PHYSICAL REVIEW D VOLUME 4, NUMBER 12 15 DECEMBER 1971

Cluster-Decomposition Properties of φ^3-Perturbation-Theory Amplitudes at High Energy. II *

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(Received 27 September 1971)

In a previous paper, to which we shall refer as I, we demonstrated that a cluster-decomposition technique similar to that used in statistical mechanics can be applied to the study of high-energy scattering processes. Specifically, we examined in detail the ladder diagrams in a φ^3 field theory. In the analysis of I we focused attention on the cluster-decomposition properties of the differential multiparticle-production cross sections expressed as functions of the "momentum-transfer variables," k_i; in terms of ladder diagrams these variables correspond to the momenta carried by the sides of the ladders. Although the k-variable cluster decomposition arises naturally in theoretical analysis, as we emphasized in I, a phenomenologically potentially more useful approach would apply a cluster decomposition in terms of the actual final-state particle momenta, q_i; these variables, of course, correspond to the momenta carried by the rungs of the ladder diagrams in the simple model. In the present paper, we investigate the validity of such a q-variable cluster decomposition in φ^3 field theories. We explore the relationship between the k- and q-variable approaches and discuss and clarify a number of subtleties involved in the introduction of q-variable clusters. A feature that distinguishes the q-variable analysis from the earlier k-variable analysis is that a complete cluster decomposition – that is, a decomposition in terms of all components of the momenta q_i – of the differential exclusive-production cross sections is not possible even in the simple model in which these cross sections are calculated from ladder diagrams in a φ^3 field theory in three space and one time dimensions. We are thus led to consider the q-variable cluster decomposition of the partially differential cross sections obtained by integrating over the transverse-momentum components, $q_i^\perp$. The resulting cluster decomposition, essentially in terms of the rapidities corresponding to the longitudinal components of the q_i, provides a direct and intuitive framework for theoretical calculations of inclusive multiparticle spectra and avoids the ambiguities of "particle ordering" which would have hindered application of the original k-variable clusters to phenomenological analysis. We illustrate the utility of the cluster approach in two brief model calculations of inclusive particle spectra.

I. INTRODUCTION

Considerable theoretical effort in the study of strong interactions at high energy is currently devoted to obtaining a better understanding of many-particle production processes. On a qualitative level, the gas analogy[1] and the distinctions between exclusive and inclusive reactions have provided a useful conceptual language in which to discuss these events.[2,3] Recently, more quantitative results concerning multiparticle spectra and correlations have been derived and presently await confrontation with experiment.

In a previous publication,[4] to which we shall henceforth refer as I, we suggested an approach which relates all these developments to a systematic framework for analyzing many-particle reactions. Based on the cluster-decomposition techniques used in statistical mechanics,[5] this approach appears capable both of supporting the qualitative analogies and of providing a direct means for calculating multiparticle spectra, correlations, and other quantitative features of these processes.[6,7] Hence, it is clearly of interest to continue the investigation of this method.

In terms of the general production process indicated schematically in Fig. 1, the analysis in I centered on the cluster-decomposition properties of the differential exclusive cross sections in the momentum-transfer variables, which are labeled k_i in the figure. While such properties prove important for theoretical analysis of Feynman diagrams, as we emphasized in I, for phenomenological applications cluster-decomposition properties in the actual final-state momenta – labeled q_i in the figure – would be more useful. Unfortunately, the relation between the k-variable and q-variable clusters is not immediately apparent. However, it is the aim of this study to establish that a q-variable cluster decomposition is indeed valid for a certain class of Feynman diagrams in a φ^3 theory and to illustrate further the techniques by which the cluster decomposition can be applied to the phenomenological analysis of multiparticle

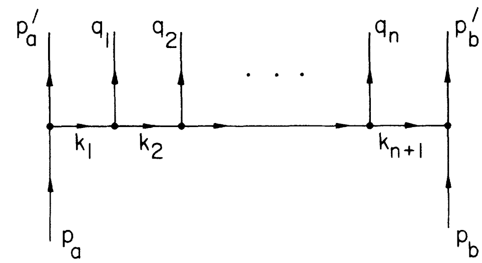

FIG. 1. The multiperipheral amplitude for the production process $2 \to n+2$.

production processes. For these purposes we have divided the investigation into two articles; the first reports a detailed examination in a simple field-theoretical model of the existence, the properties, and the application of q-variable clusters, whereas the second provides a general, model-independent framework for the utilization of cluster techniques.

In choosing a specific model for analysis in the present article, we are guided by the motivations and results of I, which suggest that simple ladder diagrams in a $(1+1)$-dimensional field theory provide an excellent indication of the cluster-decomposition properties – in terms of the k variables, in any case – exhibited by more general classes of diagrams and more realistic field theories. Thus to examine the possibility of a q-variable cluster expansion, we shall begin by discussing in Sec. II various properties of production cross sections derived from these ladder diagrams.

In Sec. III we establish the validity of a q-variable cluster decomposition for these cross sections and derive the general expression for the correlated part of a multiparticle spectrum in terms of the cluster functions. In addition, we calculate the explicit form of the first few cluster functions in the "nearest-neighbor approximation" to the ladder model.

Section IV contains calculations of two different inclusive spectra by both cluster and exact techniques; the agreement between the two methods serves as a specific verification of the cluster approach.

Finally, in Sec. V we comment on the extension of our results to a φ^3 theory in $(3+1)$ dimensions and briefly on the general applicability of a q-variable cluster decomposition.

II. THE LADDER MODEL FOR $\sigma^{(n)}(s)$

In Fig. 2 we summarize our conventions regarding φ^3 ladder diagrams. The incident particles, labeled a and b, have masses m_a and m_b, respectively, whereas the masses of all exchanged and produced particles are taken to be μ. Notice that we have defined the nth-order amplitude, $A^{(n)}(s)$, as that ladder having a discontinuity corresponding to a production process of the form $2 \to 2+n$. This somewhat unconventional definition will simplify later labeling problems. The amplitude $A^{(n)}(s)$ will correspond to that particular Feynman diagram in which the momenta are ordered naturally, that is, in the sequence $q_1, q_2, \ldots, q_n$. Since we wish to examine the case in which the q_i correspond to momenta of indistinguishable physical particles, we shall be concerned primarily with the partial inelastic cross sections, given by

$$\sigma^{(n)}(s) \equiv \sigma_{2\to n+2}(s) = \frac{1}{2s}\frac{1}{n!}\,\mathrm{Im}\,A^{s(n)}(s)\,. \tag{2.1}$$

Here the function $\mathrm{Im}\,A^{s(n)}(s)$ is obtained by summing over the imaginary parts of the $n!$ ladder diagrams corresponding to different possible orderings of $q_1, \ldots, q_n$. This summation over permutations is necessary to guarantee the required symmetry of the differential inelastic cross sections. Our investigation will consider potential cluster-decomposition properties in the variables q_i of these differential exclusive cross sections,

$$d^n\sigma(q_1, \ldots, q_n)\Big/\left(\frac{dq_1}{q_1}\cdots\frac{dq_n}{q_n}\right).$$

We shall integrate over the momenta p_a' and p_b', thus treating the corresponding particles as unobserved in an inclusive sense.

In the conventions of I, the absorptive part of $A^{(n)}(s)$ (the diagram corresponding to the labeling of the q_i as in Fig. 2) can be written as[8]

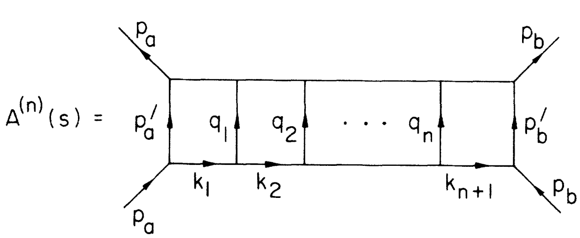

FIG. 2. The general n-rung ladder amplitude studied in this paper.

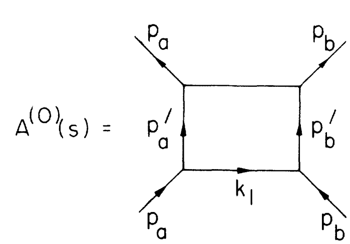

FIG. 3. The elastic scattering diagram corresponding to $A^{(0)}(s)$.

$$\mathrm{Im}A^{(n)}(s)=(g^2)^{n+2}\int\frac{d^2p_a'}{(2\pi)^2}(2\pi)\delta(p_a'^2-\mu^2)\prod_{i=1}^{n}\left(\frac{d^2q_i}{(2\pi)^2}(2\pi)\delta(q_i^{\,2}-\mu^2)\right)\frac{d^2p_b'}{(2\pi)^2}(2\pi)\delta(p_b'^2-\mu^2)$$
$$\times(2\pi)^2\delta^{(2)}\left(p_a+p_b-p_a'-p_b'-\sum_{i=1}^{n}q_i\right)\prod_{i=1}^{n+1}\frac{1}{(k_i^{\,2}-\mu^2)^2}, \tag{2.2}$$

where the k_i's are determined in terms of p_a, p_b, p_a', p_b' and the q's as indicated in Fig. 2. The equation may be transformed into an expression in terms of the variables[9]

$$q_i^{\pm}\equiv q_i^0\pm q_i^3$$

by using the results that

$$d^2q_i=\tfrac{1}{2}dq_i^+dq_i^-\,,$$
$$d^2q_i\delta(q_i^{\,2}-\mu^2)=\frac{dq_i^+}{q_i^+}\bigg|_{q_i^2=\mu^2}=\frac{dq_i^-}{q_i^-}\bigg|_{q_i^2=\mu^2},$$

and

$$\delta^{(2)}\left(p_a+p_b-p_a'-p_b'-\sum_{i=1}^{n}q_i\right)=2\delta^{(+)}\left(p_a^++p_b^+-p_a'^+-p_b'^+-\sum_{i=1}^{n}q_i^+\right)\delta^{(-)}\left(p_a^-+p_b^--p_a'^--p_b'^--\sum_{i=1}^{n}q_i^-\right).$$

Hence, if we write the result in the center-of-mass frame in which

$$p_a^++p_b^+=\sqrt{s}=p_a^-+p_b^-\,,$$

we have

$$\mathrm{Im}A^{(n)}(s)=\frac{(g^2)^{n+2}}{2^{n+1}(2\pi)^n}\int\frac{dp_a'^+}{p_a'^+}\frac{dp_b'^-}{p_b'^-}\prod_{i=1}^{n}\frac{dq_i^+}{q_i^+}\delta^{(+)}\left(\sqrt{s}-p_a'^+-p_b'^+-\sum_{i=1}^{n}q_i^+\right)\delta^{(-)}\left(\sqrt{s}-p_a'^--p_b'^--\sum_{i=1}^{n}q_i^-\right)\prod_{i=1}^{n+1}\frac{1}{(k_i^{\,2}-\mu^2)^2}, \tag{2.3}$$

where again we leave implicit the dependence of the k_i on the independent variables.

In (2.3) we have also left implicit the mass-shell restrictions on the dependent variables

$$p_a'^-=\mu^2/p_a'^+\,,\quad q_i^-=\mu^2/q_i^+\,,\ \text{and}\quad p_b'^+=\mu^2/p_b'^-\,. \tag{2.4}$$

In the high-energy limit, the exact result of (2.3) can be simplified by ignoring $p_b'^+$ and $p_a'^-$ in the functions $\delta^{(+)}$ and $\delta^{(-)}$, respectively, thereby "linearizing" these δ functions. This is a well-known approximation. That it is also a valid one can be indicated most simply by considering the diagram corresponding to $A^{(0)}(s)$ and shown in Fig. 3. The reader can see that if $p_a'^+$ becomes small (and hence $p_a'^-$ becomes large) then momentum conservation forces $k_1^+=p_a^+-p_a'^+$ to become large. At the same time, $p_a'^-$ is large and $p_a^-=m_a^{\,2}/\sqrt{s}$ is small, hence $k_1^-=p_a^--p_a'^-$ also becomes large. Thus the product $k_1^+k_1^-$ is forced to be far off the mass shell and the contribution to the amplitude from this region damps out rapidly at large s. With this simplification we may use the $\delta^{(+)}$ function to do the $p_a'^+$ integration and the $\delta^{(-)}$ function to do the $p_b'^-$ integration. At the same time we may make the replacements

$$\begin{aligned}p_a'^+&=\sqrt{s}-\sum_{i=1}^{n}q_i^+\,,\\ p_b'^-&=\sqrt{s}-\sum_{i=1}^{n}q_i^-\\ &=\sqrt{s}-\sum_{i=1}^{n}\mu^2/q_i^+\end{aligned} \tag{2.5}$$

everywhere in (2.3). Hence we obtain

$$\mathrm{Im}A^{(n)}(s)=\frac{(g^2)^{n+2}}{2^{n+1}(2\pi)^n}\int\prod_{i=1}^{n}\frac{dq_i^+}{q_i^+}\frac{1}{\sqrt{s}-\sum_{i=1}^{n}q_i^+}\frac{1}{\sqrt{s}-\sum_{i=1}^{n}q_i^-}\prod_{i=1}^{n+1}\frac{1}{(k_i^{\,2}-\mu^2)^2}. \tag{2.6}$$

Notice that to be consistent with the mass-shell restrictions (2.4) and the solutions given by (2.5) for the linearized δ functions, we must substitute

$$p_a'^-=\frac{\mu^2}{\sqrt{s}-\sum_{i=1}^{n}q_i^+}\quad\text{and}\quad p_b'^+=\frac{\mu^2}{\sqrt{s}-\sum_{i=1}^{n}q_i^-} \tag{2.7}$$

in the integrand of (2.6). To treat our approximations consistently in the denominator factors $1/(k_i^2-\mu^2)^2$, we shall write the minus components of the k_i in the form

$$k_i^- \equiv p_a^- - p_a'^- - \sum_{j=1}^{i-1} q_j^- = \frac{m_a^2}{\sqrt{s}} - \frac{\mu^2}{\sqrt{s}-\sum_{m=1}^{n} q_m^+} - \sum_{j=1}^{i-1}\frac{\mu^2}{q_j^+} \tag{2.8}$$

and the plus components as

$$k_i^+ \equiv p_b'^+ - p_b^+ + \sum_{j=i}^{n} q_j^+ = -\frac{m_b^2}{\sqrt{s}} + \frac{\mu^2}{\sqrt{s}-\sum_{m=1}^{n}\mu^2/q_m^+} + \sum_{j=i}^{n} q_j^+ . \tag{2.9}$$

Using these expressions to write out the denominators and introducing the dimensionless constant $\lambda = g^2/4\pi\mu^4$, we obtain

$$\operatorname{Im}A^{(n)}(s) \equiv 2\pi g^2 \frac{\lambda^{n+1}}{s}\int \prod_{i=1}^{n}\frac{dq_i^+}{q_i^+}\,\theta(\sqrt{s}-\textstyle\sum q_i^+)\,\theta(\sqrt{s}-\textstyle\sum q_i^-)[\operatorname{Im}a^{(n)}(q_1,\ldots,q_n)] . \tag{2.10}$$

Here,

$$\operatorname{Im}a^{(n)}(q_1,\ldots,q_n) = \frac{s(\mu^4)^{n+1}}{\left(\sqrt{s}-\sum_{i=1}^{n} q_i^+\right)\left(\sqrt{s}-\sum_{i=1}^{n} q_i^-\right)}\prod_{i=1}^{n+1}\frac{1}{d_i(q)} , \tag{2.11}$$

where

$$d_i(q) \equiv (k_i^+k_i^- - \mu^2)^2 = \left[\left(\frac{m_a^2}{\sqrt{s}} - \frac{\mu^2}{\sqrt{s}-\sum_{m=1}^{n} q_m^+} - \sum_{j=1}^{i-1} q_j^-\right)\left(-\frac{m_b^2}{\sqrt{s}} + \frac{\mu^2}{\sqrt{s}-\sum_{m=1}^{n} q_m^-} + \sum_{j=i}^{n} q_j^+\right) - \mu^2\right]^2 . \tag{2.12}$$

This result could also have been obtained directly from the full expression for $A_{n+1}(s)$ given in I in terms of the k variables simply by taking the imaginary part; however, since we wish to emphasize the q-variable approach here, clarity dictates that we derive the expression directly in terms of the q's. We shall later discuss the relation between the k-variable clusters and those in terms of the q variables.

