1] that the Dashen–Fubini–Gell-Mann sum rule,[2] conventionally derived from the equal-time algebra with the help of dispersion relations, is valid when the light-cone commutator of currents has a certain structure (described in Sec. IV). Another way of stating this is that various invariant amplitudes grow not more rapidly at infinite momentum than allowed by Bjorken's scaling laws[3] for electroproduction. Recently, a number of authors have demonstrated that the scaling functions observed in deep-inelastic electroproduction directly measure the matrix elements of light-cone current commutators.[4]

It is of the greatest importance to have some idea of the operator structure of current commutators on the light cone, for reasons given above and in order to give quantitative content to the program of operator expansions of field products at short distances.[5,6] One obvious method by which one may calculate such commutators is perturbation theory.[7] This technique suffers from two shortcomings. First, only selected matrix elements of the light-cone commutator can be computed, and the operator structure remains obscure. Second, it is prohibitively tedious to go beyond the first nontrivial order of perturbation theory. Thus it is desirable to develop alternative methods for exposing the light-cone commutator. In this paper we report a canonical, nonperturbative analysis based on the theory of quantum electrodynamics in the infinite-momentum frame, as discussed by Bjorken, Kogut, and Soper.[8] The commutators on the light-cone can be computed canonically in this framework. These commutators probe dynamics rather more deeply than conventional equal-time ones, and we find a number of sum rules, some of which are apparently new. As is inevitable, we also encounter infinities. Moreover, just as the conventional equal-time techniques are not verified in perturbation theory, so also the present results suffer from the same defect.

To discuss such things as the electroproduction experiments realistically, it is of course necessary to go beyond a model field theory like quantum electrodynamics. We make a first attempt at an approximate, phenomenological description of light-cone current commutators which can incorporate features of hadronic physics such as Regge trajectories. This attempt is complementary to the often-made observation that Bjorken's scaling functions can incorporate Regge poles.[9]

This paper is organized as follows. In Sec. II we review the theory of Bjorken, Kogut, and Soper[8] and derive the commutator of fermion fields on the light cone. Current commutators are given in Sec. III and various sum rules which follow from them are deduced in Sec. IV, along with a version of the Bjorken-Johnson-Low (BJL) limit which governs light-cone commutators. Speculation on Regge trajectories is given in Sec. V, and Sec. VI contains concluding remarks. In an Appendix we record the canonical $SU(3)\times SU(3)$ light-cone commutators.

II. CANONICAL QUANTIZATION ON THE LIGHT CONE

Bjorken, Kogut, and Soper[8] have shown that it is possible to quantize a field theory in what amounts to an infinite-momentum frame by setting up canonical commutation relations between independent fields on the light cone. In such a theory, the rules for constructing diagrams are essentially the infinite-momentum rules first developed by Weinberg.[10] There is, or course, no difference in content between this theory and the one which follows from the usual equal-time quantization, but the information is organized in a novel way, which might be of heuristic value in constructing approximations and generating insight for ultrarelativistic processes.

We use the notation of Kogut and Soper[8]: The conventional coordinates (and other four-vectors) are labeled with a caret, as in $\hat{x}^\mu = (t, x, y, z)$, while light-cone variables are denoted as $x^\mu = (\tau, x, y, \mathfrak{z})$, with $\tau = 2^{-1/2}(t+z)$, $\mathfrak{z} = 2^{-1/2}(t-z)$. These are related by

$$x^\mu = C^\mu{}_\nu \hat{x}^\nu ,\qquad C^\mu{}_\nu = \begin{pmatrix} 2^{-1/2} & 0 & 0 & 2^{-1/2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2^{-1/2} & 0 & 0 & -2^{-1/2} \end{pmatrix}. \tag{2.1}$$

In light-cone coordinates, the metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.2}$$

We consider a quark-vector gluon theory, described by the Lagrangian

$$\mathfrak{L} = \bar{\psi}[(\tfrac{1}{2}i\overleftrightarrow{\partial}_\mu - gB_\mu)\gamma^\mu - M]\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} ,\qquad F^{\mu\nu} = \partial^\nu B^\mu - \partial^\mu B^\nu . \tag{2.3}$$

For simplicity we have set the gluon mass to zero, so that we are really dealing with quantum electrodynamics, and may take over directly the results of Bjorken, Kogut, and Soper.[8] This is an important simplification, since we need to deal with only two independent degrees of freedom in B_μ, instead of three. Following Ref. 8, we use the extra gauge freedom of the massless theory to set $B^0 = B_3 = 0$, and B^3 will be expressed in terms of the independent field variables. It is our expectation that the high-energy results in which we are interested do not depend in an important way on the boson mass. Hence the present results should be valid for a massive vector-meson theory as well. We return to this point and to the question of the gauge invariance of our results in the conclusion.

The canonically independent operators are $B^i(x)$ (Latin indices run over the "perpendicular" components 1, 2, also described by the subscript $\perp$), and a certain projection ψ_+ of the four-component Dirac field ψ: $\psi_+ = P_+\psi$. The projection operator is given by $P_+ = \frac{1}{2}\gamma^3\gamma^0$; its partner is $P_- = \frac{1}{2}\gamma^0\gamma^3$, $P_+ + P_- = 1$. The Dirac adjoint field is constructed conventionally, i.e.,

$$\bar{\psi} = \psi^* \hat{\gamma}^0 = \psi^* 2^{-1/2}(\gamma^0 + \gamma^3) = 2^{-1/2}(\psi_-^* \gamma^0 + \psi_+^* \gamma^3) . \tag{2.4}$$

According to Ref. 8, the dependent operators ψ_- and B^3 are given in terms of independent quantities by

$$\psi_-(x) = -\tfrac{1}{4}i\int d\xi\, \epsilon(x^3-\xi)\{[i\partial_j - gB_j(x^0, \vec{x}_\perp, \xi)]\gamma^j + M\} \times \gamma^0\psi_+(x^0, \vec{x}_\perp, \xi) , \tag{2.5}$$

$$B^3(x) = -\tfrac{1}{2}\int d\xi\, |x^3-\xi|\{\partial_3\partial_j B^j(x^0, \vec{x}_\perp, \xi) + gJ^0(x^0, \vec{x}_\perp, \xi)\} . \tag{2.6}$$

In (2.6), we have introduced the τ component (zeroth component) of the vector current

$$J^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x) . \tag{2.7}$$

In a straightforward generalization, the usual nonet of vector currents is

$$J_a^\mu(x) = \bar{\psi}(x)\gamma^\mu \tfrac{1}{2}\lambda_a \psi(x) . \tag{2.8}$$

The canonical commutators of the independent operators are[11]

$$[B^i(x), B^j(0)]_{x^0=0} = -\tfrac{1}{4}i\delta_{ij}\epsilon(x^3)\delta^2(\vec{x}_\perp) , \tag{2.9}$$

$$\{\psi_+(x), \psi_+^*(0)\}_{x^0=0} = (1/\sqrt{2})P_+\delta(x^3)\delta^2(\vec{x}_\perp) . \tag{2.10}$$

The operators ψ_+ and ψ_+^* anticommute with themselves and commute with the B^i. The commutators of the dependent fields are complicated because the equations of motion (2.5) and (2.6) must be used.

In a straightforward way, (2.5), (2.6), and (2.10) yield the following:

$$\begin{aligned}\{\psi(x), \bar{\psi}(0)\}_{x^0=0} = \Big\{ & \tfrac{1}{2}\gamma^3\delta(x^3) - \tfrac{1}{4}i\epsilon(x^3)[i\gamma^j\partial_j + M] + \tfrac{1}{4}ig\gamma^j\epsilon(x^3)[P_+B_j(0, \vec{0}, x^3) + P_-B_j(0)] + M\} \\ & + \tfrac{1}{16}\int_{-\infty}^{\infty} d\xi\, \epsilon(x^3-\xi)\epsilon(\xi)[i\gamma^i\partial_i - g\gamma^i B_i(0, \vec{x}_\perp, \xi) + M][i\gamma^j\partial_j - g\gamma^j B_j(0, \vec{x}_\perp, \xi) - M]\gamma^0 \\ & - \tfrac{1}{64}ig^2\int_{-\infty}^{\infty} d\xi d\xi'\, \epsilon(x^3-\xi)\epsilon(\xi')\epsilon(\xi-\xi')\gamma^0\gamma_i\psi(0, \vec{0}, \xi)\bar{\psi}(0, \vec{0}, \xi')\gamma^i\gamma^0 \Big\}\delta^2(\vec{x}_\perp) .\end{aligned} \tag{2.11}$$

This lengthy expression, with its terms in g and g^2, shows the extent to which light-cone commutators incorporate the underlying dynamics. Note that there is inevitably an infinity in the terms proportional to g^2 on the right-hand side, which involve products of operators at the same point. The products of distributions that occur in (2.11) are defined in terms of their finite parts, i.e., in momentum space. Thus, for example,

$$\int_{-\infty}^{\infty} d\xi\, \epsilon(x^3-\xi)\epsilon(\xi) = \frac{-2}{\pi}\int_{-\infty}^{\infty}\frac{dk}{k^2}e^{ikx^3} = 2|x^3|.$$

III. CURRENT COMMUTATORS

We are able to derive all our results by considering the two equal-τ commutators $[J_a^0, J_b^0]$ and $[J_a^0, J_b^3]$. The first commutator is extremely simple, since J_a^0 is expressible in terms of ψ_+, i.e.,

$$J_a^0(x) = \sqrt{2}\ \psi_+^*(x)\tfrac{1}{2}\lambda_a\psi_+(x). \tag{3.1}$$

With the aid of the canonical commutator (2.11), it is easy to find

$$[J_a^0(x), J_b^0(0)]_{x^0=0} = if_{abc}J_c^0(x)\delta(x^3)\delta(\vec{x}_\perp). \tag{3.2}$$

This has precisely the same structure as the conventional equal-time commutator, but its meaning is quite different. It also contains the commutator of an ordinary time component with an ordinary space component, and therefore should carry information about Schwinger terms. In the sum rules to be given below, it will be seen that (3.2) gives a vanishing Schwinger term, as does the usual naive, formal derivation of equal-time quark current commutators. This indicates that (3.2) must be modified to accommodate the necessarily nonvanishing Schwinger term [see (4.13)].

The other commutator is rather complicated since J_a^3 involves dependent fields, i.e.,

$$J_a^3(x) = \sqrt{2}\ \psi_-^*(x)\tfrac{1}{2}\lambda_a\psi_-(x). \tag{3.3}$$

For the sake of simplicity, we suppress $SU(3)$ indices in the following expression, which therefore should be understood as the commutator of two electromagnetic currents. In the Appendix we list the full $SU(3)\times SU(3)$ commutators. After a not too lengthy calculation, we find

$$\begin{aligned}[][J^0(x), J^3(0)]_{x^0=0} = &-\tfrac{1}{2}i\epsilon(x^3)\delta^2(\vec{x}_\perp)\overline{\psi}(0,\vec{0},x^3)\\ &\times\{\gamma^j[i\overleftrightarrow{\partial}_j + gB_j(0,\vec{0},x^3)]+M\}P_-\psi(0)\\ &-\tfrac{1}{2}\epsilon(x^3)\partial_i\delta^2(\vec{x}_\perp)\overline{\psi}(0,\vec{0},x^3)\gamma^iP_-\psi(0)\\ &-\text{H.c}.\end{aligned} \tag{3.4}$$

The commutator is not expressible in terms of the current itself, but rather it contains the product $\overline{\psi}\psi$ at different space-time points. Like the commutator of the fields (2.11), it appears to depend explicitly on the coupling to gB^μ. However, we now demonstrate that this dependence may be eliminated. By use of the equations of motion, we find

$$\begin{aligned}&\overline{\psi}(x)\{\gamma^\mu[i\overleftrightarrow{\partial}_\mu + gB_\mu(x)]+M\}=0,\\ &\overline{\psi}(x)\{\gamma^i[i\overleftrightarrow{\partial}_i + gB_i(x)]+M\}P_- = -\overline{\psi}(x)\gamma^3[i\overleftrightarrow{\partial}_3 + gB_3(x)]\\ &\qquad = -i\partial_3\overline{\psi}(x)\gamma^3.\end{aligned} \tag{3.5}$$

The last equality is true in the special gauge $B^0 = B_3 = 0$. Hence (3.4) becomes

$$\begin{aligned}[][J^0(x), J^3(0)]_{x^0=0} = &-\tfrac{1}{2}\epsilon(x^3)\delta^2(\vec{x}_\perp)\partial_3[\overline{\psi}(0,\vec{0},x^3)\gamma^3\psi(0)]\\ &-\tfrac{1}{4}\epsilon(x^3)\partial_i\delta^2(\vec{x}_\perp)\overline{\psi}(0,\vec{0},x^3)\gamma^i\psi(0)\\ &-\tfrac{1}{4}\epsilon(x^3)\partial_i\delta^2(\vec{x}_\perp)\epsilon^{ij}\overline{\psi}(0,\vec{0},x^3)\gamma_j\gamma_5\psi(0)\\ &-\text{H.c}.\end{aligned} \tag{3.6}$$

In offering (3.6), we have rewritten the gradient term with the help of the identity $\gamma^iP_- = \frac{1}{2}\gamma^i + \frac{1}{2}\epsilon^{ij}\gamma_j\gamma_5$, where ϵ^{ij} is the antisymmetric tensor in two dimensions and γ_5 is given by the conventional formula $\gamma_5 = \hat{\gamma}^0\hat{\gamma}^1\hat{\gamma}^2\hat{\gamma}^3$. The remarkable feature of (3.6) is that all reference to the gluon field, and to the interaction has disappeared. The commutator is completely described by bilocal generalizations of the vector and axial-vector current, i.e., $\overline{\psi}(x)\gamma^\mu\psi(0)$, $i\overline{\psi}(x)\gamma^\mu\gamma_5\psi(0)$.

The commutators (3.2) and (3.6) can also be obtained from Schwinger's action principle,[12] extended to light-cone quantization. According to that technique, a commutator can be represented by a variational formula. For conserved currents the appropriate expression is

$$\int d^2z_\perp dz^3[J^0(x^0,\vec{x}_\perp,x^3), \delta\mathcal{L}(x^0,\vec{z}_\perp,z^3)] = i\partial_\mu\delta J^\mu(x). \tag{3.7}$$

The variation is performed with respect to external fields which are introduced for that purpose. For example, to generate the $[J^0, J^\mu]$ commutator, one varies an external field A_μ, which is taken to couple to $-J_\mu$.

$$\begin{aligned}-[J^0(x), J^\mu(y)]_{x^0=y^0} &= i\partial_\alpha\frac{\delta J^\alpha(x)}{\delta A_\mu(y)}\\ &= i\partial_\alpha\left[\overline{\psi}(x)\gamma^\alpha\frac{\delta\psi(x)}{\delta A_\mu(y)}\right]-\text{H.c}.\end{aligned} \tag{3.8}$$

In the gluon model which we are considering, the dependence of ψ on A_μ may be computed. We have, in the limit of zero external field,

$$\frac{\delta\psi(x)}{\delta A_\mu(y)} = -\tfrac{1}{4}i\epsilon(x^3-y^3)\delta^2(\vec{x}_\perp-\vec{y}_\perp)\gamma^0\gamma^\mu\psi(y). \tag{3.9}$$

Here the variation is performed, as it should be, with the independent canonical variable ψ_+ fixed, while the dependent variable ψ_-, when expressed in terms of ψ_+, contains the indicated A_μ dependence. Consequently, (3.8) becomes

$$[J^0(x), J^\mu(y)]_{x^0=y^0} = \tfrac{1}{4}\partial_\alpha[\overline{\psi}(x)\gamma^\alpha\gamma^0\gamma^\mu\psi(y)\epsilon(x^3-y^3)\delta^2(\vec{x}_\perp-\vec{y}_\perp)] - \text{H.c.} \tag{3.10}$$

This verifies (3.2) in the Abelian case, reproduces (3.6), and yields a new result: the commutator $[J^0, J^i]$.