Much of the remainder of our analysis hinges on several properties of $\operatorname{Im}a^{(n)}(s)$; hence, we shall study this function in detail. The first essential attribute of $\operatorname{Im}a^{(n)}(q_i)$ is factorization, which from the results of I we know to be crucial for the existence of a cluster decomposition; for[10]

$$\sqrt{s} > (q_1^+,\ldots,q_l^+) \gg (q_{l+1}^+,\ldots,q_m^+) \gg (q_{m+1}^+,\ldots,q_n^+) > \mu^2/\sqrt{s} , \tag{2.13}$$

a careful analysis establishes that $\operatorname{Im}a^{(n)}(s)$ becomes the product of three functions,

$$\operatorname{Im}a^{(n)}(q_1,\ldots,q_n) \longrightarrow f_1^{(l)}(q_1,\ldots,q_l)f_2^{(m-l)}(q_{l+1},\ldots,q_m)f_3^{(n-m)}(q_{m+1},\ldots,q_n) + O(\{q_l/q_m\},\{q_m/q_n\}) , \tag{2.14}$$

where in the expression $O(\{q_l/q_m\},\{q_m/q_n\})$ the terms q_l, q_m, and q_n represent generic elements of the sets $q_1,\ldots,q_l$, $q_{l+1},\ldots,q_m$, and $q_{m+1},\ldots,q_n$, respectively. The explicit forms of the three functions are

$$f_1^{(l)}(q_1,\ldots,q_l) = \left[\frac{\sqrt{s}}{\sqrt{s}-\sum_{j=1}^{l} q_j^+}(\mu^4)^l\left(\sum_{j=1}^{l} q_j^+\right)^{-2}\left(\frac{m_a^2}{\sqrt{s}} - \frac{\mu^2}{\sqrt{s}-\sum_{j=1}^{l} q_j^+} - \frac{\mu^2}{\sum_{j=1}^{l} q_j^+}\right)^{-2}\right.$$
$$\left.\times\left(\sum_{j=2}^{l} q_j^+\right)^{-2}\left(\frac{m_a^2}{\sqrt{s}} - \frac{\mu^2}{\sqrt{s}-\sum_{j=1}^{l} q_j^+} - \frac{\mu^2}{q_1^+} - \frac{\mu^2}{\sum_{j=2}^{l} q_j^+}\right)^{-2}\cdots(q_l^+)^{-2}\left(\frac{m_a^2}{\sqrt{s}} - \frac{\mu^2}{\sqrt{s}-\sum_{j=1}^{l} q_j^+} - \frac{\mu^2}{q_1^+} - \cdots - \frac{\mu^2}{q_l^+}\right)^{-2}\right] , \tag{2.15a}$$

$$f_2^{(m-l)}(q_{l+1},\ldots,q_m) = \left[(\mu^4)^{m-l-1}\left(\sum_{j=l+2}^{m} q_j^+\right)^{-2}\left(\frac{\mu^2}{q_{l+1}^+} + \frac{\mu^2}{\sum_{j=l+2}^{m} q_j^+}\right)^{-2}\cdots(q_m^+)^{-2}\left(\frac{\mu^2}{q_{l+1}^+} + \cdots + \frac{\mu^2}{q_m^+}\right)^{-2}\right] , \tag{2.15b}$$

and

$$f_3^{(n-m)}(q_{m+1},\ldots,q_n)=\Bigg[(\mu^4)^{n-m}\frac{\sqrt{s}}{\sqrt{s}-\sum_{i=m+1}^{n}q_i^-}(q_{m+1}^-)^{-2}\left(\frac{m_b^2}{\sqrt{s}}-\frac{\mu^2}{\sqrt{s}-\sum_{j=m+1}^{n}q_j^-}-\frac{\mu^2}{q_n^-}-\frac{\mu^2}{q_{n-1}^-}-\cdots-\frac{\mu^2}{q_{m+1}^-}\right)^{-2}$$

$$\times\left(\sum_{j=m+1}^{m+2}q_j^-\right)^{-2}\left(\frac{m_b^2}{\sqrt{s}}-\frac{\mu^2}{\sqrt{s}-\sum_{j=m+1}^{n}q_j^-}-\frac{\mu^2}{q_n^-}-\cdots-\frac{\mu^2}{q_{m+2}^-}-\frac{\mu^2}{\sum_{j=m+1}^{m+2}q_j^-}\right)^{-2}\cdots\left(\sum_{j=m+1}^{n}q_j^-\right)$$

$$\times\left(\frac{m_b^2}{\sqrt{s}}-\frac{\mu^2}{\sqrt{s}-\sum_{j=m+1}^{n}q_j^-}-\frac{\mu^2}{\sum_{j=m+1}^{n}q_j^-}\right)^{-2}\Bigg]. \tag{2.15c}$$

To obtain these results we have used several simplifications which follow from the large differences in magnitude among the q_i. First, in the denominators d_1 to d_l in Ima_n we have ignored

$$p_b^+=m_b^2/\sqrt{s} \quad\text{and}\quad p_b'^+=\frac{\mu^2}{\sqrt{s}-\sum_{j=1}^{n}\mu^2/q_j^+}$$

relative to the (large) momenta $q_1^+,\ldots,q_l^+$. In the case of p_b^+, this approximation is clear; for $p_b'^+$, we know it is also valid because if this momentum becomes large – that is, $p_b'^+\gg\mu^2/\sqrt{s}$ – our previous arguments show that the amplitude is damped. Similarly, for the denominators d_{m+1} to d_n we have ignored

$$p_a^-=m_a^2/\sqrt{s} \quad\text{and}\quad p_a'^-=\frac{\mu^2}{\sqrt{s}-\sum_{j=1}^{n}q_j^+}.$$

Further, in the expressions d_{l+1} to d_m we have ignored all of p_a^-, $p_a'^-$, p_b^+, and $p_b'^+$ for the reasons indicated.

Finally, we remark that Im$a^{(n)}(s)$ reduces to the $f_i^{(n)}$ in the appropriate regions: That is,

$$\text{Im}a^{(n)}(q_1,\ldots,q_n)\longrightarrow f_1^{(n)}(q_1,\ldots,q_n)+O\left(\frac{\mu^2}{\sqrt{s}\,q_i^+}\right), \tag{2.16a}$$

when

$$\sqrt{s}>q_1^+,\ldots,q_n^+\gg\mu^2/\sqrt{s};$$

$$\text{Im}a^{(n)}(q_1,\ldots,q_n)\longrightarrow f_2^{(n)}(q_1,\ldots,q_n)+O\left(\frac{\mu^2}{\sqrt{s}\,q_i^+},\frac{q_i^+}{\sqrt{s}}\right), \tag{2.16b}$$

when

$$\sqrt{s}\gg q_1^+,\ldots,q_n^+\gg\mu^2/\sqrt{s};$$

and

$$\text{Im}a^{(n)}(q_1,\ldots,q_n)\longrightarrow f_3^{(n)}(q_1,\ldots,q_n)+O\left(\frac{\mu^2}{\sqrt{s}\,q_i^-}\right), \tag{2.16c}$$

when

$$\sqrt{s}>q_1^-,\ldots,q_n^-\gg\mu^2/\sqrt{s}.$$

Hence, the factorization property (2.1a) may also be written as

$$\text{Im}a^{(n)}(q_1,\ldots,q_n)\longrightarrow\{\text{Im}a^{(l)}(q_1,\ldots,q_l)\,|\,\sqrt{s}>q_1^+,\ldots,q_l^+\gg\mu^2/\sqrt{s}\}$$

$$\times\{\text{Im}a^{(m-l)}(q_{l+1},\ldots,q_m)\,|\,\sqrt{s}\gg q_{l+1}^+,\ldots,q_m^+\gg\mu^2/\sqrt{s}\}$$

$$\times\{\text{Im}a^{(n-m)}(q_{m+1},\ldots,q_n)\,|\,\sqrt{s}>q_{m+1}^-,\ldots,q_n^-\gg\mu^2/\sqrt{s}\}. \tag{2.14'}$$

Returning to the factorization property in the form (2.14), we observe that the three functions f_1, f_2, and f_3 are related to the functions f_L, b, and f_R introduced in I to describe left fragmentation, pionization, and right fragmentation events, respectively. Before we establish the exact form of those relations, however, we wish to clarify two points.

First, let us recall the definitions[4] we use for the three kinematic regions mentioned above. A particle is a left fragment – or, a fragment of particle a – if its momentum satisfies $(p_a^+ \geq)\ q_i^+ > \eta p_a^+$, where η is a fixed constant, much less than unity and s independent. Similarly, a particle is a right fragment – or, a fragment of particle b – if its momentum obeys $(p_b^- \geq)\ q_i^- > \eta' p_b^-$; in terms of plus components, a right fragment is defined by

$$\frac{\mu^2 p_a^+}{s} \leq q_i^+ < \frac{\mu^2 p_a^+}{\eta' s},$$

where again $\eta' \ll 1$ and is s-independent. Finally, pionization particles are defined by $\eta p_a^+ \geq q_i^+ \geq \mu^2 p_a^+/s\eta'$. In the center-of-mass system, we may replace p_b^- and p_a^+ by $\sqrt{s}$ everywhere in the above equations.

Second, notice the symmetry between f_1, expressed as a function of q^+, and f_3 as a function of q^-. This symmetry underlies the result, stated in I, that in this simple model left fragmentation in terms of q^+ is the same as right fragmentation in terms of q^-. Hence, if we wish to isolate the fragmentation region for separate analysis, we do not have to study both f_1 and f_3, as their properties – in terms of the appropriate variables – are identical. For purposes of comparison with the results of I, we shall analyze $f_1(q^+)$.

Introducing scaled momentum variables $x_i = q_i^+/\sqrt{s}$, we see from (2.15a) and (2.15b) that $f_1^{(l)}$ and $f_2^{(k)}$ become

$$f_1^{(l)}(x_1, \ldots, x_l) = \frac{1}{1-\sum_{i=1}^{l} x_i}\left(\sum_{i=1}^{l} x_i\right)^{-2}\left(-\frac{m_a^2}{\mu^2} + \frac{1}{1-\sum_{i=1}^{l} x_i} + \frac{1}{\sum_{i=1}^{l} x_i}\right)^{-2}\left(\sum_{i=2}^{l} x_i\right)^{-2}$$
$$\times\left(-\frac{m_a^2}{\mu^2} + \frac{1}{1-\sum_{i=1}^{l} x_i} + \frac{1}{x_1} + \frac{1}{\sum_{i=2}^{l} x_i}\right)^{-2} \cdots (x_l)^{-2}\left(-\frac{m_a^2}{\mu^2} + \frac{1}{1-\sum_{i=1}^{l} x_i} + \frac{1}{x_1} + \cdots + \frac{1}{x_l}\right)^{-2} \tag{2.17a}$$

and

$$f_2^{(k)}(x_1, \ldots, x_k) = \frac{1}{\left(\sum_{i=2}^{k} x_i\right)^2}\left(\frac{1}{x_1} + \frac{1}{\sum_{i=2}^{k} x_i}\right)^{-2}\left(\sum_{i=3}^{k} x_i\right)^{-2}\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{\sum_{i=3}^{k} x_i}\right)^{-2} \cdots (x_k)^{-2}\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k}\right)^{-2}. \tag{2.17b}$$

Referring to the analysis of I we recall that the functions $f_L^{(l)}(y_1, \ldots, y_l)$ and $b^{(m)}(y_1, \ldots, y_m)$ are defined by

$$f_L^{(l)}(y_1, \ldots, y_l) = [(1-y_1)y_1(y_1-y_2)\cdots(y_{l-1}-y_l)y_l^2]^{-1}$$
$$\times\left(-\frac{m_a^2}{\mu^2} + \frac{1}{1-y_1} + \frac{1}{y_1}\right)^{-2}\left(-\frac{m_a^2}{\mu^2} + \frac{1}{1-y_1} + \frac{1}{y_1-y_2} + \frac{1}{y_2}\right)^{-2} \cdots \left(-\frac{m_a^2}{\mu^2} + \frac{1}{1-y_1} + \frac{1}{y_1-y_2} + \cdots + \frac{1}{y_l}\right)^{-2} \tag{2.18a}$$

and

$$b^{(m)}(y_1, \ldots, y_m) = \frac{y_1/y_m}{(y_1-y_2)y_2(y_2-y_3)\cdots y_m}\left(\frac{1}{y_1-y_2} + \frac{1}{y_2}\right)^{-2}\left(\frac{1}{y_1-y_2} + \frac{1}{y_2-y_3} + \frac{1}{y_3}\right)^{-2} \cdots \left(\frac{1}{y_1-y_2} + \frac{1}{y_2-y_3} + \cdots + \frac{1}{y_m}\right)^{-2}. \tag{2.18b}$$

Hence, we see that

$$f_1^{(l)}(x_1, \ldots, x_l) = \frac{x_1 x_2 \cdots x_l}{\left(\sum_{i=1}^{l} x_i\right)\left(\sum_{i=2}^{l} x_i\right)\cdots x_l} f_L^{(l)}\left(\sum_{i=1}^{l} x_i, \sum_{i=2}^{l} x_i, \ldots, x_l\right) \tag{2.19a}$$

and

$$f_2^{(m)}(x_1,\ldots,x_m)=\frac{x_1x_2\cdots x_m}{\left(\sum_{i=1}^m x_i\right)\left(\sum_{i=2}^m x_i\right)\cdots x_m}\,b^{(m)}\left(\sum_{i=1}^m x_i,\ \sum_{i=2}^m x_i,\ldots,x_m\right). \tag{2.19b}$$

These relations make apparent, in view of the known factorization properties of $f_L^{(l)}$ and $b^{(m)}$, the factorization of $f_1^{(l)}$ and $f_2^{(m)}$, which for brevity we may indicate symbolically

$$f_1^{(l)}\longrightarrow f_1^{(k)}f_2^{(l-k)},\qquad x_1,\ldots,x_k\gg x_{k+1},\ldots,x_l \tag{2.20a}$$

and

$$f_2^{(m)}\longrightarrow f_2^{(k)}f_2^{(m-k)},\qquad x_1,\ldots,x_k\gg x_{k+1},\ldots,x_m. \tag{2.20b}$$

Of course these properties are in any event indicated by the expressions for f_1 and f_2 in (2.17). More significantly, Eqs. (2.19) express the relation between functions introduced to study k-variable clusters and those used to study q-variable clusters; in fact, this type of equation expressing the q-variable amplitude as a factorizable term times a k-variable amplitude holds quite generally for multiperipheral-type diagrams in a $(1+1)$-dimensional φ^3 theory. This means that if there exists k-variable factorization and therefore a k-variable cluster decomposition, then there must exist similar q-variable factorization and cluster-decomposition properties. We shall comment further on this point later.