We emphasize that these results are obtained by a *formal* argument; one cannot expect them to be valid in perturbation theory [see also (4.13) and (4.15), below].

IV. SUM RULES

In the standard fashion of deriving sum rules, we first write down a spectral representation and then deduce the appropriate equal-τ commutators and compare them to the matrix elements of the canonical commutators of Sec. III. There are two equivalent ways of going to the equal-τ limit from the spectral representation: the first works with the space-time coordinates directly, and the second is an obvious modification of the BJL limit in momentum space. The latter is useful in extracting light-cone commutators from conventional Feynman diagrams. This will be discussed in Sec. IV C, below.

A. Vacuum Expectation Value of Field Commutator

Consider the spectral representation for the anticommutator of unrenormalized Heisenberg fields

$$S(x) \equiv \langle 0|\{\psi(x), \psi(0)\}|0\rangle = \int_0^\infty d\lambda^2[\rho_1(\lambda^2)i\gamma^\mu\partial_\mu + M\rho_2(\lambda^2)]\Delta(x,\lambda^2). \tag{4.1}$$

Here Δ is the usual free-field commutator function, with the representation

$$\Delta(x,\lambda^2) = \frac{1}{(2\pi)^3}\int d^4k\,\epsilon(k^0+k^3)\delta(2k^0k^3 - \vec{k}_\perp^2 - \lambda^2)e^{-ikx}$$
$$= -i\epsilon(x^0+x^3)\frac{\delta(x^2)}{2\pi} + i\epsilon(x^0+x^3)\theta(x^2)\frac{\lambda}{4\pi(x^2)^{1/2}}J_1(\lambda(x^2)^{1/2}). \tag{4.2}$$

From (4.2), we learn the following useful results at $x^0=0$:

$$\Delta(x,\lambda^2)|_{x^0=0} = -\tfrac{1}{4}i\epsilon(x^3)\delta^2(\vec{x}_\perp), \tag{4.3}$$

$$\partial_0\Delta(x,\lambda^2)|_{x^0=0} = \tfrac{1}{8}i|x^3|(\partial_i\partial^i+\lambda^2)\delta^2(\vec{x}_\perp). \tag{4.4}$$

Equation (4.4) can also be derived from (4.2) by integrating the equation

$$(\Box+\lambda^2)\Delta = (2\partial_0\partial_3 + \partial_i\partial^i + \lambda^2)\Delta = 0$$

over x^3, and setting $x^0=0$.

The two *conventional* sum rules for the spectral functions are simply

$$\int_0^\infty d\lambda^2\rho_i(\lambda^2) = 1, \quad i=1,2 \tag{4.5}$$

if, in (4.1), M has the significance of the bare mass. For $i=1$, this is the usual sum rule based on canonical anticommutators at equal t, while for $i=2$ one uses the equations of motion to determine $\{\partial_t\psi(x), \overline{\psi}(0)\}$ at $t=0$. *Both* of these plus another relation appear as canonical in the light-cone theory. We now demonstrate this fact.

With the aid of (4.1), (4.3), and (4.4), we find

$$S(x)|_{x^0=0} = \int d\lambda^2\{\rho_\perp(\lambda^2)[-\tfrac{1}{8}\gamma^0|x^3|(\partial_i\partial^i+\lambda^2) + \tfrac{1}{2}\gamma^3\delta(x^3) + \tfrac{1}{4}\epsilon(x^3)\gamma^i\partial_i] - \tfrac{1}{4}iM\rho_2(\lambda^2)\epsilon(x^3)\}\delta^2(\vec{x}_\perp). \tag{4.6}$$

To derive the sum rules, we compare (4.6) with the vacuum expectation value of the canonical anticommutator (2.11):

$$\langle 0|\{\psi(x), \overline{\psi}(0)\}|0\rangle_{x^0=0} = [-\tfrac{1}{8}\gamma^0|x^3|(\partial_i\partial^i+M^2) + \tfrac{1}{2}\gamma^3\delta(x^3) - \tfrac{1}{4}i(i\gamma^i\partial_i+M)\epsilon(x^3) + \tfrac{1}{8}g^2\gamma^0|x^3|\langle 0|B_i(0)B^i(0)|0\rangle]\delta^2(\vec{x}_\perp) + \cdots. \tag{4.7}$$

In addition to dropping terms linear in B^μ, whose vacuum expectation value vanishes, we have omitted in (4.7) a messy expression, coming from the last term of (2.11), which must be kept to maintain gauge covariance but does not materially assist our understanding of the nature of the sum rules. This omitted term is proportional to γ^0, and is represented by the dots in (4.7).

By comparing the coefficients of γ^3 and 1, we recover the sum rules (4.5). Then the coefficients of γ^0 yields

$$\int_0^\infty d\lambda^2\rho_1(\lambda^2)(M^2-\lambda^2) = g^2\langle 0|B_i(0)B^i(0)|0\rangle + \cdots. \tag{4.8}$$

This is the naive form of a sum rule which can be derived by a double application of the field equations, plus the canonical equal-time anticommutators. We do not pursue the discussion of this relation any further, since it is extremely unlikely that the spectral function ρ_1 is sufficiently convergent to give meaning to (4.8). However, the present results are interesting because they indicate that (at least for vacuum expectation values) the *equal-time* commutators are valid simultaneously with the *equal-τ* commutators. That this should be true is not evident *a priori*.

B. Vacuum Expectation Value of Current Commutators

We begin with the vacuum expectation value of the current commutator, with $SU(3)$ indices suppressed. The spectral form for this object is

$$\langle 0|[J^\mu(x),J^\nu(0)]|0\rangle=(g^{\mu\nu}\Box-\partial^\mu\partial^\nu)\int_0^\infty d\lambda^2\rho(\lambda^2)\Delta(x;\lambda^2). \tag{4.9}$$

With the aid of (4.3) and (4.4), we find

$$\langle 0|[J^0(x),J^0(0)]|0\rangle_{x^0=0}=\tfrac{1}{2}i\int_0^\infty d\lambda^2\rho(\lambda^2)\delta'(x^3)\delta^2(\vec{x}_\perp), \tag{4.10}$$

$$\langle 0|[J^0(x),J^3(0)]|0\rangle_{x^0=0}=\tfrac{1}{8}i\int_0^\infty d\lambda^2(\lambda^2-\partial_i\partial^i)\rho(\lambda^2)\epsilon(x^3)\delta^2(\vec{x}_\perp). \tag{4.11}$$

The canonical commutator (3.2) would lead us to expect zero for the right-hand side of (4.10), which cannot vanish because of the positivity of ρ. Of course, this is just the ancient problem that naive canonical commutators do not yield the Schwinger term in fermion theories. Recall that the conventional Schwinger term has precisely the same form as (4.10), i.e.,

$$\langle 0|[\hat{J}^0(x),\hat{J}^i(0)]|0\rangle_{t=0}=i\int_0^\infty d\lambda^2\rho(\lambda^2)\partial^i\delta^3(\hat{x}). \tag{4.12}$$

Hence we learn that the equal-τ commutator (3.2) must be modified by a gradient of the δ function.

$$[J_a^0(x),J_b^0(0)]_{x^0=0}=if_{abc}J_c^0(x)\delta(x^3)\delta^2(\vec{x}_\perp)+\tfrac{1}{2}iS_{ab}\delta'(x^3)\delta^2(\vec{x}_\perp). \tag{4.13}$$

Of course more general forms are possible [see (4.23), below]. However the exhibited expression is the minimal generalization of the formal results which does not suffer from manifest contradictions.

The integral in (4.11) is the same as would be found for the conventional equal-time commutator (the dot means differentiation with respect to t)

$$\langle 0|[\dot{J}^i(x),J^j(0)]|0\rangle_{t=0}=-i\int_0^\infty d\lambda^2\rho(\lambda^2)[\delta^{ij}\lambda^2-\partial^i\partial^j]\delta^3(\hat{x}). \tag{4.14}$$

Therefore the sum rules for ρ which follow from (3.6) and (4.11) are equivalent to the results of the equal-time algebra and the equations of motion. Such theorems are only formal, due to divergences of the integrals over ρ, and we do not elaborate on them any further, beyond remarking that the Schwinger term anomaly is present in the $[J^0,J^3]$ commutator as well. Note that (4.11) indicates the presence of a second derivative of a δ function, whose coefficient is the *same* Schwinger term as in (4.10), i.e.,

$$-\tfrac{1}{8}iS\epsilon(x^3)\partial_i\partial^i\delta^2(\vec{x}_\perp).$$

No such second derivative terms are present in our operator formula (3.6). Hence we must also modify that expression in the indicated fashion. This parallels the fact that the $[\dot{J}^i,J^j]$ canonical equal-*time* commutator also misses this Schwinger term [compare (4.14)]. Hence we replace (3.6) by

$$\begin{aligned}[J^0(x),J^3(0)]_{x^0=0}&=-\tfrac{1}{2}\epsilon(x^3)\delta^2(\vec{x}_\perp)\partial_3[\bar\psi(0,\vec{0},x^3)\gamma^3\psi(0)]\\&\quad-\tfrac{1}{4}\epsilon(x^3)\partial_i\delta^2(\vec{x}_\perp)\bar\psi(0,\vec{0},x^3)\gamma^i\psi(0)\\&\quad-\tfrac{1}{4}\epsilon(x^3)\partial_i\delta^2(\vec{x}_\perp)\epsilon^{ij}\bar\psi(0,\vec{0},x^3)\gamma_j\gamma_5\psi(0)-\text{H.c.}\\&\quad-\tfrac{1}{8}iS\epsilon(x^3)\partial_i\partial^i\delta^2(\vec{x}_\perp).\end{aligned} \tag{4.15}$$

C. BJL Theorem on the Light Cone

The results discussed in the previous sections can be gotten quite easily in momentum space, by a version of the BJL limit[13] which we now describe. Let $J(x)$ and $K(0)$ be any two operators, and consider the matrix element between states (A) and (B),

$$M(q)=i\int d^4x\,e^{iqx}\langle A|T^*J(x)K(0)|B\rangle. \tag{4.16}$$

Here T^* indicates a covariant product, time-ordered with respect to the usual t. We claim that, in the limit $q^3=2^{-1/2}(\hat{q}^0-\hat{q}^3)\to\infty$ with $q^0=2^{-1/2}(\hat{q}^0+\hat{q}^3)$ and q^i fixed, $M(q)$ is expressible in an inverse power series whose coefficients are equal-τ commutators, i.e.,

$$M(q)\xrightarrow[q\to\infty]{}\text{polynomials}-\frac{1}{q^3}\int dx^3d^2x_\perp\exp[i(q^0x^3-\vec{q}_\perp\cdot\vec{x}_\perp)]\langle A|[J(x),K(0)]|B\rangle_{x^0=0}+\cdots. \tag{4.17}$$

This is not at all surprising, since in the exponent of (4.16), $q\cdot x = q^3x^0+q^0x^3-\vec{q}_\perp\cdot\vec{x}_\perp$, and the large-$q^3$ behavior comes from the region of small x^0. The only nontrivial aspect of (4.17) is that starting with a t-ordered product leads to an *equal-τ commutator*. This is a consequence of causality. To establish our theorem, we need only to deal with the case where A and B are the vacuum, which amounts to some demonstrations about the free-field causal functions. Since these same free-field functions appear in the representation when A and B are not the vacuum (e.g., the DGS representation), the argument will be established in general.

For definiteness, consider J and K to be conserved currents.

$$M^{\mu\nu}(q) = i\int d^4x\, e^{iqx}\langle 0|T^*J^\mu(x)J^\nu(0)|0\rangle = (q^\mu q^\nu - g^{\mu\nu}q^2)\int_0^\infty d\lambda^2\rho(\lambda^2)\frac{1}{q^2-\lambda^2+i\epsilon}\,. \tag{4.18}$$

In the limit $q^3\to\infty$, it is simple to find

$$M^{00}(q)\xrightarrow[q\to\infty]{}\left(-\frac{1}{q^3}\right)\left[-\tfrac{1}{2}q^0\int_0^\infty d\lambda^2\rho(\lambda^2)\right], \tag{4.19}$$

$$M^{03}(q)\xrightarrow[q^3\to\infty]{}\text{const}+\left(-\frac{1}{q^3}\right)\left[\frac{1}{4q^0}\int_0^\infty d\lambda^2\rho(\lambda^2)(\lambda^2-q_\perp{}^2)\right]. \tag{4.20}$$

The coefficients of $-1/q^3$ in (4.19) and (4.20) are, respectively, the Fourier transforms (with respect to x^3 and $\vec{x}_\perp$) of (4.10) and (4.11).

Another way to see the validity of the present version of the BJL theorem, is to observe that the *time-ordered* product coincides with a *τ-ordered* product, apart from seagulls. The reason for this is that a step function in time (which is relevant for the time-ordered product) differs from a step function in $\tau=(t+z)/\sqrt{2}$ (which is relevant for the τ-ordered product) only for spacelike separations. But in that region causality renders ordering immaterial. Since the discontinuity of τ-ordered product is the light-cone commutator, the theorem follows directly.[14]

D. Light-Cone Commutators in Electroproduction

Several authors[4] have observed that the scaling functions introduced by Bjorken[3] to describe the MIT-SLAC electroproduction experiments[15] are intimately related to the behavior of current commutators near the light cone. We shall here derive sum rules for Fourier transforms of these scaling functions which follow from the light-cone algebra. Consider the spin-averaged nucleon matrix element of the electromagnetic-current commutator. From the observed scaling of the cross sections, it follows that this object must have the following form[4]:

$$\begin{aligned} i\langle p\|[J^\mu(x),J^\nu(0)]|p\rangle &= [g^{\mu\nu}\Box-\partial^\mu\partial^\nu]\epsilon(x^0+x^3)\left[\delta(x^2)\frac{1}{8\pi^2}\int_{-1}^1 d\omega\,\frac{\cos(\omega x\cdot p)}{\omega^2}F_L(\omega)+\theta(x^2)f_1(x^2,x\cdot p)\right]\\ &\quad+[p^\mu p^\nu\Box - p\cdot\partial(\partial^\mu p^\nu+\partial^\nu p^\mu)+g^{\mu\nu}(p\cdot\partial)^2]\epsilon(x^0+x^3)\theta(x^2)\left[\frac{1}{8\pi^2}\int_{-1}^1 d\omega\,\frac{\sin(\omega x\cdot p)}{\omega x\cdot p}F_2(\omega)+f_2(x^2,x\cdot p)\right], \end{aligned} \tag{4.21a}$$

$$x^2f_1(x^2,x\cdot p)\xrightarrow[x^2\to 0]{}0\,, \tag{4.21b}$$

$$f_2(x^2,x\cdot p)\xrightarrow[x^2\to 0]{}0\,. \tag{4.21c}$$

The corresponding covariant time-ordered product in momentum space is

$$\begin{aligned} T^{\mu\nu}(q) &\equiv i\int d^4x\,e^{iqx}\langle p|T^*J^\mu(x)J^\nu(0)|p\rangle\\ &=\frac{1}{4\pi}[g^{\mu\nu}q^2-q^\mu q^\nu]\int_{-1}^1\frac{d\omega}{\omega^2}\frac{F_L(\omega)}{q^2+2\omega\nu}+\frac{1}{2\pi\nu}[q^2p^\mu p^\nu-p\cdot q(p^\mu q^\nu+p^\nu q^\mu)+g^{\mu\nu}(p\cdot q)^2]\int_{-1}^1\frac{d\omega}{\omega}\frac{F_2(\omega)}{q^2+2\omega\nu}+\cdots. \end{aligned} \tag{4.22}$$

In the above, $F_L(\omega)=F_2(\omega)-2\omega F_1(\omega)$, $\omega=-q^2/2\nu$, $\nu=p\cdot q$, and F_1 and F_2 are Bjorken's scaling functions.[3] The dots in (4.22) represent the contribution to $T^{\mu\nu}$ which arises from the nonscaling portion of the absorptive parts. {In position space this is given by $f_i(x^2,x\cdot p)$ [see (4.21)].} The results which follow can be derived either by taking the limit $x^0\to 0$ in (4.21), or by considering the limit $q^3\to\infty$ in (4.22). The two procedures are equivalent; we shall follow the first method.