We close this subsection with three observations. First, we note that the inelastic cross section is related to $\mathrm{Im}\,a^{(n)}(x_1,\ldots,x_n)$ by the equation

$$\sigma^{(n)}(s)=\frac{1}{2s^2}(2\pi g^2)\frac{\lambda^{n+1}}{n!}\int\prod_{i=1}^n\frac{dx_i}{x_i}\,\theta\left(1-\sum_i x_i\right)\theta\left(1-\sum_i\frac{\mu^2}{sx_i}\right)\mathrm{Im}\,a^{s(n)}(x_1,\ldots,x_n). \tag{2.21}$$

For compactness of notation, we have introduced the completely symmetric function of the x_i, $\mathrm{Im}\,a^{s(n)}(x_1,\ldots,x_n)$, defined by

$$\mathrm{Im}\,a^{s(n)}(x_1,\ldots,x_n)\equiv\sum_{\nu_n}\mathrm{Im}\,a^{(n)}(x_{\nu_1},\ldots,x_{\nu_n}), \tag{2.22}$$

where the sum over ν_n runs over the $n!$ orderings of $\{q_1,\ldots,q_n\}$. It is then straightforward to establish that the crucial factorization property derived for $\mathrm{Im}\,a^{(n)}(q_i)$ in (2.14) or (2.14′) generalizes when all permutations are summed over to the form

$$\mathrm{Im}\,a^{s(n)}(x_1,\ldots,x_n)\equiv\sum_{\nu_n}\mathrm{Im}\,a^{(n)}(x_{\nu_1},\ldots,x_{\nu_n})\to f_1^{s(l)}f_2^{s(n-l)}f_3^{s(n-m)}+O(\{x_l/x_m\},\{x_m/x_n\}) \tag{2.23}$$

for $(x_1,\ldots,x_l)\gg(x_{l+1},\ldots,x_m)\gg(x_{m+1},\ldots,x_n)$. Here each of the f_i^s is a symmetrical function of its arguments obtained from the corresponding f_i simply by summing over all possible permutations of the x_i; that is,

$$f_i^{s(l)}\equiv\sum_{\nu_l}f_i^{(l)}(x_{\nu_1},\ldots,x_{\nu_l}), \tag{2.24}$$

where the sum over ν_l runs over the $l!$ permutations of $x_1,\ldots x_l$. It is simple to verify that the f_i^s satisfy equations analogous to (2.19) and (2.20). We have

$$f_i^{s(l)}=\sum_{\nu_l}\left(\frac{x_{\nu_1}\cdots x_{\nu_l}}{\left(\sum_{i=1}^l x_{\nu_i}\right)\left(\sum_{i=2}^l x_{\nu_i}\right)\cdots x_{\nu_l}}\right)f_L^{(l)}\left(\sum_{i=1}^l x_{\nu_i},\ \sum_{i=2}^l x_{\nu_i},\ldots,x_{\nu_l}\right) \tag{2.25a}$$

and

$$f_2^{s(m)}=\sum_{\nu_m}\left(\frac{x_{\nu_1}\cdots x_{\nu_m}}{\left(\sum_{i=1}^m x_{\nu_i}\right)\left(\sum_{i=2}^m x_{\nu_i}\right)\cdots x_{\nu_m}}\right)b^{(m)}\left(\sum_{i=1}^m x_{\nu_i},\ \sum_{i=2}^m x_{\nu_i},\ldots,x_{\nu_m}\right). \tag{2.25b}$$

In addition

$$f_1^{s(l)}\to f_1^{s(k)}f_2^{s(l-k)}+O(\{x_k/x_l\}) \tag{2.26a}$$

and

$$f_2^{s(l)} \longrightarrow f_2^{s(k)} f_2^{s(l-k)} + O(\{x_k/x_l\}) , \tag{2.26b}$$

whenever the elements of any set of k of the x_i – say, $\{x_{\nu_1}, \ldots, x_{\nu_k}\}$ – are much larger than the remaining $(l-k)$ x_i.

Second, to introduce cluster functions which are of order 1 as $s \to \infty$ for the differential inelastic cross sections, we normalize the integrated inelastic cross section by the total elastic cross section,

$$\frac{\sigma^{(n)}(s)}{\sigma^{(0)}(s)} \equiv \frac{\sigma^{(n)}(s)}{\sigma_E(s)} . \tag{2.27}$$

From (2.23) and (2.24) it follows that

$$\frac{\sigma^{(n)}(s)}{\sigma_E(s)} = \frac{\lambda^n}{n!} \int \prod_{i=1}^{n} \frac{dx_i}{x_i} \operatorname{Im} a^{s(n)}(x_1, \ldots, x_n) \theta\left(1 - \sum_i x_i\right) \theta\left(1 - \sum_i \frac{\mu^2}{s x_i}\right) , \tag{2.28}$$

since

$$\sigma_E(s) = \sigma^{(0)}(s) = \frac{1}{2s} A^{(0)}(s) = \frac{1}{2s^2} (2\pi g^2)\lambda . \tag{2.29}$$

Finally, we remark that Eq. (2.27) indicates that the normalized differential exclusive cross sections are simply related to $\operatorname{Im} a^{s(n)}(x_1, \ldots, x_n)$. Hence, the factorization property given in (2.14) or (2.14′) motivates the introduction of a cluster decomposition for these cross sections. In the first part of Sec. III, we shall investigate the general properties of such a decomposition. Later, however, we shall wish to distinguish particle distribution properties in the fragmentation region (described by f_1^s) from those in the pionization region (described by f_2^s). Since the factorization properties of f_1^s and f_2^s separately are somewhat simpler than those of $\operatorname{Im} a^s$, for the purpose of explicit calculations, we shall introduce separate cluster decompositions for f_1^s and f_2^s.

III. THE CLUSTER DECOMPOSITION IN TERMS OF THE q VARIABLES

A. General Considerations

The motivations for introducing a cluster decomposition for functions satisfying factorization properties like (2.21) or (2.25) are detailed in I; for brevity we shall not repeat them here. However, since a subtlety will arise in the q-variable cluster expansion that was not present in the k clusters, we shall review briefly the general method of introducing and applying a cluster decomposition to clarify the nature of this new effect. For this purpose we begin with Eq. (2.14′) which expresses the factorization property of the normalized differential inelastic cross sections in the form

$$\operatorname{Im} a^{s(n)}(q_1, \ldots, q_n) \longrightarrow \operatorname{Im} a^{s(m)}(q_1, \ldots q_m) \operatorname{Im} a^{s(n-m)}(q_{m+1}, \ldots, q_n) + O(\{q_m/q_n\}) \tag{2.14′}$$

when

$$q_1^+, \ldots, q_m^+ \gg q_{m+1}^+, \ldots, q_n^+ .$$

If one defines cluster functions according to

$$\begin{aligned} g^{(1)}(q_1) &\equiv \operatorname{Im} a^{s(1)}(q_1) , \\ g^{(2)}(q_1, q_2) &\equiv \operatorname{Im} a^{s(2)}(q_1, q_2) - g^{(1)}(q_1) g^{(1)}(q_2) , \\ g^{(3)}(q_1, q_2, q_3) &\equiv \operatorname{Im} a^{s(3)}(q_1, q_2, q_3) - g^{(1)}(q_1) g^{(2)}(q_2, q_3) \\ &\quad - g^{(1)}(q_2) g^{(2)}(q_1, q_3) - g^{(1)}(q_3) g^{(2)}(q_1, q_2) - g^{(1)}(q_1) g^{(1)}(q_2) g^{(1)}(q_3) , \end{aligned} \tag{3.1}$$

and similarly for the higher $g^{(n)}$, then several results follow immediately. First, the definitions of the $g^{(n)}$ imply that

$$\operatorname{Im} a^{s(n)}(q_1, \ldots, q_n) = \sum \left(\prod_{n_i} g^{(n_i)} \right) , \tag{3.2}$$

where the sum runs over all sets of integers n_i such that $\sum n_i = n$. An equivalent form of (3.2) is

$$\operatorname{Im} a^{s(n)}(q_1, \ldots, q_n) = g^{(1)}(q_1) \operatorname{Im} a^{s(n-1)}(q_2, \ldots, q_n) + \sum_{i=2}^{n} g^{(2)}(q_1, q_i) \operatorname{Im} a^{s(n-2)}(\cdots) + \cdots + g^{(n)}(q_1, \ldots, q_n) . \tag{3.3}$$

The complete symmetry of the $\mathrm{Im}a^{s(n)}(q_1, \ldots, q_n)$ in their arguments implies that the $g^{(n)}$ are also totally symmetric functions. Further, from (2.14′) one can see that when $q_i^+ \gg q_j^+$,

$$g^{(n)}(q_1, \ldots, q_n) \longrightarrow O(q_i^+/q_j^+) \longrightarrow 0 . \tag{3.4}$$

In addition, from (3.4) one can establish that

$$G^{(n)}(s) \equiv \int_{\sum(\mu^2/q_i^+)=\sqrt{s}}^{\sum q_i^+=\sqrt{s}} \frac{dq_1}{q_1} \cdots \frac{dq_n}{q_n} g^{(n)}(q_1, \ldots q_n) = \alpha_n \ln s + \beta_n + O(1/s) , \tag{3.5}$$

where hereafter dq/q is a shorthand version of dq^+/q^+. These results all follow directly from (2.14′) and (3.1) and hence certainly apply in our present case. From these equations one proceeds to the more interesting relations between the $g^{(n)}$ and, for example, the total cross section or the multiparticle inclusive spectra. It is in this step that the subtlety appears; let us concentrate on the expression for the total cross section in terms of the $g^{(n)}$ in order to illustrate the problem. Recall that

$$\frac{\sigma^{(n)}(s)}{\sigma_E(s)} = \frac{\lambda^n}{n!} \int \frac{dq_1}{q_1} \cdots \frac{dq_n}{q_n} \mathrm{Im}a^{s(n)}(q_1, \ldots, q_n)$$

and

$$\frac{\sigma_T(s)}{\sigma_E(s)} \equiv \frac{\sigma_T(s)}{\sigma^{(0)}(s)} = \sum_{n=1}^{\infty} \frac{\sigma^{(n)}(s)}{\sigma^{(0)}(s)} . \tag{3.6}$$

Hence since the $\mathrm{Im}a^{s(n)}(q_1, \ldots, q_n)$ admit the introduction of cluster functions $g^{(n)}(q_1, \ldots, q_n)$, reference to the arguments of I might appear to suggest that the Mayer cluster-decomposition result holds[5]:

$$\frac{\sigma_T(s)}{\sigma_E(s)} \equiv \sum \frac{\sigma^{(n)}}{\sigma_E} = \exp\left(\sum \frac{\lambda^n}{n!} \int \frac{dq_1}{q_1} \cdots \frac{dq_n}{q_n} g^{(n)}(q_1, \ldots, q_n) \right) . \tag{3.7}$$

However, a closer analysis of the derivation given in I of the k-variable analog of this cluster-decomposition theorem shows that in the present case (3.7) is not valid in terms of the $g^{(n)}$ defined above for the simple reason that the integration region in terms of the q_i^+ is itself not factorizable. Later, in our discussion of the one-particle inclusive spectrum, we shall illustrate explicitly the crucial nature of this lack of factorization; for the moment, however, we shall seek a qualitative understanding of the effect to enable us to redefine the cluster functions in order to preserve some variant of the cluster-decomposition theorem.

It is clear that the origin of the lack of factorization of the integration region lies in the kinematic coupling of the q_i required by momentum conservation. We can also readily establish that this coupling implies that the contributions to the total cross section of the clusters as defined above are not independent. To demonstrate this, we consider the contribution of $g^{(2)}$ to the total cross section,

$$G^{(2)}(s) \equiv \int_{\mu^2/q_1^+ + \mu^2/q_2^+ = \sqrt{s}}^{q_1^+ + q_2^+ = \sqrt{s}} g^{(2)}(q_1, q_2) \frac{dq_1}{q_1} \frac{dq_2}{q_2}$$

$$= \int_{\mu^2/q_1^+ + \mu^2/q_1^+ = \sqrt{s}}^{q_1^+ + q_2^+ = \sqrt{s}} [\mathrm{Im}a^{s(2)}(q_1, q_2) - g^{(1)}(q_1) g^{(1)}(q_2)] \frac{dq_1}{q_1} \frac{dq_2}{q_2} . \tag{3.8}$$

The term $g^{(1)}(q_1)\, g^{(1)}(q_2)$ integrated over the phase space is supposed to represent the independent contributions of the two one-particle cluster functions $g^{(1)}$. However, since the phase space for a single $g^{(1)}(q)$ is $\mu^2/\sqrt{s} \le q^+ \le \sqrt{s}$, it is clear that when the product $g^{(1)}(q_1) g^{(1)}(q_2)$ is integrated over the phase space

$$\sqrt{s} \ge \frac{\mu^2}{q_1^+} + \frac{\mu^2}{q_2^+}, \quad q_1^+ + q_2^+ \le \sqrt{s} ,$$

then neither cluster function is separately integrated over the full phase space appropriate to its definition; in short, the phase space available to $g^{(1)}(q_2)$ depends on the value of q_1, and hence, the contributions of the two one-particle cluster functions are not mutually independent.