Note that, because of the possibility of Regge poles with intercepts $\alpha(t)$ at $t=0$ contributing to the small-ω behavior in the F_i (for small ω, F_2, $F_L\sim\omega^{1-\alpha(0)}$), certain integrals in (4.22) appear to be superficially divergent if $\alpha(0)\geq 0$. A careful investigation of how to treat this problem results in the simple answer: The contributions of a Regge pole with $\alpha(0)>0$ is to be treated as if $\alpha(0)$ were less than zero, and an analytic continuation to the proper value is done after integrating. This is entirely equivalent to the truncation procedure which has been discussed elsewhere.[16] If $\alpha(0)=0$, this method is inapplicable and logarithmic infinities arise which must be separately treated. For purposes of this investigation, we assume such terms are not present.

To facilitate the computation of the $x^0 \to 0$ limit is (4.21), we record the form that various distributions assume at this point.

$$
\begin{aligned}
&\epsilon(x^0+x^3)\delta(x^2)\big|_{x^0=0}=\tfrac{1}{2}\pi\epsilon(x^3)\delta^2(\vec{x}_\perp)\,,\\
&\frac{\partial}{\partial x^0}[\epsilon(x^0+x^3)\delta(x^2)]_{x^0=0}=\tfrac{1}{4}\pi|x^3|\nabla_\perp{}^2\delta^2(\vec{x}_\perp)\,,\\
&\epsilon(x^0+x^3)\theta(x^2)\big|_{x^0=0}=0\,,\\
&\frac{\partial}{\partial x^0}[\epsilon(x^0+x^3)\theta(x^2)]_{x^0=0}=\pi|x^3|\delta^2(\vec{x}_\perp)\,.
\end{aligned}
\tag{4.23}
$$

We now consider the zero-zero component of (4.20a). From (4.22) it follows that

$$
\begin{aligned}
i\langle p|[J^0(x),J^0(0)]|p\rangle_{x^0=0}&=-\partial_3\partial_3\left[\epsilon(x^3)\delta^2(\vec{x}_\perp)\frac{1}{16\pi}\int_{-1}^{1}d\omega\,\frac{\cos(\omega x^3p^0)}{\omega^2}F_L(\omega)\right]\\
&=-\frac{1}{8\pi}\delta'(x^3)\delta^2(\vec{x}_\perp)\int_{-1}^{1}\frac{d\omega}{\omega^2}F_L(\omega)+\epsilon(x^3)\delta^2(\vec{x}_\perp)\frac{(p^0)^2}{16\pi}\int_{-1}^{1}d\omega\,\cos(\omega x^3p^0)\,F_L(\omega)\,.
\end{aligned}
\tag{4.24}
$$

Comparing this with the corresponding matrix element of (4.13) (with internal symmetry indices suppressed), we find that the connected matrix element of the Schwinger term S is given by

$$
\langle p|S|p\rangle=\frac{1}{4\pi}\int_{-1}^{1}\frac{d\omega}{\omega^2}F_L(\omega)\,.
\tag{4.25}
$$

Also the second term on the right-hand side of (4.24) is absent; according to (4.13),

$$
\int_{-1}^{1}d\omega\,\cos(\omega x^3p^0)F_L(\omega)=0\,.
\tag{4.26}
$$

Equation (4.25) is recognized as the Schwinger-term sum rule,[17] while (4.26) is the analog of the Callan-Gross sum rule for this model.[18] Note that while Callan and Gross deduce $\int_{-1}^{1}d\omega F_L(\omega)=0$, we obtain the more general result (4.26). Another difference between the Callan-Gross derivation and the present one is that whereas the former required the equations of motion and the $[\dot{J}^i,J^j]$ equal-time commutator, the latter uses the *interaction-independent* light-cone commutator between the zeroth components. The validity of these results will be discussed in Sec. VI.

The remaining sum rules which we shall discuss follow from the 03 components of (4.21a). To simplify the tedious calculation, $F_L(\omega)$ is set equal to zero, as is indicated by (4.26), by (4.25) when the Schwinger term is a c-number, and by the experimental data. From (4.23) we deduce

$$
\begin{aligned}
i\langle p|[J^0(x),J^3(0)]|p\rangle_{x^0=0}&=\pi\delta^2(\vec{x}_\perp)\frac{\partial}{\partial x^3}[|x^3|f_1(0,x^3p^0)]+\left(2p^3\frac{\partial}{\partial x^3}+p^i\partial_i\right)\epsilon(x^3)\delta^2(\vec{x}_\perp)\frac{1}{8\pi}\int_{-1}^{1}d\omega\,\frac{\sin(\omega x^3p^0)}{\omega}F_2(\omega)\\
&=\delta^2(\vec{x}_\perp)\epsilon(x^3)\left[\frac{\vec{p}_\perp{}^2}{8\pi}\int_{-1}^{1}d\omega\,\cos(\omega x^3p^0)\,F_2(\omega)+f(x^3p^0)\right]\\
&\quad+p^i\partial_i\delta^2(\vec{x}_\perp)\epsilon(x^3)\frac{1}{8\pi}\int_{-1}^{1}d\omega\,\frac{\sin(\omega x^3p^0)}{\omega}F_2(\omega)\,.
\end{aligned}
\tag{4.27}
$$

We have introduced the function f which is defined by

$$
f(\alpha)=\frac{m^2}{8\pi}\int_{-1}^{1}d\omega\,\cos(\omega\alpha)F_2(\omega)+\frac{\partial}{\partial\alpha}[\alpha f_1(0,\alpha)]\,.
$$

In order to obtain a sum rule for $F_2(\omega)$, we now equate (4.27) to the spin-averaged matrix element of (4.15). Since we are assuming a c-number Schwinger term, the double derivative of the δ function is absent. Also, parity conservation prevents the bilocal axial-vector in (4.15) from having a matrix element. Thus we are left with

$$i\langle p\,\|[J^0(x),J^3(0)]\,|p\rangle_{x^0=0} = -\tfrac{1}{2}\epsilon(x^3)\delta^2(\vec{x}_\perp)\partial_3\langle p\,|\,i\overline{\psi}(0,\vec{0},x^3)\gamma^3\psi(0)+\text{H.c.}\,|p\rangle$$
$$-\tfrac{1}{4}\epsilon(x^3)\partial_i\delta^2(\vec{x}_\perp)\langle p\,|\,i\overline{\psi}(0,\vec{0},x^3)\gamma^i\psi(0)+\text{H.c.}\,|p\rangle\,. \tag{4.28}$$

Equating coefficients of the gradient of the δ function in (4.27) and (4.28) gives

$$p^i\Lambda(x^3p^0) \equiv \langle p\,|\,i\overline{\psi}(x)\gamma^i\psi(0)+\text{H.c.}\,|p\rangle|_{x^2=0}$$
$$= -\frac{p_i}{2\pi}\int_{-1}^{1}d\omega\,\frac{\sin(\omega x^3p^0)}{\omega}F_2(\omega)\,. \tag{4.29}$$

On the left-hand side of (4.29), $x^2=0$ is achieved by letting x^0 and $\vec{x}_\perp$ vanish, but $x^3\neq 0$. The sum rule which follows from an identification of the remaining terms in (4.27) and (4.28) is not independent of (4.29); one merely arrives at a derivative of (4.29) with respect to x^3.

Equation (4.29) is considerably more general than the formula obtained by Callan and Gross.[18] It will be remembered that they related $\int_{-1}^{1}d\omega\,F_2(\omega)$ to the $[J^i,J^j]$ equal-time commutator, which involves among other things an explicit dependence on the gluon field. Within our formalism it is possible, though unnecessary, to obtain this result. To do so, one needs to consider the alternate formula for $[J^0,J^3]$ which *does* involve the gluon field, (3.4). Equating coefficients of the δ function reproduces the Callan-Gross relation as $x^3\rightarrow 0$.

Though $\Lambda(x^3p^0)$, defined by (4.29), cannot at present be evaluated, the sum rule for $F_2(\omega)$ is sufficiently striking in its simplicity to warrant further study. It is interesting to remark that the operator, whose matrix element defines Λ, becomes related to the *fermion* part of the energy-momentum tensor when a differentiation with respect to x is performed; then x is set to zero.[19] Repeating this procedure relates moments of F_2 weighted by powers of ω to matrix elements of operators of the form $\overline{\psi}\gamma^j\overleftrightarrow{\partial}^{i_1}\overleftrightarrow{\partial}^{i_2}\cdots\psi$.

It is seen that the main difference between the present results and the analogous ones which follow from the equal-time algebra together with the equations of motion is the following. The conventional technique provides relations between moments of the F_i and matrix elements of local operators. The present analysis yields formulas for Fourier transforms of the F_i in terms of "bilocal" operators, i.e., products of operators defined at *different* points. The difference between these points is the conjugate variable in the Fourier transform.

Another interesting aspect of (4.29) is that it may be used to circumvent some of the divergences encountered in perturbation theory. Although our results are not verified in perturbative calculations (see the discussion in Sec. VI), we may adopt the (somewhat inconsistent) viewpoint that (4.29) is postulated to be true, and Λ, defined by (4.29), is calculated in perturbation theory, thus determining $F_2(\omega)$. A preliminary investigation indicates that it may be possible to give a calculational prescription which renders Λ finite. [The sources of divergences in $\langle p|i\overline{\psi}(x)\gamma^\mu\psi(0)|p\rangle$ are twofold: (1) The operators ψ and $\overline{\psi}$ are unrenormalized, hence logarithmically divergent. (2) There is a logarithmic singularity as $x^2\rightarrow 0$. To obtain finite results one must arrange that these two divergences cancel.]

In conclusion, we remark that the formula (4.13) is such that the Dashen – Fubini – Gell-Mann sum rule is satisfied. The validity of that relation may be summarized by the equation[1]

$$\int dx^3\langle p\,\|[J_a^0(x),J_b^0(0)]\,|p'\rangle_{x^0=0}$$
$$= if_{abc}\langle p\,|\,J_c^0(0)\,|p'\rangle\delta^2(\vec{x}_\perp)\,. \tag{4.30}$$

It is seen that (4.32) is satisfied by (4.13). It is also true that (4.32) is true, even if the $[J^0,J^0]$ commutator is of a form such that F_L is nonvanishing as in (4.24).

V. COMPARISON OF EQUAL-t AND EQUAL-τ COMMUTATORS: REGGE TRAJECTORIES

An equal-τ commutator can be expressed as a sum of an infinite number of equal-t commutators, involving time derivatives of the basic currents. This expression is a special case of the general expression of operator products at short distances, first introduced by Wilson[5] for small x, and recently generalized by various people[20] to the case of small x^2. We show here how to incorporate an entire Regge trajectory with a single equal-τ commutator, by appropriate choice of equal-t commutators.

It has already been argued[21] that the equal-time commutator structure

$$[\partial_t{}^nJ_i(x),J_j(0)]^{(+)}_{t=0}$$
$$\equiv \tfrac{1}{2}\{[\partial_t{}^nJ_i(x),J_j(0)]_{t=0}+(i\leftrightarrow j)\}$$
$$=[\beta_{n+1}\hat{T}^{(n+1)}_{ij0\ldots0}(0)-\gamma_{n+1}\delta_{ij}\hat{T}^{(n-1)}_{0\ldots0}(0)]\delta^3(\vec{x})+\cdots\,,$$

where the $\hat{T}^{(n+1)}_{\mu\nu\ldots}$ are local quantum fields of spin $n+1$, expresses Bjorken's scaling laws as an operator equation. When the two currents J_i,J_j bear the same $SU(3)$ labels, only odd-n terms contribute to (5.1), because of the symmetrization on i and j. (Recall that the careted operators $\hat{T}$ are described with the conventional index structure: 0 indices are *time* components.) The omitted terms in (5.1) contain gradients or are local fields of spin less than $n+1$, and are of no interest for describing scaling. The normalization of these fields is, at this point, arbitrary; we choose it such that

$$\langle p' | T^{(n)}_{\mu\nu\ldots}(0) | p\rangle = \frac{(P_\mu P_\nu \ldots) + \cdots}{t - M_n^{\,2}}\,, \qquad P_\mu = \tfrac{1}{2}(p+p')_\mu, \quad t = (p-p')^2\,, \tag{5.2}$$

where p' and p are the momenta of two hadrons of equal mass. We suppose that the fields are traceless, symmetric, and conserved, and that they create one-particle states of mass $M_n^{\,2}$ from the vacuum. Thus these fields might be used in the Van Hove[22] model to construct a Regge trajectory out of an infinite number of integral-spin fields. We shall now incorporate this entire Regge trajectory into a single equal-τ commutator.

For the sake of simplicity, set all the γ_n in (5.1) to zero; their contribution can easily be added at the end. The expression of the equal-τ commutator in terms of the infinite set of equal-t commutators (5.1) is

$$[J_i(x), J_j(0)]^{(+)}_{x^0=0} = \tfrac{1}{4}\epsilon(x^3)\delta^2(\vec{x}_\perp) \sum_{n=1,3,\ldots}^{\infty} (\tfrac{1}{2}x^3)^{n-1} \frac{\beta_{n+1}}{(n-1)!}\, T^{(n+1)}_{ij3\ldots3}(0) + \cdots. \tag{5.3}$$

(The uncareted fields have unconventional indices.) Again, the omitted part contains gradients and other uninteresting terms. Equation (5.3) is most easily established by taking its matrix elements for various states, using appropriate causal representations and the properties of $\Delta(x, \lambda^2 = 0)$.

We take the single-particle matrix elements of (5.3), i.e.,

$$\langle p' \| [J_i(x), J_j(0)]^{(+)} | p\rangle_{x^0=0} = \tfrac{1}{4}\epsilon(x^3)\delta^2(\vec{x}_\perp) P_i P_j \sum_{n=0,2,\ldots}^{\infty} \frac{1}{n!}\left(\frac{x\cdot P}{2}\right)^n \frac{\beta_{n+2}}{t - M_{n+2}^{\,2}}\,, \tag{5.4}$$

where we have used (5.2) and the fact that $x\cdot P = x^3 P_3$ at $x^0 = 0$, $\vec{x}_\perp = 0$. Just as in the Van Hove model, the sum in (5.4) may be replaced by a Regge pole and a background term, by smoothly interpolating a trajectory function $\alpha(t)$ and residue function $\beta(t)$ between integral points. The interpolations obey

$$\alpha(t = M_n^{\,2}) = n, \quad n = 2, 4, \ldots, \tag{5.5a}$$

$$\beta(t = M_n^{\,2}) = 2^{2-n}\beta_n, \quad n = 2, 4, \ldots. \tag{5.5b}$$

To save writing, we take a linear trajectory function

$$\alpha(t) = a + bt\,, \tag{5.6}$$

but this entails no loss of generality in principle.