To restore the independence of the $g^{(n)}$ and simultaneously to introduce a factorizable integration region, we can adopt the following convention: We integrate each of the variables q_i^+ over the region $\mu^2/\sqrt{s} < q_i^+ < \sqrt{s}$ and associate with each integrand function $\mathrm{Im}a^{s(n)}(q_1, \ldots, q_n)$ the appropriate θ functions necessary to specify the kinematic limits of the phase space. Thus we define

$$\mathrm{Im}\bar{a}^{s(n)}(q_1,\ldots,q_n) \equiv \mathrm{Im}a^{s(n)}(q_1,\ldots,q_n)\,\theta\Big(\sqrt{s}-\sum_{i=1}^{n}q_i^+\Big)\theta\Big(\sqrt{s}-\sum_{i=1}^{n}q_i^-\Big). \tag{3.9}$$

Clearly we have

$$\frac{\sigma^{(n)}(s)}{\sigma^{(0)}(s)}=\frac{\lambda^n}{n!}\Big(\prod_{i=1}^{n}\int_{\mu^2/\sqrt{s}}^{\sqrt{s}}\frac{dq_i^+}{q_i^+}\Big)\mathrm{Im}\bar{a}^{s(n)}(q_1,\ldots,q_n) \tag{3.10}$$

and the factorization of the integration region is explicit. It is equally clear, however, that we have not solved the difficulties of a lack of factorization, for now the functions $\mathrm{Im}\bar{a}^{s(n)}$ themselves do not factorize exactly because of the θ functions they contain. Instead of (2.14′) we find for $(q_1^+,\ldots,q_m^+)\gg(q_{m+1}^+,\ldots,q_n^+)$

$$\begin{aligned}\mathrm{Im}\bar{a}^{s(n)}(q_1,\ldots,q_n)\longrightarrow{}&\mathrm{Im}\bar{a}^{s(m)}(q_1,\ldots,q_m)\mathrm{Im}\bar{a}^{s(n-m)}(q_{m+1},\ldots,q_n)+O(\{q_m/q_n\})\\&+\mathrm{Im}a^{s(m)}(q_1,\ldots,q_m)\mathrm{Im}a^{s(n-m)}(q_{m+1},\ldots,q_n)\\&\times\Big[\theta\Big(\sqrt{s}-\sum_{i=1}^{n}q_i^+\Big)\theta\Big(\sqrt{s}-\sum_{i=1}^{n}q_i^-\Big)-\theta\Big(\sqrt{s}-\sum_{i=1}^{m}q_i^+\Big)\theta\Big(\sqrt{s}-\sum_{i=1}^{m}q_i^-\Big)\theta\Big(\sqrt{s}-\sum_{m+1}^{n}q_i^+\Big)\theta\Big(\sqrt{s}-\sum_{m+1}^{n}q_i^-\Big)\Big].\end{aligned} \tag{3.11}$$

In the phase-space region appropriate to $\mathrm{Im}a^{s(n)}(q_1,\ldots,q_n)$, since $\mathrm{Im}\bar{a}^{s(n)}=\mathrm{Im}a^{s(n)}$, the final term in (3.11) vanishes, and this equation becomes equivalent to (2.14′). However, when we define $\bar{g}^{(n)}$ in terms of the $\mathrm{Im}\bar{a}^{s(n)}$, the lack of exact factorization in (3.11) will create additional problems which, although surmountable, will require some care to resolve.

We define $\bar{g}^{(n)}(q_1,\ldots,q_n)$ according to

$$\begin{aligned}&\bar{g}^{(1)}(q_1)=\mathrm{Im}\bar{a}^{(1)}(q_1),\\&\bar{g}^{(2)}(q_1,q_2)=\mathrm{Im}\bar{a}^{(2)}(q_1,q_2)-\bar{g}^{(1)}(q_1)\bar{g}^{(1)}(q_2),\end{aligned} \tag{3.12}$$

and similarly for the higher $\bar{g}^{(n)}$: In the phase-space region $\sum_{i=1}^{n}q_i^{\pm}<\sqrt{s}$,

$$\bar{g}^{(n)}(q_1,\ldots,q_n)=g^{(n)}(q_1,\ldots,q_n).$$

However, we see that

$$\bar{g}^{(n)}(q_1,\ldots,q_n)\neq g^{(n)}(q_1,\ldots,q_n)\times\theta\Big(\sqrt{s}-\sum_{i=1}^{n}q_i^+\Big)\theta\Big(\sqrt{s}-\sum_{i=1}^{n}q_i^-\Big)$$

and further that the integrated cluster functions, defined by

$$G^{(n)}(s)\equiv\int_{\Sigma q_i^-=\sqrt{s}}^{\Sigma q_i^+=\sqrt{s}}\frac{dq_1^+}{q_1^+}\cdots\frac{dq_n^+}{q_n^+}g^{(n)}(q_1,\ldots,q_n) \tag{3.13a}$$

and

$$\bar{G}^{(n)}(s)=\int_{\mu^2/\sqrt{s}}^{\sqrt{s}}\cdots\int_{\mu^2/\sqrt{s}}^{\sqrt{s}}\frac{dq_1^+}{q_1^+}\cdots\frac{dq_n^+}{q_n^+}\bar{g}^{(n)}(q_1,\ldots,q_n), \tag{3.13b}$$

are also not equal.

From the definitions (3.12) we have immediately that

$$\mathrm{Im}\bar{a}^{s(n)}(q_1,\ldots,q_n)=\sum\Big(\prod_{n_i}\bar{g}^{(n_i)}\Big), \tag{3.14}$$

where $\sum n_i=n$ and

$$\mathrm{Im}\bar{a}^{s(n)}(q_1,\ldots,q_n)=\bar{g}^{(1)}(q_1)\,\mathrm{Im}\bar{a}^{s(n-1)}(q_2,\ldots,q_n)+\sum_{i=2}^{n}\bar{g}^{(2)}(q_1,q_i)\mathrm{Im}\bar{a}^{s(n-2)}(\cdots)+\cdots+\bar{g}^{(n)}(q_1,\ldots q_n). \tag{3.15}$$

Further, since we may now express $\sigma_T(s)$ as

$$\frac{\sigma_T(s)}{\sigma_E(s)}=\sum_n\frac{\lambda^n}{n!}\prod_{i=1}^{n}\int_{\mu^2/\sqrt{s}}^{\sqrt{s}}\frac{dq_i^+}{q_i^+}\,\mathrm{Im}\bar{a}^{s(n)}(q_1,\ldots,q_n), \tag{3.16}$$

where the integration region is manifestly factorizable, the method of I can be applied to derive the Mayer cluster-decomposition theorem from (3.15).

We obtain

$$\frac{\sigma_T(s)}{\sigma_E(s)} = \exp\left(\sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \overline{G}^{(n)}(s) \right)$$

$$= \exp\left(\sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \prod_{i=1}^{n} \int_{\mu^2/\sqrt{s}}^{\sqrt{s}} \frac{dq_i^+}{q_i^+} \overline{g}^{(n)}(q_1, \ldots, q_n) \right). \tag{3.17}$$

Hence the $\overline{g}^{(n)}(q_1, \ldots, q_n)$ seem to represent a proper set of cluster functions in that their contributions to the total cross section – and, as we shall later verify, to the multiparticle spectra – are mutually independent. However, in redefining the cluster functions, we have not completely escaped the problems engendered by the kinematic coupling of the q_i; since (3.11) does not represent exact factorization for $\mathrm{Im}\overline{a}^{s(n)}$, the result that

$$\overline{g}^{(n)}(q_1, \ldots, q_n) \longrightarrow O(\{q_m/q_n\}) \tag{3.18}$$

when $(q_1, \ldots, q_m) \gg (q_{m+1}, \ldots, q_n)$ does not follow except in the kinematic region $\sqrt{s} > \sum q_i^+$ and $\sqrt{s} > \sum q_i^-$, in which $\overline{g}^{(n)} = g^{(n)}$. But to obtain the integrated cluster function, $\overline{G}^{(n)}(s)$, from $\overline{g}^{(n)}(q_1, \ldots, q_n)$ we integrate each q_i over the region $\mu^2/\sqrt{s} < q_i^+ < \sqrt{s}$ and hence for at least part of this region the validity of (3.18) is not clear. This means that the important result that

$$\overline{G}^{(n)}(s) = \overline{\alpha}_n \ln s + \overline{\beta}_n, \tag{3.19}$$

which together with (3.17) would establish the Regge behavior of the amplitude, does not follow immediately. Further, it casts some doubt on the validity of our interpretation of clusters as representing the effects of short-range correlations among the particles in rapidity space.

To clarify this difficulty, we can consider the simple instance of $\overline{g}^{(2)}(q_1, q_2)$. In terms of functions with known factorization properties we have

$$\begin{aligned}\overline{g}^{(2)}(q_1, q_2) = \mathrm{Im}a^{(2)}\, &\theta(\sqrt{s} - q_1^+ - q_2^+)\theta(\sqrt{s} - q_1^- - q_2^-) \\ &- g^{(1)}(q_1)\theta(\sqrt{s} - q_1^+)\theta(\sqrt{s} - q_1^-) \\ &\times g^{(1)}(q_2)\theta(\sqrt{s} - q_2^+)\theta(\sqrt{s} - q_2^-).\end{aligned} \tag{3.20}$$

As noted previously, provided $q_1^+ + q_2^+ < \sqrt{s}$ and $q_1^- + q_2^- < \sqrt{s}$, $\overline{g}^{(2)}(q_1, q_2) \to O(q_2/q_1)$ when $q_1 \gg q_2$. However, when $q_2^+ = 2\epsilon$ and $q_1^+ = \sqrt{s} - \epsilon$, for example – this represents a point contained in the full integration region – we have $q_1^+ \gg q_2^+$ but

$$\overline{g}^{(2)}(q_1, q_2) \to -g^{(1)}(q_1) g^{(1)}(q_2); \tag{3.21}$$

clearly, the behavior of this expression, and, in particular, whether or not it approaches zero, depends on the nature of $g^{(1)}(q_1)$ and hence on the dynamics. In the ladder model under consideration, we shall see that

$$g^{(1)}(q_1) \longrightarrow O(\sqrt{s} - q_1^+) \tag{3.22}$$

as $q_1^+ \to \sqrt{s}$ and hence that

$$\overline{g}^{(2)}(q_1, q_2) \longrightarrow 0 \tag{3.23}$$

whenever $q_1^+ \gg q_2^+$. More importantly, we can establish in a general way that even if the detailed dynamics does not imply an equation of the form (3.18), the integrated clusters do satisfy (3.19). To separate this general result clearly from the model-dependent result (3.22) we shall reserve the general proof for the Appendix. However, a simple physical argument can make plausible that (3.19) should remain valid. The kinematic region discussed above in which $\overline{g}^{(2)} \not\to 0$ is a very limited one in that at least one of the variables must be near the kinematic boundary $q_i^+ = \sqrt{s}$. Since these regions represent essentially the "surface" of the phase space, it is plausible that the $\overline{G}^{(n)}(s)$, being functions integrated over the full volume, will not be affected in leading order by these "surface effects." A proof based on this intuition is presented in the Appendix. For the moment we shall proceed with our explication of the cluster decomposition in terms of the $\overline{g}^{(n)}(x_1, \ldots, x_n)$.

B. Multiparticle Inclusive Spectra

Perhaps the most significant application of the concept of q-variable clusters lies in their potential for providing a systematic method of analysis of multiparticle spectra and correlation effects. In this regard it is almost essential that one use q-variable clusters, rather than those expressed in k variables, for, as discussed in I, the ordering ambiguities and the kinematic complications in the fragmentation regions hamper a meaningful k-variable analysis.

To clarify the manner in which the cluster functions may be used to determine particle spectra we wish to consider the general case of the contribution of an nth-order exclusive cluster to the correlated part of the m-particle inclusive spectrum ($n \geq m$). We shall call this contribution $(d^m\sigma_c)_n$. Let us begin, however, by studying in detail the one-particle spectrum.

In Sec. II we established that the cross section for the production of N secondary particles has the form

$$\frac{\sigma^{(N)}}{\sigma^{(0)}} = \frac{\sigma^{(N)}}{\sigma_E} = \frac{\lambda^N}{N!} \int_{\mu^2/\sqrt{s}}^{\sqrt{s}} \frac{dq_1}{q_1} \cdots \frac{dq_N}{q_N} \mathrm{Im}\overline{a}^{s(N)}(q_1, \ldots, q_N), \tag{3.24}$$

where $\mathrm{Im}\overline{a}^{s(n)}(q_1, \ldots, q_n)$ has the cluster decomposition indicated symbolically by

$$\mathrm{Im}\bar{a}^{s(N)}(q_1,\ldots,q_N)=\sum\Big(\prod_{n_i}\bar{g}^{(n_i)}(q_i)\Big). \tag{3.25}$$

Here the sum runs over all sets of integers n_i such that $\sum n_i=N$. Since the observed particle must originate in a definite cluster, we must start by calculating the differential cross section for producing a single, specific nth-order cluster with momenta $q_1,\ldots,q_n$ plus any additional clusters. From (3.25) we see that this cross section is

$$\begin{aligned}\frac{(d\sigma^N)_n}{\sigma_E}&=\frac{\lambda^N}{N!}n!C_n^N\bar{g}^{(n)}(q_1,\ldots,q_n)\frac{dq_1}{q_1}\cdots\frac{dq_n}{q_n}\int_{\mu^2/\sqrt{s}}^{\sqrt{s}}\mathrm{Im}\bar{a}^{s(N-n)}(q_{n+1},\ldots,q_N)\frac{dq_{n+1}}{q_{n+1}}\cdots\frac{dq_N}{q_N}\\&=\lambda^n\bar{g}^{(n)}(q_1,\ldots,q_n)\frac{dq_1}{q_1}\cdots\frac{dq_n}{q_n}\frac{\lambda^{N-n}}{(N-n)!}\int_{\mu^2/\sqrt{s}}^{\sqrt{s}}\mathrm{Im}\bar{a}^{s(N-n)}(q_{n+1},\ldots,q_N)\frac{dq_{n+1}}{q_{n+1}}\cdots\frac{dq_N}{q_N},\end{aligned} \tag{3.26}$$

where clearly $N>n$, and all q's stand for q^+'s unless otherwise specified. The combinatorial factors can be understood by the following:

(1) From the original N particles one has selected n and this can be done in $C_n^N=N/(N-n)!n!$ ways.