With the aid of (5.5) and (5.6), the sum in (5.4) is transformed to

$$-\tfrac{1}{4}\epsilon(x^3)\delta^2(\vec{x}_\perp)p^i p^j \int_0^1 d\omega \cos(\omega x\cdot P)\frac{\beta(t)}{b}\,\omega^{1-\alpha(t)} \tag{5.7}$$

plus a background term with no particle poles, which we omit. This is precisely the form one expects for the contribution of a scaling Regge pole to a nonforward matrix element of an equal-τ commutator. Although we have not discussed nonforward matrix elements in this paper, it should be clear that (5.7) is correct from the discussion of Ref. 21 and from the forward matrix element (4.27). In particular, the coefficient of $p^i p^j \epsilon(x^3) \times \delta^2(\vec{x}_\perp)$ in this latter equation is (up to an over-all constant) the integral

$$\int_0^1 d\omega\, F_2(\omega)\cos(\omega x\cdot P)\,, \tag{5.8}$$

while at $t=0$ a Regge pole contributes a term proportional to $\omega^{1-\alpha(0)}$ to $F_2(\omega)$. Clearly, the scheme given here can be extended to linear combinations of Regge poles, to satellites (of the form $\omega^{N+1-\alpha(t)}$, $N = 1, 2, \ldots$) and daughters, etc.

Finally, note that in the vacuum expectation value of the equal-τ commutator (5.3), only the $N=1$ term can possibly survive on the right (and then only if the $T^{(n)}_{\mu\nu\ldots}$ are allowed to have nonvanishing traces). The reason is that the vacuum expectation value must be constructed from products of $g_{\mu\nu}$, but $g_{i3} = g_{33} = 0$. In this case, the matrix element of an equal-τ commutator reduces to a single equal-t commutator (not counting gradient terms). This is consistent with our earlier discussion of (4.10) and (4.11).

VI. CONCLUSION

The light-cone algebra, which we have deduced canonically, organizes in an elegant and compact fashion whole families of sum rules which follow from the usual equal-time algebra and the equations of motion. For example all the conventional moment sum rules, relating $\int_{-1}^{1} d\omega\, \omega^{2n} F_2(\omega)$ to appropriate matrix elements, are contained in the single relation (4.29) which involves only the particularly simple bilocal generalization of the local current.

In addition to the economical rendition of conventional results, the present techniques yield also apparently new relations such as (4.29). This suggests that there is dynamical content in light-cone commutators which exceeds that of the equal-time commutators.

It is clear that the forms we present for the various commutators are no more reliable predictors of perturbative calculations than the conventional equal-time relations. We have already remarked on our failure to calculate the Schwinger term. It is also true that in the model in question, $F_L(\omega) \neq 0$, contrary to (4.26).[23] Nevertheless, the left-hand side of (4.25) vanishes in perturbation theory. This happens because $F_L(\omega)/\omega^2 \propto 1 - 2\delta(\omega)$ (see Ref. 7).

In spite of these shortcomings, we believe that light-cone commutators provide useful relations for summarizing various aspects of hadron physics. Clearly one needs to go beyond the model computation presented here; a step in this direction was taken in Sec. V. It shall be most interesting to explore the constraints that Regge phenomenology imposes on the structure of the light-cone commutator. A related inquiry will expose the relation between fixed q^2 dispersion relations and the light-cone commutators. That such a relation exists is seen from the modified BJL theorem which we established: The light-cone commutator is related to the properties of amplitudes as $q^3 \to \infty$ at fixed q^0 and $\vec{q}_\perp$; while the large momentum limit at fixed q^2, which is relevant to dispersion relations, may be arrived at by passing to $q^3 \to \infty$ at fixed $\vec{q}_\perp$ and vanishing q^0. These and related topics are now under investigation.

It should be stressed that our results, even when they are interaction-independent, remain model dependent. For example, the equal-τ commutator between components of the electromagnetic current constructed from charged boson fields is not of the form (4.13) and (4.15). We leave it as an exercise to deduce this commutator, whose form also implies that $F_L \neq 0$ in that model.

Finally we wish to discuss the gauge invariance of our results. Since the quantization was performed in a special gauge $B^0 = B_3 = 0$, our results are not manifestly gauge invariant. This means that all calculations must be performed with a gluon propagator which respects this gauge:

$$D^{\mu\nu}(k) = D(k)\left(g^{\mu\nu} - \frac{n^\mu k^\nu}{n\cdot k} - \frac{n^\nu k^\mu}{n\cdot k}\right), \quad n^2 = 0,\ n^3 = 1\,.$$

This poses no problems for quantum electrodynamics, where the gluon (photon) is massless. In the massive gluon model we believe that the option of using the above propagator still remains, since the interaction is through a conserved current. The complete verification of this heuristic argument must await a thorough examination of the quantization of the massive gluon theory, a topic which is under present investigation. In any case, it is easy to make our bilocal operators manifestly gauge invariant. One merely replaces

$$\overline{\psi}(0, \vec{0}, x^3)\gamma^\mu\psi(0)$$

by

$$\overline{\psi}(0, \vec{0}, x^3)\exp\left[ig\int_0^{x^3} dy^3 B^0(0, \vec{0}, y^3)\right]\gamma^\mu\psi(0)\,.$$

That such a replacement is indeed correct follows also from the work of Gross and Treiman.[24]

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APPENDIX

We record here the generalization of the light-cone commutators which we have computed to the full $SU(3)\times SU(3)$ group. The vector and axial-vector currents are defined as follows:

$$V_a^\mu(x) = \tfrac{1}{2}\overline{\psi}(x)\gamma^\mu\lambda_a\psi(y)\,, \tag{A1a}$$

$$A_a^\mu(x) = \tfrac{1}{2}i\overline{\psi}(x)\gamma^\mu\gamma_5\lambda_a\psi(y)\,. \tag{A1b}$$

We shall also need bilocal generalizations of these objects.

$$V_a^\mu(x\,|\,y) = \tfrac{1}{2}\overline{\psi}(x)\gamma^\mu\lambda_a\psi(y)\,, \tag{A2a}$$

$$A_a^\mu(x\,|\,y) = \tfrac{1}{2}i\overline{\psi}(x)\gamma^\mu\gamma_5\lambda_a\psi(y)\,. \tag{A2b}$$

The commutators between the zeroth components are as in (3.2).

$$[V_a^0(x), V_b^0(y)]_{x^0=y^0} = if_{abc}V_c^0(x)\delta(x^3-y^3)\delta^2(\vec{x}_\perp - \vec{y}_\perp)\,, \tag{A3a}$$

$$[V_a^0(x), A_b^0(y)]_{x^0=y^0} = if_{abc}A_c^0(x)\delta(x^3-y^3)\delta^2(\vec{x}_\perp - \vec{y}_\perp)\,, \tag{A3b}$$

$$[A_a^0(x), A_b^0(y)]_{x^0=y^0} = if_{abc}V_c^0(x)\delta(x^3-y^3)\delta^2(\vec{x}_\perp - \vec{y}_\perp)\,. \tag{A3c}$$

The commutator between the 03 components are complicated.