(2) The labeling of the particle momenta in $\bar{g}^{(n)}$ can be done in $n!$ ways. Alternatively, one may follow the convention of associating a factor $1/r!$ with any integration[11] over a phase space of r particles. To obtain the contribution of the nth-particles cluster to the full one-particle spectrum, we first sum over all N in (3.3) to obtain

$$\frac{(d\sigma)_n}{\sigma_E}=\lambda^n\bar{g}^{(n)}(q_1,\ldots,q_n)\frac{dq_1}{q_1}\cdots\frac{dq_n}{q_n}\frac{\sigma_T}{\sigma_E}. \tag{3.27}$$

Here we have used the result that

$$\frac{\sigma_T}{\sigma_E}=\sum\frac{\lambda^m}{m!}\int_{\mu^2/\sqrt{s}}^{\sqrt{s}}\mathrm{Im}\bar{a}^{s(m)}(q_1,\ldots,q_m)\frac{dq_1}{q_1}\cdots\frac{dq_m}{q_m}. \tag{3.28}$$

Notice that in deriving Eqs. (3.26)–(3.28) the factorizability of the integration region plays an essential role. One simply cannot consider the contributions of the clusters as independent if they are coupled kinematically via the integration region.

Recalling that we should associate a factor of $1/(n-1)!$ with the phase-space integrations over q_2 to q_n, we find that the contribution of the n-particle cluster to the one-particle spectrum is

$$\frac{(d\sigma)_n}{\sigma_E}=\frac{dq_1}{q_1}\frac{\sigma_T}{\sigma_E}\frac{\lambda^n}{(n-1)!}\int_{\mu^2/\sqrt{s}}^{\sqrt{s}}\cdots\int_{\mu^2/\sqrt{s}}^{\sqrt{s}}\frac{dq_2}{q_2}\cdots\frac{dq_n}{q_n}\bar{g}^{(n)}(q_1,\ldots q_n). \tag{3.29}$$

The full normalized particle spectrum follows from (3.29) by summing over the contributions of all the clusters; the result is

$$\left(\frac{d\sigma}{\sigma_T}\right)_{\text{one particle}}=\frac{dq_1}{q_1}\sum\frac{\lambda^n}{(n-1)!}\int_{\mu^2/\sqrt{s}}^{\sqrt{s}}\cdots\int_{\mu^2/\sqrt{s}}^{\sqrt{s}}\frac{dq_2}{q_2}\cdots\frac{dq_n}{q_n}\bar{g}^{(n)}(q_1,\ldots,q_n). \tag{3.30}$$

The significant properties of this equation are the independence of the contributions of different clusters and the related result that the nth-order cluster contributes only to the term of order λ^n in (3.6).

To emphasize further the importance of introducing independent clusters, let us discuss briefly the form of the one-particle spectrum in terms of the clusters $g^{(n)}$; our analysis will parallel, insofar as is possible, that given previously for the $\bar{g}^{(n)}$. From (3.2) we see that the differential cross section for the production of N particles, including a cluster $g^{(n)}$ and anything else is

$$\begin{aligned}\frac{(d\sigma^N)_n}{\sigma_E}=&\frac{\lambda^N}{N!}n!C_n^Ng^{(n)}(q_1,\ldots,q_n)\frac{dq_1}{q_1}\cdots\frac{dq_n}{q_n}\\&\times\int\frac{dq_{n+1}}{q_{n+1}}\cdots\frac{dq_N}{q_N}\mathrm{Im}a^{s(N-n)}(q_{n+1},\ldots,q_N)\theta\Big(\sqrt{s}-\sum_1^n q_i^+-\sum_{n+1}^N q_i^+\Big)\theta\Big(\sqrt{s}-\sum_1^n q_i^- -\sum_{n+1}^N q_i^-\Big).\end{aligned} \tag{3.31}$$

Simplifying the combinational factors and summing over all N gives

$$\frac{(d\sigma)_n}{\sigma_E(s)}=\lambda^n g^{(n)}(q_1,\ldots,q_n)\frac{dq_1}{q_1}\cdots\frac{dq_n}{q_n}\frac{\bar{\sigma}_T\Big(\sqrt{s}-\sum_{i=1}^n q_i^+,\sqrt{s}-\sum_1^n q_i^-\Big)}{\sigma_E(s)}, \tag{3.32}$$

where

$$\frac{\tilde{\sigma}_T(\sqrt{s}-x,\sqrt{s}-y)}{\sigma_E(s)} \equiv \sum \frac{\lambda^m}{m!}\int \frac{dq_1}{q_1}\cdots\frac{dq_m}{q_m}\,\mathrm{Im}a^{s(m)}(q_1,\ldots q_m)\theta\left(\sqrt{s}-x-\sum_1^m q_i^+\right)\theta\left(\sqrt{s}-y-\sum_1^m q_i^-\right) \tag{3.33}$$

and represents a type of inclusive cross section in which the incoming particles have total $p^\pm = \sqrt{s}$ and the unobserved particles have total $p^\pm = \sqrt{s} - \sum_{i=1}^n q_i^\pm$. Here $\sum_{i=1}^n q_i^+$ represents the total plus component of the observed cluster. If we now ask to observe only one particle in the particular cluster and sum over all clusters we obtain finally

$$\frac{d\sigma}{\sigma_T(s)} = \frac{dq_1}{q_1}\sum\int\frac{dq_2}{q_2}\cdots\frac{dq_n}{q_n}\,\theta(\sqrt{s}-\sum q_i^+)\theta(\sqrt{s}-\sum q_i^-)\frac{\lambda^n}{(n-1)!}g^{(n)}(q_1,q_2,\ldots,q_n)\frac{\tilde{\sigma}_T\left(\sqrt{s}-\sum_1^n q_i^+,\sqrt{s}-\sum_1^n q_i^-\right)}{\sigma_T(s)}\,. \tag{3.34}$$

From a theoretical viewpoint the interdependence of the contributions of different clusters renders (3.34) considerably less elegant and useful than (3.30). Further, the existence of $\tilde{\sigma}_T$ linked to each cluster makes this equation unattractive for phenomenological applications.

We return to the analysis of multiparticle spectra by considering the two-particle spectrum. In this case we note that there are two mutually exclusive possibilities for the manner in which the production of the two particles occurs: Either both particles originate in a single cluster, or they come from two separate clusters. The two-particle spectrum will be the sum over all clusters of the contributions of these two types of possibilities.

Consider first the possibility that the two particles came from the same cluster, which we take to be of nth order. Then we have, directly from (3.27),

$$\frac{1}{\sigma_T}(d^2\sigma_1)_n \equiv \frac{dq_1}{q_1}\frac{dq_2}{q_2}\lambda^n\frac{1}{(n-2)!}\int\cdots\int\frac{dq_3}{q_3}\cdots\frac{dq_n}{q_n}\bar{g}^{(n)}(q_1,q_2,q_3,\ldots,q_n)\,. \tag{3.35}$$

The subscript here simply reminds us that this is the first of two contributions to the two-particle spectrum. To discuss the case in which the particles come from different clusters, we begin with the generalization of (3.26) which applies when two clusters, say $\bar{g}^{(k)}$ and $\bar{g}^{(l)}$, are observed:

$$\begin{aligned}(d\sigma_2^N)_{k,l} &\equiv \frac{\lambda^N}{N!}k!\,C_k^N\bar{g}^{(k)}(q_1,\ldots,q_k)\frac{dq_1}{q_1}\cdots\frac{dq_k}{q_k}\,l!\,C_l^{N-k}\bar{g}^{(l)}(q_{k+1},\ldots,q_{k+l})\\ &\quad\times\frac{dq_{k+1}}{q_{k+1}}\cdots\frac{dq_{k+l}}{q_{k+l}}\int\cdots\int\mathrm{Im}\bar{a}^{s(N-k-l)}(q_{l+k+1},\ldots,q_N)\frac{dq_{k+l+1}}{q_{k+l+1}}\cdots\frac{dq_N}{q_N}\\ &=\lambda^k\bar{g}^{(k)}(q_1,\ldots,q_k)\frac{dq_1}{q_1}\cdots\frac{dq_k}{q_k}\lambda^l\bar{g}^{(l)}(q_{k+1},\ldots,q_{k+l})\frac{dq_{k+1}}{q_{k+1}}\cdots\frac{dq_{k+l}}{q_{k+l}}\\ &\quad\times\frac{\lambda^{N-l-k}}{(N-l-k)!}\int\cdots\int\mathrm{Im}\bar{a}^{s(N-k-l)}(q_{l+k+1},\ldots,q_N)\frac{dq_{l+k+1}}{q_{l+k+1}}\cdots\frac{dq_N}{q_N}\,.\end{aligned} \tag{3.36}$$

Summing over N gives

$$(d^2\sigma_2)_{k,l} = \left(\lambda^k\bar{g}^{(k)}(q_1,\ldots,q_k)\frac{dq_1}{q_1}\cdots\frac{dq_k}{q_k}\right)\left(\lambda^l\bar{g}^{(l)}(q_{k+1},\ldots,q_{k+l})\frac{dq_{k+1}}{q_{k+1}}\cdots\frac{dq_{k+l}}{q_{k+l}}\right)\sigma_T\,. \tag{3.37}$$

Using (3.4) we see that this can be written

$$\frac{(d^2\sigma_2)_{k,l}}{\sigma_T} = \frac{(d\sigma)_k}{\sigma_T}\frac{(d\sigma)_l}{\sigma_T}\,, \tag{3.38}$$

where $(d\sigma)_k/\sigma_T$ is the normalized contribution of the kth-order cluster to the one-particle spectrum. The full two-particle spectrum is obtained by adding (3.35), summed over n, to (3.38), summed over k and l. Defining

$$\frac{1}{\sigma_T}d^2\sigma_1 \equiv \sum_n\frac{1}{\sigma_T}(d^2\sigma_1)_n \tag{3.39}$$

and

$$\frac{1}{\sigma_T}\,d^2\sigma_2 \equiv \sum_{k,\,l}\frac{1}{\sigma_T}(d^2\sigma_2)_{k,\,l}\,. \tag{3.40}$$

We see that the full two-particle spectrum is given by

$$\frac{1}{\sigma_T}\,d^2\sigma = \frac{1}{\sigma_T}(d^2\sigma_1 + d^2\sigma_2)\,. \tag{3.41}$$

But since, by (3.38),

$$\frac{d^2\sigma_2}{\sigma_T} = \left(\frac{d\sigma}{\sigma_T}\right)\left(\frac{d\sigma}{\sigma_T}\right), \tag{3.42}$$

we have that

$$\frac{d^2\sigma_1}{\sigma_T} = \frac{d^2\sigma}{\sigma_T} - \left(\frac{d\sigma}{\sigma_T}\right)\left(\frac{d\sigma}{\sigma_T}\right). \tag{3.43}$$

But by definition, the right-hand side of (3.43) is equal to the correlated part of the two-particle spectrum and accordingly

$$\frac{d^2\sigma_1}{\sigma_T} = \frac{d^2\sigma_c}{\sigma_T}. \tag{3.44}$$

From the preceding we observe the important result that in the two-particle spectrum, particles can be correlated only if they come from the same cluster. It is intuitively clear that, if we wish to interpret the clusters as independent of each other and as describing correlations among the particles they contain, it is essential for consistency that whenever particles are correlated they must originate in the same cluster.

It is relatively straightforward to extend this analysis to the general case of m-particle spectra. The same arguments as given above suffice to establish that to calculate the correlated part of the m-particle spectrum we simply consider the contribution of each cluster $\bar{g}^{(n)}(q_1, \ldots, q_n)$ $(n \geq m)$ independently and then sum over all allowed n. From (3.27) we see that in the general case the contribution of the nth-order exclusive cluster to the correlated part of the m-particle inclusive spectrum gives

$$\frac{1}{\sigma_T}(d^m\sigma_c)_n = \frac{\lambda^n}{(n-m)!}\,\frac{dq_1}{q_1}\cdots\frac{dq_m}{q_m}\int_{\mu^2/\sqrt{s}}^{\sqrt{s}}\cdots\int_{\mu^2/\sqrt{s}}^{\sqrt{s}}\frac{dq_{m+1}}{q_{m+1}}\cdots\frac{dq_n}{q_n}\,\bar{g}^{(n)}(q_1, \ldots, q_n)\,. \tag{3.45}$$

C. The Nearest-Neighbor Approximation

The analysis of the previous subsection provides a general understanding of the introduction and application of the q-variable cluster decomposition. To complement this approach, we wish to give a sample calculation illustrating the technique directly. In Sec. IV we shall calculate the contributions of the first few cluster functions to the one-particle spectrum in the fragmentation region and to the two-particle pionization spectrum. For these purposes, as an explicit check on the validity of the cluster techniques, it is useful to be able to compare the cluster calculation with a known exact result. Hence we shall implement the k-space "nearest-neighbor approximation," introduced in I, as this will enable us to calculate both spectra in closed form. In the present subsection, in preparation for the calculations of Sec. IV, we shall derive the form of the first few cluster functions in the nearest-neighbor approximation in both the fragmentation and pionization regions.

Consider first the pionization region in which $\sqrt{s} \gg q_i^+$, $q_i^- \gg \mu^2/\sqrt{s}$. In this region, we see that (2.11), (2.15b), and (2.24) imply

$$\mathrm{Im}\,\bar{a}^{s(n)}(q_1, \ldots, q_n) \longrightarrow \mathrm{Im}\,a^{s(n)}(q_1, \ldots, q_n) \longrightarrow f_2^{s(n)}(q_1, \ldots, q_n)\,. \tag{3.46}$$

Here the first reduction follows because for $q_i^+ \ll \sqrt{s}$ and $q_i^- \ll \sqrt{s}$, $\theta(\sqrt{s} - \sum q_i^+)$ and $\theta(\sqrt{s} - \sum q_i^-) \to 1$. Thus in the pionization region the troublesome θ functions can be ignored. From (2.25b) we have that, in terms of the scaled momenta $x_i \equiv q_i^+/\sqrt{s}$,

$$f_2^{s(n)} = \sum_{\gamma_n}\left(\frac{x_{\gamma_1}\cdots x_{\gamma_n}}{\left(\sum_{i=1}^{n} x_{\gamma_i}\right)\left(\sum_{i=2}^{n} x_{\gamma_i}\right)\cdots x_{\gamma_n}}\right) b^{(n)}\left(\sum_{i=1}^{n} x_{\gamma_i}, \sum_{i=2}^{n} x_{\gamma_i}, \ldots, x_{\gamma_n}\right). \tag{3.47}$$

The k-space nearest-neighbor approximation to $b^{(n)}$ is given by Eq. (3.27) of I. This is

$$b^{(n)}_{\rm nna}(y_1, \ldots, y_n) = \left(1 - \frac{y_2}{y_1}\right)\left(1 - \frac{y_3}{y_2}\right) \cdots \left(1 - \frac{y_n}{y_{n-1}}\right). \tag{3.48}$$

Hence the first few $f_2^{s(n)}$ can be calculated explicitly; we obtain

$$f_2^{s(1)}(x_1) = 1\,, \tag{3.49a}$$

$$f_2^{s(2)}(x_1, x_2) = \frac{x_1^{\,2}}{x_1 + x_2} + \frac{x_2^{\,2}}{(x_1 + x_2)^2}\,, \tag{3.49b}$$

$$f_2^{s(3)}(x_1, x_2, x_3) = \frac{x_1^{\,2}x_2^{\,2}x_3^{\,2}}{(x_1+x_2+x_3)^2}\left(\frac{1}{x_1^{\,2}(x_1+x_2)^2} + \frac{1}{x_1^{\,2}(x_1+x_3)^2} + \frac{1}{x_2^{\,2}(x_1+x_2)^2} + \frac{1}{x_2^{\,2}(x_2+x_3)^2} + \frac{1}{x_3^{\,2}(x_1+x_3)^2} + \frac{1}{x_3^{\,2}(x_2+x_3)^2}\right). \tag{3.49c}$$

The corresponding cluster functions, defined in terms of these $f_2^{s(i)}$ by equations analogous to (3.1), are

$$g_2^{(1)}(x_1) = 1\,, \tag{3.50a}$$

$$g_2^{(2)}(x_1, x_2) = \frac{-2x_1x_2}{(x_1 + x_2)^2}\,, \tag{3.50b}$$

and

$$g_2^{(3)}(x_1, x_2, x_3) = \frac{4x_1x_2x_3}{(x_1+x_2+x_3)^2}\left(\frac{1}{x_1+x_2} + \frac{1}{x_2+x_3} + \frac{1}{x_1+x_3}\right). \tag{3.50c}$$

Higher cluster functions can be calculated in the same manner, albeit with rapidly increasing effort.

In the fragmentation region of particle a the momenta satisfy $\sqrt{s} \geq q_i^+ \gg \mu^2/\sqrt{s}$, $\mu^2/\sqrt{s} \leq q_i^- \ll \sqrt{s}$. Hence in this region the θ functions involving plus components cannot be ignored and the full expression for $\mathrm{Im}\bar{a}^{s(n)}$ reduces to

$$\mathrm{Im}a^{s(n)}(q_1, \ldots, q_n)\theta\left(\sqrt{s} - \sum_{i=1}^n q_i^+\right) \longrightarrow f_1^{s(n)}(q_1, \ldots, q_n)\theta\left(\sqrt{s} - \sum_{i=1}^n q_i^+\right) \equiv \bar{f}_1^{s(n)}(q_1, \ldots, q_n)\,. \tag{3.51}$$

Equation (2.25a) indicates that

$$f_1^{s(n)}(x_1, \ldots, x_n) = \left(\sum_{\gamma_n} \frac{x_{\gamma_1} \cdots x_{\gamma_n}}{\left(\sum_{i=1}^n x_{\gamma_i}\right)\left(\sum_{i=2}^n x_{\gamma_i}\right) \cdots x_{\gamma_n}}\right) f_L^{(n)}\left(\sum_{i=1}^n x_{\gamma_i}, \sum_{i=2}^n x_{\gamma_i}, \ldots, x_{\gamma_n}\right). \tag{3.52}$$

Since our calculation is intended primarily for verification and clarification of the cluster techniques, in addition to introducing the nearest-neighbor approximation to f_L we shall take $m_a = 0$ for further simplicity. Then Eq. (4.13) of I yields

$$f_L^{(n)}(y_1, \ldots, y_n) = (1 - y_1)\left(1 - \frac{y_2}{y_1}\right) \cdots \left(1 - \frac{y_n}{y_{n-1}}\right). \tag{3.53}$$

Combining (3.52) and (3.53), we obtain

$$\bar{f}_1^{s(1)}(x_1) = (1 - x_1)\theta(1 - x_1)\,, \tag{3.54a}$$

$$\bar{f}_1^{s(2)}(x_1, x_2) = (1 - x_1 - x_2)\theta(1 - x_1 - x_2)\left(\frac{x_1^{\,2}}{(x_1+x_2)^2} + \frac{x_2^{\,2}}{(x_1+x_2)^2}\right), \tag{3.54b}$$

and similarly for higher $\bar{f}_1^{s(n)}$. Notice the "dynamical vanishing" of these functions as any of the $x_i \to 1$. It is this damping which, reflected in the $\bar{g}^{(n)}(q_1, \ldots, q_n)$, guarantees in this simple model the validity of (3.18) even near the "surface" of the kinematic region. From (3.54) we obtain for the first two fragmentation cluster functions,

$$\bar{g}_1^{(1)}(x_1) = (1 - x_1)\theta(1 - x_1) \tag{3.55a}$$

and

$$g_1^{(2)}(x_1, x_2) = \theta(1 - x_1 - x_2)(1 - x_1 - x_2)\left(\frac{x_1^{\,2} + x_2^{\,2}}{(x_1 + x_2)^2}\right) - (1 - x_1)\theta(1 - x_1)(1 - x_2)\theta(1 - x_2)\,. \tag{3.55b}$$

Notice that if in (3.54) and (3.55) we consider the case $1 \gg x_i$, in which the fragmentation region approaches the pionization region, the fragmentation amplitudes and cluster functions reduce correctly to the corresponding pionization functions.

IV. EXPLICIT CLUSTER CALCULATIONS IN THE LADDER MODEL

A. The One-Particle Spectrum in the Fragmentation Region

In I, to demonstrate the smooth transition of the spectrum between the fragmentation and pionization regions, we calculated the full one-particle spectrum in the fragmentation region in the "nearest-neighbor approximation." In that analysis we did not attempt to compare the complete result with a cluster-expansion calculation of the one-particle spectrum because of the kinematic complications that arose in the fragmentation region in the relation between the phase space in terms of the k variables and that in terms of the q variables; that is, the simple relation $dk^+/k^+ = dq^+/q^+$, which holds in the pionization region, does not apply to fragmentation events.

Now that we have established the validity of a q-variable cluster decomposition, it is a simple matter to check the first few orders in λ of the cluster result with the corresponding orders in the complete solution given in I. From Sec. III, we have that in the fragmentation region the first two cluster functions are, in terms of the scaled momentum variables $x_i \equiv q_i^+/\sqrt{s}$,

$$\bar{g}^{(1)}(x_1) = (1 - x_1)\theta(1 - x_1)\,, \tag{4.1a}$$

$$\bar{g}^{(2)}(x_1, x_2) = \theta(1 - x_1 - x_2)(1 - x_1 - x_2)\left(\frac{x_1^{\,2} + x_2^{\,2}}{(x_1 + x_2)^2}\right) - (1 - x_1)\theta(1 - x_1)\,(1 - x_2)\theta(1 - x_2)\,. \tag{4.1b}$$

Hence we see from (3.30) that the contributions of $\bar{g}^{(1)}(x_1)$ and $\bar{g}^{(2)}(x_1, x_2)$ as given in (4.1) to the one-particle fragmentation spectrum are

$$\frac{1}{\sigma_T}(d\sigma)_1 = \lambda(1 - x_1)\frac{dx_1}{x_1} \tag{4.2a}$$

and

$$\frac{1}{\sigma_T}(d\sigma)_2 = \left[\lambda^2 \int_{\mu^2/s}^{1} \frac{dx_2}{x_2}\, \bar{g}^{(2)}(x_1, x_2)\right]\frac{dx_1}{x_1}$$
$$= \lambda^2[\,2(x_1 - 1) - 2x_1 \ln x_1 + (1 - x_1)\ln(1 - x_1)]\frac{dx_1}{x_1}\,, \tag{4.2b}$$

to within powers of s.

Notice that, as a consequence of our dynamical model, both terms approach zero smoothly as $x \to 1$.

To compare expressions (4.2) with the appropriate terms in the complete result for the fragmentation spectrum in the nearest-neighbor approximation, we refer to Eqs. (5.34) and (5.35) of I which give, in our present notation,

$$\frac{d\sigma_{\text{frag}}}{\sigma_T} = \frac{\lambda}{(1 + 4\lambda)^{1/2}} M(\lambda, x)\frac{dx}{x}\,, \tag{4.3}$$

with

$$M(x, \lambda) \equiv \int_1^{1/x - 1} du\, u^{\alpha - 1}\left(\frac{1}{(u + 1)^{\alpha + 2}} - (u + 1)^{\alpha - 1} x^{1 + 2\alpha}\right),$$

where

$$\alpha = \tfrac{1}{2}[-1 + (1 + 4\lambda)^{1/2}]\,.$$

By a careful expansion of M about the point $\lambda = 0$, we obtain

$$\frac{d\sigma_{\text{frag}}}{\sigma_T} = \{\lambda(1 - x) + \lambda^2[(1 - x)\ln(1 - x) - 2x\ln x + 2(x - 1)] + \cdots\}dx/x\,, \tag{4.4}$$

which as required agrees with (4.2). For a more detailed discussion of the qualitative features of the one-particle fragmentation spectrum in this simple model we refer the reader to I.

B. The Two-Particle Spectrum in the Pionization Region

As a second illustration of the validity and usefulness of the q-cluster decomposition, we shall consider the two-particle spectrum in the pionization region, $\sqrt{s} \gg q_1^+,\ q_2^+ \gg \mu^2/\sqrt{s}$. To permit a comparison of the cluster expansion with an exact result, we shall again make the nearest-neighbor approximation in both calculations.

In the cluster-expansion approach the two lowest-order contributions in λ to the correlated part of this spectrum come from the cluster functions $\bar{g}^{(2)}(q_1, q_2)$ and $\bar{g}^{(3)}(q_1, q_2, q_3)$. Using the simplification that in the pionization region $\bar{g}^{(n)}(q_1, \ldots, q_n) \to g^{(n)}(q_1, \ldots, q_n)$, we see that in terms of $x_i \equiv q_i^+/\sqrt{s}$ the contributions of $g^{(2)}$ and $g^{(3)}$ are, respectively,

$$\frac{1}{\sigma_T}(d^2\sigma_c)_2 = \lambda^2 g^{(2)}(x_1, x_2)\frac{dx_1}{x_1}\frac{dx_2}{x_2}$$
$$= -2\lambda^2\frac{x_1 x_2}{(x_1 + x_2)^2}\frac{dx_1}{x_1}\frac{dx_2}{x_2}\,, \tag{4.5}$$

where we have used the explicit form of $g^{(2)}(x_1, x_2)$ given in (3.50) and[12]

$$\frac{1}{\sigma_T}(d^2\sigma_c)_3 = \lambda^3 \frac{dx_1}{x_1}\frac{dx_2}{x_2}\int_{\mu^2/s}^{1} g^{(3)}(x_1, x_2, x_3)\frac{dx_3}{x_3}$$

$$= \lambda^3 \frac{dx_1}{x_1}\frac{dx_2}{x_2}\, 4\left[\frac{x_1x_2}{(x_1+x_2)^2} + \frac{x_1}{x_2}\ln\left(\frac{x_1+x_2}{x_1}\right) + \frac{x_2}{x_1}\ln\left(\frac{x_1+x_2}{x_2}\right) - 1\right] + O\left(\frac{1}{s}\right). \tag{4.6}$$

The last equation follows from the form of $g^{(3)}(x_1, x_2, x_3)$ given in (3.50).

By a direct calculation using the method of I, we establish that the full two-particle spectrum in the pionization region is, in the nearest-neighbor approximation,

$$\frac{d^2\sigma(x_1, x_2)}{\sigma_T} = \frac{\lambda^2}{1+4\lambda}\frac{dx_1}{x_1}\frac{dx_2}{x_2}[H(x_1, x_2) + H(x_2, x_1)], \tag{4.7}$$

where

$$H(x_1, x_2) = \lambda^2 \int_{x_2+\mu^2/s}^{1} \frac{dw}{w}\frac{x_1^2}{(w+x_1)^{2+\alpha}} \int_{\mu^2/s}^{w-x_2} \frac{dz}{z} z^{\alpha} \frac{x_2^2}{(z+x_2)^2}\left[\left(\frac{w}{x_2+z}\right)^{\alpha} - \left(\frac{w}{x_2+z}\right)^{-1-\alpha}\right]$$

$$+ \lambda(1+4\lambda)^{1/2}\int_{\mu^2/s}^{1} dw\, w^{\alpha-1}\frac{x_1^2}{(w+x_1+x_2)^{2+\alpha}}\frac{x_2^2}{(w+x_2)^2}, \tag{4.8}$$

where

$$\alpha = \tfrac{1}{2}[-1 + (1+4\lambda)^{1/2}].$$

Recall that the correlated part of the two-particle spectrum is given by

$$\frac{d^2\sigma_c(x_1, x_2)}{\sigma_T} = \frac{d^2\sigma(x_1, x_2)}{\sigma_T} - \frac{d\sigma(x_1)}{\sigma_T}\frac{d\sigma(x_2)}{\sigma_T} = \frac{\lambda^2}{1+4\lambda}\frac{dx_1}{x_1}\frac{dx_2}{x_2}[H(x_1, x_2) + H(x_2, x_1) - 1], \tag{4.9}$$

where we have used the result, established in I, that in the pionization region

$$\frac{d\sigma(x_1)}{\sigma_T} = \frac{\lambda}{(1+4\lambda)^{1/2}}\frac{dx_1}{x_1}. \tag{4.10}$$

A careful Taylor's expansion of (4.9) about $\lambda = 0$ yields

$$\frac{d^2\sigma_c(x_1, x_2)}{\sigma_T} = -2\lambda^2\frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{x_1x_2}{(x_1+x_2)^2} + 4\lambda^3\frac{dx_1}{x_1}\frac{dx_2}{x_2}\left[\frac{x_2}{x_1}\ln\left(\frac{x_1+x_2}{x_2}\right) + \frac{x_1}{x_2}\ln\left(\frac{x_1+x_2}{x_1}\right) + \frac{x_1x_2}{(x_1+x_2)^2} - 1\right] + \cdots \tag{4.11}$$

which again agrees with the cluster results as given by (4.5) and (4.6). Interested readers are invited to reproduce the calculations.

There are several qualitative features of this two-particle spectrum which merit comment, since they appear to be more general than our extremely simple model[13] and further since some of them clarify certain aspects of cluster techniques.

First we note that the normalized spectrum is independent of s (to order $1/s$) and indeed depends only on the magnitude of the difference of the rapidities z_1 and z_2 of the two particles; that is,

$$\frac{d^2\sigma_c}{\sigma_T} = f(|z_1 - z_2|),$$

where $z_1 - z_2 = \ln(q_1/q_2)$.