$$[V_a^0(x), V_b^3(y)]_{x^0=y^0} - if_{abc}V_c^3(x)\delta(x^3-y^3)\delta^2(\vec{x}_\perp-\vec{y}_\perp)$$
$$= -\tfrac{1}{4}(if_{abc}+d_{abc})\{\partial_3[\epsilon(x^3-y^3)\delta^2(\vec{x}_\perp-\vec{y}_\perp)V_c^3(x|y)] + \tfrac{1}{2}\partial_i[\epsilon(x^3-y^3)\delta^2(\vec{x}_\perp-\vec{y}_\perp)(V_c^i(x|y)-i\epsilon^{ij}A_{jc}(x|y))]\}$$
$$+\tfrac{1}{8}i\epsilon(x^3-y^3)\delta^2(\vec{x}_\perp-\vec{y}_\perp)\overline{\psi}(x)P_-\Lambda_{ab}^-\psi(y) - \text{H.c}. \tag{A4a}$$

$$[V_a^0(x), A_b^3(y)]_{x^0=y^0} - if_{abc}A_c^3(x)\delta(x^3-y^3)\delta^2(\vec{x}_\perp-\vec{y}_\perp)$$
$$= -\tfrac{1}{4}(if_{abc}+d_{abc})\{\partial_3[\epsilon(x^3-y^3)\delta^2(\vec{x}_\perp-\vec{y}_\perp)A_c^3(x|y)] + \tfrac{1}{2}\partial_i[\epsilon(x^3-y^3)\delta^2(\vec{x}_\perp-\vec{y}_\perp)(A_c^i(x|y)-i\epsilon^{ij}V_{jc}(x|y))]\}$$
$$+\tfrac{1}{8}i\epsilon(x^3-y^3)\delta^2(\vec{x}_\perp-\vec{y}_\perp)\overline{\psi}(x)P_-i\gamma_5\Lambda_{ab}^-\psi(y) - \text{H.c}. \tag{A4b}$$

$$[A_a^0(x), V_b^3(y)]_{x^0=y^0} - if_{abc}A_c^3(x)\delta(x^3-y^3)\delta^2(\vec{x}_\perp-\vec{y}_\perp)$$
$$= -\tfrac{1}{4}(if_{abc}+d_{abc})\{\partial_3[\epsilon(x^3-y^3)\delta^2(\vec{x}_\perp-\vec{y}_\perp)A_c^3(x|y)] + \tfrac{1}{2}\partial_i[\epsilon(x^3-y^3)\delta^2(\vec{x}_\perp-\vec{y}_\perp)(A_c^i(x|y)-i\epsilon^{ij}V_{jc}(x|y))]\}$$
$$+\tfrac{1}{8}i\epsilon(x^3-y^3)\delta^2(\vec{x}_\perp-\vec{y}_\perp)\overline{\psi}(x)P_-i\gamma_5\Lambda_{ab}^+\psi(y) - \text{H.c}. \tag{A4c}$$

$$[A_a^0(x), A_b^3(y)]_{x^0=y^0} - if_{abc}V_c^3(x)\delta(x^3-y^3)\delta^2(\vec{x}_\perp-\vec{y}_\perp)$$
$$= -\tfrac{1}{4}(if_{abc}+d_{abc})\{\partial_3[\epsilon(x^3-y^3)\delta^2(\vec{x}_\perp-\vec{y}_\perp)V_c^3(x|y)] + \tfrac{1}{2}\partial_i[\epsilon(x^3-y^3)\delta^2(\vec{x}_\perp-\vec{y}_\perp)(V_c^i(x|y)-i\epsilon^{ij}A_{jc}(x|y))]\}$$
$$+\tfrac{1}{8}i\epsilon(x^3-y^3)\delta^2(\vec{x}_\perp-\vec{y}_\perp)\overline{\psi}(x)P_-\Lambda_{ab}^+\psi(y) - \text{H.c}. \tag{A4d}$$

In order to allow for current nonconservation, we have introduced a mass matrix M, and $\Lambda_{ab}^\pm \equiv [M,\lambda_a]_\pm\lambda_b$. It is seen that commutators of the (0, 3) components possess terms not seen at equal times. These terms are of two kinds: (1) There are gradient terms which vanish upon integrating the 0 component. (2) There are symmetry-breaking terms which are present even in the integrated commutator. The latter terms are a consequence of the fact that $Q_a = \int dx^3 d^2x_\perp V_a^0(x)$ and $Q_a^5 = \int dx^3 d^2x_\perp A_a^0(x)$ generate the appropriate transformation only when the symmetry is exact.

We emphasize that the above are the *canonical* commutators. In particular, noncanonical Schwinger terms *must* be added to the $[V_a^0, V_a^0]$, $[A_a^0, A_a^0]$, $[V_a^0, V_a^3]$, and $[A_a^0, A_a^3]$ commutators.

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$$\sum_{a=1}^{8}[J_a^i(\vec{x},0),J_a^j(0)] = -i\times\tfrac{16}{3}\Theta^{ij}(0)\delta(\vec{x}) \text{ plus terms in } \delta^{ij},$$

where $\Theta^{\mu\nu}$ is the fermion stress-energy tensor, and the commutator is taken at equal time. In the Sugawara model, the coefficient $\frac{16}{3}$ is replaced by 3, and $\Theta^{\mu\nu}$ is the full stress-energy tensor. If the equal-τ commutator (4.27) is expanded in powers of $\frac{1}{2}x^3$, and summed over $SU(3)$ indices, the coefficient of $-\frac{1}{4}|\frac{1}{2}x^3|\partial_j\delta(\vec{x}_\perp)$ is $\frac{16}{3}\Theta^{j0}(0)$, plus a term involving the gradient of the axial-vector current. The relationship between the expansion in powers of $\frac{1}{2}x^3$ and equal-time commutators is discussed in Sec. V.

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Feynman Rules for the Yang-Mills Field: A Canonical Quantization Approach. I*

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An attempt has been made to derive covariant Feynman rules for the massless Yang-Mills field, starting with canonical methods of quantization. In this paper we will summarize the techniques involved in such a program, along with a few preliminary results. Working in the radiation gauge ($\partial_i\vec{b}_i=0$), we find that there is an infinity of noncovariant vertices. We obtain a noncovariant set of rules to describe them to any order. Working with the suggested set of rules, we first prove that all tree diagrams can be described by a covariant set of Feynman rules. Secondly, to order g^2, we find that the one-loop diagram can also be made covariant. However, apart from the usual three-vector and four-vector vertices, the covariant loop contains an extra vertex of vector-scalar-scalar type and the scalar loop occurs with a weight factor of -2 with respect to the vector loop.

I. INTRODUCTION

In recent years, considerable attention has been given to the problem of obtaining covariant Feynman rules for the Yang-Mills field.[1] Because the Lagrangian for the massless Yang-Mills field obeys non-Abelian gauge symmetry, canonical methods of quantization are complicated due to the nonlinear nature of the constraints on the independent dynamical variables. Therefore, other less conventional methods were employed in deriving the Feynman rules for the field and accurate rules were suggested, first by Feynman,[2,2a] and later by Fadeev and Popov,[3] Mandelstam,[4] DeWitt,[5] and Fradkin and Tyutin.[6] Massless limits of massive gauge fields have also been studied[7] in this connection, but the resulting rules are found to violate both unitarity and Lorentz invariance and hence are incorrect. The method of canonical quantization, though complicated, is an unambiguous and more conventional procedure, and it serves to elucidate rather clearly the role of constraint equations in the derivation of the rules. The present paper is devoted to a study of this procedure. In this first of a series of papers, we will summarize the techniques involved in such a program, including a treatment of the constraint equations, and we will report a few preliminary results for tree and one-loop diagrams.

We will work in the radiation gauge [$\partial_i b_i^a(x)=0$, i.e., the field is transverse] and first isolate the independent dynamical variables, which will be postulated to satisfy the canonical commutation relations (CCR). In this gauge the interaction Hamiltonian is an infinite series in the coupling constant, each term of the series being noncovariant. Since we are working in a noncovariant gauge, the propagator is also noncovariant and contains the so-called normal-dependent terms.[8] First of all, we will suggest noncovariant rules for tree diagrams. However, we will prove them only up to order g^4 since all the essential elements in the proof are exhausted by fourth order. Going to higher order requires only a very complicated combinatorial analysis. We solve this problem in Appendix B. Using these noncovariant rules,