Second, we observe that for integral values of α the function $H(x_1, x_2)$ of (4.8), which determines the two-particle spectrum, can be evaluated exactly analytically. For $\alpha = 1$ and $\alpha = 2$ (so that $\lambda = 2$ and $\lambda = 6$) – recall that our definition of α is $\alpha = \alpha_c + 1$, where α_c is the conventional Regge trajectory function – the resulting forms of the correlated part of two-particle spectrum, as given by (4.9), are plotted in Fig. 4 as functions of $z_1 - z_2 \equiv \ln(q_1/q_2)$. The general shape of these curves agrees with that found by numerical methods in a (3 + 1)-dimensional φ^3 model.[13] For purposes of comparison, the individual contributions of the two- and three-particle clusters only, normalized to their respective magnitudes at $z_1 - z_2 = 0$, are plotted in Figs. 5(a) and 5(b). From (4.5) and (4.6) we see that to obtain the actual contribution to the spectrum for a given λ from these curves, we must multiply by $2\lambda^2$ for the two-particle cluster and by $(8\ln 2 - 3)\lambda^3$ for the three-particle cluster. Clearly for $\lambda = 2$ and $\lambda = 6$

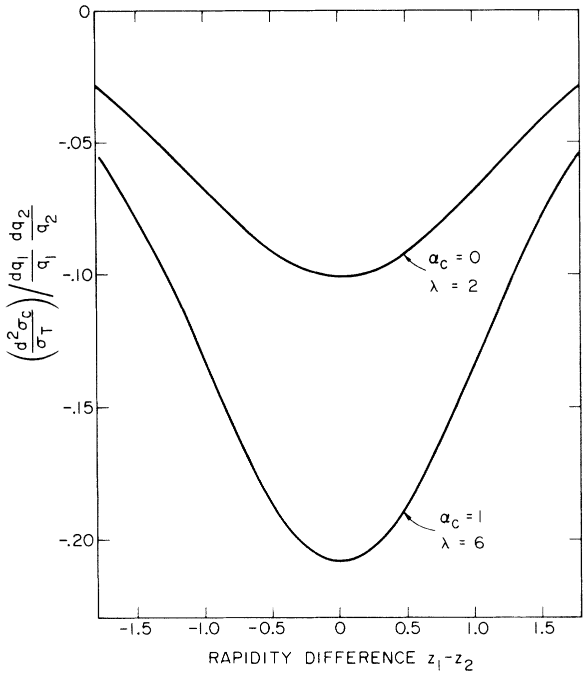

FIG. 4. The over-all correlated part of the two-particle spectrum in the nearest-neighbor approximation for $\alpha_c = 0$ and 1.

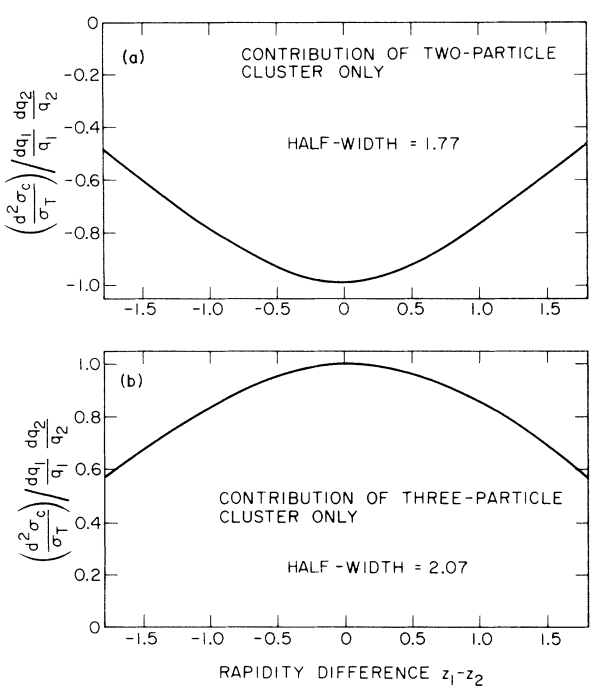

FIG. 5. (a) The contribution of two-particle clusters to the correlated part of the two-particle spectrum. (b) The contribution of three-particle clusters to the correlated part of the two-particle spectrum.

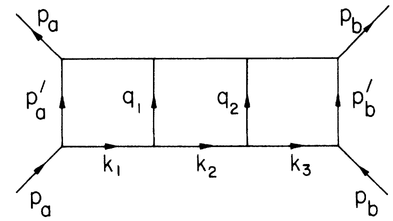

FIG. 6. The ladder amplitude corresponding to $A^{(2)}(s,t=0)$ in a (3+1)-dimensional theory.

these separate contributions are of much greater magnitude than the final result and hence important cancellations must occur. We shall return to this point later.

From Figs. 4 and 5 we can also comment on another important qualitative feature of the spectrum, namely, the total width and the widths of the individual contributions. In Fig. 5 the half width at half height of the two-particle cluster contribution is shown to be 1.77, whereas that of the three-particle cluster is 2.07. We believe this reflects the general result that higher-order clusters give rise to correlations which have longer-range effects and which therefore produce contributions of greater width to the two-particle spectrum. This agrees with the physical intuition which suggests that in higher-cluster contributions the many unmeasured particles can lie between the two observed ones in rapidity space, linking them even for large separations.

In view of the increasing widths of the higher-cluster contributions, it is very interesting to note that the widths of the full spectra for $\alpha = 1$ ($\lambda = 2$) and $\alpha = 2$ ($\lambda = 6$) are 1.34 and 1.25, respectively;

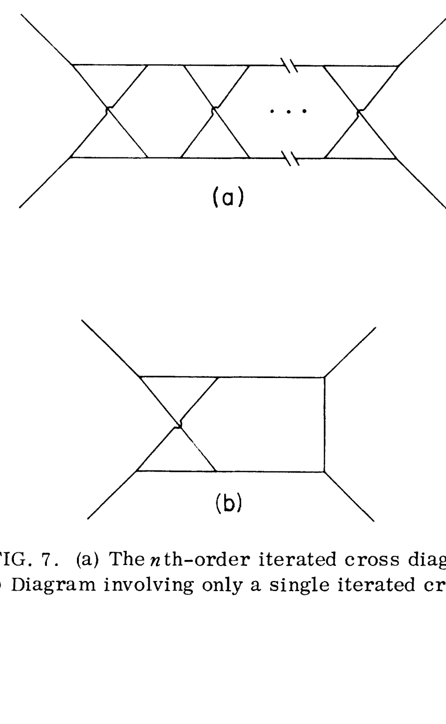

FIG. 7. (a) The nth-order iterated cross diagram. (b) Diagram involving only a single iterated cross.

further, in the limit $\alpha \to 0$ ($\lambda \to 0$), the width of the full spectrum approaches that of the two-particle cluster. Thus in our model we find that the sum over all cluster contributions has a narrower width than any of the individual terms. This result is made possible only by cancellations occurring between different terms.

Finally, let us return to the question raised above regarding cancellations between terms and the size of individual cluster contribution in our model. The form of the cluster functions $\bar{g}^{(n)}$ is such that after integrating over q_3 to q_n, the remaining expression, as a function of q_1/q_2, is roughly of $O(1)$ at $q_1/q_2=1$ ($z_1-z_2=0$). Hence for large values of λ, the individual contribution of the nth-order cluster to the two-particle spectrum will grow like λ^n. In this limit, then, any approximation which consisted of keeping only a finite number of clusters would be very inadequate. It is relatively easy to see that this difficulty is not intrinsic to the cluster technique but arises from the specific form of the φ^3 theory we are considering and, in particular, from the nearest-neighbor approximation to this theory. To verify this we can recall that for our model in the nearest-neighbor approximation the one-particle spectrum in the pionization region takes the form

$$\frac{d\sigma}{\sigma_T}=\frac{\lambda}{(1+4\lambda)^{1/2}}\frac{dx}{x}\,. \tag{4.10'}$$

Considered as a function of λ, this expression has a power-series expansion about $\lambda=0$ which converges only for $|\lambda|<\frac{1}{4}$. That the power series in λ for the two-particle spectrum also converges only for $|\lambda|<\frac{1}{4}$ follows from (4.9). Since by construction each cluster function $\bar{g}^{(n)}$ is associated with the factor λ^n, the increasing size of $\lambda^n\bar{g}^{(n)}$ with n merely reflects the divergence of the power series for large coupling in our model and does not alter the possibility that in using the cluster approach to analyze data one may be able to keep the contributions of only the first few clusters.

V. EXTENSION TO LADDER DIAGRAMS IN (3+1) DIMENSIONS AND FURTHER COMMENTS AND SPECULATIONS

The extension of our results concerning q-variable factorization and cluster decomposition to a φ^3 theory in $(3+1)$ dimensions offers some important insights into the crucial differences between the k- and the q-variable approaches. It develops that many of the desirable properties of perturbation-theory amplitudes expressed in terms of the k variables simply do not apply in terms of the q variables. In particular, complete factorization of the differential *exclusive* cross sections – that is, factorization in terms of both plus and transverse components of momentum – in the k variables does not imply complete factorization in terms of the q variables. In fact, as we shall now demonstrate, the assumption of complete factorization of the differential *exclusive* cross sections in terms of the q variables is unrealistic in the sense that it is definitely not motivated by the behavior of the model field theory we consider.

To gain some understanding of the difficulties associated with q-variable factorization and clusters in $(3+1)$-dimensional theories let us begin by considering the simple ladder diagram corresponding to Fig. 6. The imaginary part of the amplitude in the forward direction is

$$\begin{aligned}\operatorname{Im}A^{(2)}(s,t=0)=(g^2)^4\int&\frac{d^4p_a'}{(2\pi)^4}(2\pi)\delta(p_a'^2-\mu^2)\frac{d^4q_1}{(2\pi)^4}(2\pi)\delta(q_1^2-\mu^2)\frac{d^4q_2}{(2\pi)^4}(2\pi)\delta(q_2^2-\mu^2)\frac{d^4p_b'}{(2\pi)^4}\\&\times(2\pi)\delta(p_b'^2-\mu^2)\frac{1}{(k_1^2-\mu^2+i\epsilon)^2}\frac{1}{(k_2^2-\mu^2+i\epsilon)^2}\\&\times\frac{1}{(k_3^2-\mu^2+i\epsilon)^2}(2\pi)^4\delta^{(4)}(p_a+p_b-p_a'-p_b'-q_1-q_2)\,.\end{aligned}\tag{5.1}$$

If we introduce plus and minus variables and proceed as in Sec. II, we find that at large s

$$\operatorname{Im}A^{(2)}(s,t=0)\cong\frac{(g^2)^4}{2p_a'^+p_b'^-}\int\frac{d^2p_b'}{(2\pi)^2}\frac{d^2q_1dq_1^+}{(2\pi)^3 2q_1^+}\frac{d^2q_2}{(2\pi)^3}\frac{dq_2^+}{2q_2^+}\theta(\sqrt{s}-q_1^+-q_2^+)\theta(\sqrt{s}-q_1^--q_2^-)\prod_{i=1}^{3}\frac{1}{(k_i^2-\mu^2+i\epsilon)^2}\,,\tag{5.2}$$

where $p_a'^+\cong\sqrt{s}-q_1^+-q_2^-$, $p_b'^-\cong\sqrt{s}-q_1^--q_2^-$, and $\vec{p}$ denotes the two transverse components of a 4-vector p. For simplicity let us restrict our considerations to the pionization region $\sqrt{s}\gg q_1^+$, $q_2^+\gg\mu^2/\sqrt{s}$, as this will serve to illustrate our point. Then using arguments similar to those preceding (2.10), we can deduce that

$$(k_1^-) = O(\mu^2/\sqrt{s}), \quad k_1^+ = p_a^+ - p_a'^+ \cong (q_1+q_2)^+ \ll \sqrt{s},$$
$$(k_3^+) = O(\mu^2/\sqrt{s}), \quad k_3^- = p_b'^- - p_b^- \cong -(q_1+q_2)^- \ll \sqrt{s}, \tag{5.3}$$

and hence that

$$(k_1^2 - \mu^2 + i\epsilon)^2 \longrightarrow (\vec{k}_1^2 + \mu^2)^2 \quad \text{and} \quad (k_3^2 - \mu^2 + i\epsilon)^2 \rightarrow (\vec{k}_3^2 + \mu^2)^2 . \tag{5.4}$$

Similarly, we can establish that

$$k_2^+ = q_2^+ + O(\mu^2/\sqrt{s}), \quad k_2^- = k_1^- - q_1^- \cong -(\vec{q}_1^2 + \mu^2)/q_1^+ \tag{5.5}$$

since $k_1^- \cong O(\mu^2/\sqrt{s})$. Thus

$$(k_2^2 - \mu^2 + i\epsilon)^2 \longrightarrow [(q_2^+/q_1^+) \times (\vec{q}_1^2 + \mu^2) + \vec{k}_2^2 + \mu^2]^2 . \tag{5.6}$$

Then (5.2) becomes

$$\operatorname{Im} A^{(2)}(s, t=0) \cong \frac{(g^2)^4}{2s} \int \frac{dq_1^+}{4\pi q_1^+} \frac{dq_2^+}{4\pi q_2^+} \theta(\sqrt{s} - q_1^+ - q_2^+)\theta(\sqrt{s} - q_1^- - q_2^-) \frac{d^2q_1}{(2\pi)^2} \frac{d^2q_2}{(2\pi)^2} \frac{d^2p_b'}{(2\pi)^2} \frac{1}{(\vec{p}_b'^2 + \mu^2)^2} \operatorname{Im} a^{(2)}(q_1, q_2, \vec{p}_b') . \tag{5.7}$$

Here

$$\operatorname{Im} a^{(2)}(q_1, q_2, \vec{p}_b') = (\vec{k}_1^2 + \mu^2)^{-2}[(q_2^+/q_1^+)[(\vec{k}_1 - \vec{k}_2)^2 + \mu^2] + \vec{k}_2^2 + \mu^2]^{-2} \tag{5.8}$$

$$= [(\vec{q}_1 + \vec{q}_2 + \vec{p}_b')^2 + \mu^2]^{-2} \, [(q_2^+/q_1^+)(\vec{q}_1^2 + \mu^2) + (\vec{q}_2 + \vec{p}_b')^2 + \mu^2]^{-2} , \tag{5.9}$$

where we have used the fact that $\vec{p}_b' = -\vec{k}_3$.

Notice that in the limit $q_1^+ \gg q_2^+$, (5.8), which expresses $\operatorname{Im} a^{(2)}$ in terms of the k variables, factorizes simply in terms of these variables,

$$\operatorname{Im} a^{(2)} \longrightarrow (\vec{k}_1^2 + \mu^2)^{-2}(\vec{k}_2^2 + \mu^2)^{-2} + O(q_2^+/q_1^+) = f(\vec{k}_1) f(\vec{k}_2) + O(q_2^+/q_1^+) . \tag{5.10}$$

On the other hand, in the same limit (5.9) becomes

$$\operatorname{Im} a^{(2)} \longrightarrow [(\vec{q}_1 + \vec{q}_2 + \vec{p}_b')^2 + \mu^2]^{-2}[(\vec{q}_2 + \vec{p}_b')^2 + \mu^2]^{-2} + O(q_2^+/q_1^+) \neq f(\vec{q}_1) f(\vec{q}_2) + O(q_2^+/q_1^+) . \tag{5.11}$$

Thus $\operatorname{Im} a^{(2)}$ does not factorize in terms of the $\vec{q}_i$. This simple example is therefore sufficient to demonstrate that complete factorization in the k variables does not imply complete factorization in the q variables for exclusive spectra; further, the failure of factorization in $\vec{q}$ in ladder diagrams strongly suggests that one should not assume such factorization in phenomenological analyses of exclusive cross sections. In addition to illustrating the lack of $\vec{q}$ factorization, however, equations (5.7)–(5.9) indicate one possible resolution to the q-factorization problem. Namely, let us simply not measure the $\vec{q}$ but rather integrate over them to obtain an expression of the form

$$\operatorname{Im} A^{(2)}(s, t=0) \cong \frac{(g^2)^4}{2s} \int \frac{dq_1^+}{4\pi q_1^+} \int \frac{dq_2^+}{4\pi q_2^+} \theta(\sqrt{s} - q_1^+ - q_2^+) \times (\theta \text{ function due to } q^-) \times \operatorname{Im} a^{(2)}(q_1^+, q_2^+) . \tag{5.12}$$

Then $\operatorname{Im} a^{(2)}(q_1^+, q_2^+)$ does satisfy the factorization property in q_i^+. This specific example is indicative of a general result valid for multiperipheral-type diagrams: If a given diagram corresponding to an exclusive cross section satisfies complete factorization in the k variables, in general complete q-variable factorization does not follow. However, if one integrates over the transverse momenta, $\vec{q}_i$, the result factorizes in terms of the q_i^+ (or, of course, the q_i^-).

The lack of $\vec{q}$ factorization in the *exclusive* spectra derived from the ladder model appears somewhat paradoxical in view of the anticipated complete q-variable factorization properties, suggested by the Mueller analysis[14] and recently established explicitly in dual models,[15] of the *inclusive* spectra. These *inclusive* factorization properties appear to rest on the single assumption that certain multiparticle forward amplitudes are dominated by simple Regge poles; hence, one would expect that φ^3 ladder diagrams, which give rise to a Regge pole, should also yield *inclusive* spectra satisfying complete q-variable factorization. Thus we see the apparent paradox: The *exclusive* spectra do not factorize in $\vec{q}$ but the *inclusive* spectra do. It turns out that the cluster-decomposition technique enables one to demonstrate, to all orders in perturbation theory, that any set of multiperipheral diagrams which satisfy complete k-variable factorization gives rise to inclusive spectra which are completely q factorizable. Hence there is indeed no paradox; it is perfectly consistent to have non-$\vec{q}$-factorizable exclusive spectra and $\vec{q}$-factorizable inclusive spectra. We shall discuss the details of this result in a separate article. For the present, we observe simply that the integrations over the momenta of unobserved particles implicit in any inclusive spectrum effectively remove the dynamical correlations among the transverse momenta which destroy complete q-variable factorization in exclusive spectra.

A natural sequel to the previous discussion is an analysis of the case in which the diagrams considered do not have complete factorization in the k variables. In I we discussed the iterated cross diagrams Fig. 7(a) as prototypes of this class. In the case of the diagrams of Fig. 7(b) one can show that even after integration over the transverse momenta the resulting expression does not satisfy q_1^+ factorization and hence one cannot introduce q-variable clusters for these diagrams. The skeptical reader is invited to verify this result explicitly in the simpler case of the diagram of Fig. 7(b).

In conclusion, let us summarize our results succinctly. We have examined in detail a $(1+1)$-dimensional φ^3 ladder model in order to study the possibility of introducing a cluster decomposition for the differential exclusive-production cross sections in terms of the final-state momenta, q_i^+. We have established the factorization of the integrand corresponding to these cross sections in this model; however, we have also found that the kinematic coupling in the integration limits between the q_i^+ resulting from momentum conservation implies that the standard application of the cluster-decomposition technique leads to cluster functions which are not independent. By modifying suitably the definition of the integrand – thereby destroying exact factorization – we have been able to introduce cluster functions which do contribute independently to the total cross section and multiparticle spectra and thus to establish the existence of a q-variable cluster decomposition for a certain class of diagrams at high energy. The validity of this somewhat unconventional cluster expansion has been demonstrated explicitly in the $(1+1)$-dimensional ladder model by comparing with the exact results the cluster predictions for the one-particle spectrum in the fragmentation region and for the two-particle pionization spectrum. A more general discussion of the utility of this type of cluster expansion will appear in the sequel. Finally, we have commented on the extension of the q-variable cluster decomposition to $(3+1)$-dimensional models. Here the conclusions corroborate our earlier contention that the most natural variables, from a theorist's standpoint, in which to study cluster properties of perturbation-theory amplitudes are the "momentum transfer" variables, k_i. Unfortunately, as we discussed in I, the usefulness to phenomenological analysis of a k-variable cluster expansion is limited; it is the q variables which are observed experimentally. In terms of the q variables we found that factorization of exclusive spectra in terms of both $\vec{q}$ and q^+ is not a general feature; even the ladder amplitudes do not satisfy factorization properties in terms of the $\vec{q}_i$. By integrating over the transverse momenta, however, we have established the factorization property of the exclusive spectra derived from ladder diagrams in terms of the q_i^+ alone. The limited validity of the assumption of q^+ factorizability has been illustrated by a brief discussion of the iterated cross diagrams, for which such factorizability fails. We have discussed the relation between the nonfactorizability of *exclusive* spectra in terms of $\vec{q}$ and the anticipated[14, 15] complete q-variable factorization of *inclusive* spectra. Here we asserted that these apparently contradictory properties are in fact mutually consistent. Our analysis allowed us to establish that if the underlying field-theory diagrams satisfy complete k-variable factorization, then the exclusive spectra satisfy q^+ factorization, and the inclusive spectra satisfy complete q-variable factorization. Finally, we remark that in the following paper we shall take q^+ factorization of exclusive spectra as an ansatz and explore the consequences for the phenomenological description of high-energy scattering.

ACKNOWLEDGMENTS

The authors are indebted to Professor S. Treiman for many stimulating discussions.

APPENDIX ADDED IN PROOF

In this appendix, we wish to demonstrate that (1) the fully integrated exclusive cluster functions $\overline{G}^{(n)}$ are proportional to $\ln s$,

$$\overline{G}^{(n)} \equiv \frac{1}{n!}\int \prod_{i=1}^{n} \frac{dq_i^+}{q_i^+}\, \overline{g}^{(n)}(q_1, q_2, \ldots, q_n)$$

$$= \alpha_n \ln s + \beta_n + O(s^{-1}), \qquad \text{(A1)}$$

and (2) away from the kinematical boundary, the partially integrated exclusive cluster functions,

$$\bar{\tau}^n_m(q_1, q_2, \ldots, q_m) \equiv \frac{1}{(n-m)!} \int \prod_{i=m+1}^{n} \frac{dq_i^+}{q_i^+} \bar{g}^{(n)}(q_1, q_2, \ldots, q_n), \qquad \text{(A2)}$$

have finite correlation lengths, that is, for $(q_1^+, q_2^+, \ldots, q_l^+) \gg (q_{l+1}^+, \ldots, q_n^+)$,

$$\bar{\tau}^n_m \to O(q_i^+/q_j^+),$$

with $i \in (l+1, \ldots, n)$, $j \in (1, 2, \ldots, l)$. Notice that (3.45) implies that $\bar{\tau}^n_m$ represents the contribution of the nth-order exclusive cluster function $\bar{g}^{(n)}$ to the correlated part of the m-particle inclusive spectrum. To demonstrate this explicitly, we observe that with $\bar{\tau}^n_m$ defined by (A2), we have from (3.45)

$$\frac{1}{\sigma_T}(d^m\sigma_c)_n = \lambda^n \bar{\tau}^n_m(q_1, \ldots, q_m) \frac{dq_1^+}{q_1^+} \cdots \frac{dq_m^+}{q_m^+}. \qquad \text{(A3)}$$

For completeness we define the full correlated part of the m-particle inclusive spectrum by

$$\frac{1}{\sigma_T}(d^m\sigma_c) \equiv \bar{\tau}_m(q_1, \ldots, q_m) \frac{dq_1^+}{q_1^+} \cdots \frac{dq_m^+}{q_m^+}. \qquad \text{(A4)}$$

Hence

$$\bar{\tau}_m(q_1, \ldots, q_m) \equiv \sum \lambda^n \bar{\tau}^n_m(q_1, \ldots, q_m). \qquad \text{(A5)}$$

For clarity of presentation, the properties of the $\bar{\tau}^n_m$ near the kinematical boundary will be treated separately in Sec. A 1.

We recall that the exclusive distribution function $\bar{f}^{(n)}$ can be written as

$$\bar{f}^{(n)}(q_1, q_2, \ldots, q_n) = f^{(n)}(q_1, q_2, \ldots, q_n) \times \theta(\sqrt{s} - \sum q_i^+)\theta(\sqrt{s} - \sum q_i^-), \qquad \text{(A6)}$$

where the θ functions in (A6) describe the kinematical constraints on the final-particle momenta due to energy-momentum conservation. All dynamical information is contained in the $f^{(n)}(q_1, q_2, \ldots, q_n)$, which are assumed to satisfy (i) the factorization property that, for

$$(q_1^+, \ldots, q_m^+) \gg (q_{m+1}^+, \ldots, q_n^+),$$

$$f^{(n)}(q_1, q_2, \ldots, q_n) = f^{(m)}(q_1, q_2, \ldots, q_m) f^{(n-m)}(q_{m+1}, \ldots, q_n) + O(q_j^+/q_i^+) \qquad \text{(A7)}$$

with $i \in (1, 2, \ldots, m)$, $j \in (m+1, \ldots, n)$ and (ii) the scaling properties as outlined in the text. We assume further that as a $q^\pm$, or as the sum of certain $q^\pm$, approaches the kinematical boundary $\sqrt{s}$, the amplitude is not more singular than $(\sqrt{s} - q^\pm)^{-1}$. A less divergent amplitude, for example, $(\sqrt{s} - q^\pm)^{-a}$, $1 > a > 0$, is acceptable. This last property ensures that the integrated amplitude does not diverge (except possibly from the small $q^\pm$ region). This restriction is generally accepted for a multiperipheral amplitude to guarantee that the total cross section does not violate the Froissart bound.

1. Kinematical Correlations

To approach the problem, we first separate the correlation effects due to purely kinematical constraints from those arising from the dynamics. For simplicity, we restrict ourselves to the kinematical region in which only one θ function, say $\theta(\sqrt{s} - \sum_i q_i^+)$, is important. The other θ function is assumed to give property (i) identically. The generalization to include both θ functions is straightforward.

We begin by analyzing the correlation effects that arise solely from the kinematical constraints; that is, we take

$$\bar{f}^{(n)}(q_1, q_2, \ldots, q_n) = \theta(\sqrt{s} - \sum q_i^+) \qquad \text{(A8)}$$

and

$$f^{(n)} = 1. \qquad \text{(A9)}$$

It is convenient here to use the scaled momentum variables

$$x_i = q_i^+/\sqrt{s}, \quad 1 \geq x_i \geq 0 \qquad \text{(A10)}$$

and to introduce

$$\theta_n(x_1, \ldots, x_n) \equiv \theta\left(1 - \sum_{i=1}^{n} x_i\right). \qquad \text{(A11)}$$

We now define a set of "cluster functions," $\varphi_n(x_1, \ldots, x_n)$, from the θ_n by

$$\varphi_1(x) \equiv \theta_1(x) = \theta(1-x), \qquad \text{(A12a)}$$

$$\varphi_2(x_1, x_2) \equiv \theta_2(x_1, x_2) - \theta_1(x_1)\theta_1(x_2) = \theta(1 - x_1 - x_2) - \theta(1 - x_1)\theta(1 - x_2), \qquad \text{(A12b)}$$

and, in general,

$$\theta_n(x_1, \ldots, x_n) = \sum_{\text{all partitions}} \varphi_{n_1}\varphi_{n_2} \cdots \varphi_{n_r}. \qquad \text{(A12c)}$$

The cluster functions φ_n have only kinematical correlations and vanish identically if $\sum_{i=1}^{n} x_i < 1$.

The following properties of φ_n are crucial to our analysis:

(1) For $0 \leq x_i < 1$, we have

$$\varphi_1(0) = 1, \qquad \text{(A13)}$$

$$\varphi_n(0, x_2, \ldots, x_n) = 0 \text{ for all } n \geq 1. \qquad \text{(A14)}$$

(2) For small x_1, $\varphi_n(x_1, \ldots, x_n)$ remains zero

until x_1 passes through the root of one of the equations

$$x_1+\sum_{\text{some } i} x_i=1\,. \tag{A15}$$

(3) Let $f(x)$ be an arbitrary continuous test function and let $x_1>0$ be small. Since we are explicitly excluding the kinematical boundary from our present discussion, we may assume that none of the differences, $1-\sum_{\text{some } i} x_i$, $i\in(2,3,\ldots,n-1)$, is both positive and of the order of x_1. Then we have

$$\lim_{x_1\to 0}\int_0^1 \frac{dx_n}{x_n}\varphi_n(x_1,x_2,\ldots,x_n)f(x_n)=O(x_1)\,. \tag{A16}$$

Similarly, if both x_1 and x_2 are small [and since none of the differences, $1-\sum_{\text{some } i} x_i$, $i=(3,\ldots,n-1)$, is permitted to be positive and of the order of x_1 or x_2], we have

$$\lim_{x_1,x_2\to 0}\int_0^1 \frac{dx_n}{x_n}\varphi_n(x_1,x_2,\ldots,x_n)f(x_n)$$

$$=\text{smaller of } O(x_1) \text{ and } O(x_2)\,. \tag{A17}$$

Property (1) can be verified easily for the first few orders and can be established in general by induction. Property (2) follows from the fact that φ_n is a function of $\theta(1-\sum_{\text{some } i} x_i)$ only. Thus, φ_n will remain constant (in this case, zero) until one of the θ functions changes its value. To prove (A16), we note that for small x_1 the only regions of the x_n integration in which $\varphi_n\neq 0$ are given by

$$x_1+x_n+\sum_{\text{some } i} x_i>1>x_n+\sum_{\text{some } i} x_i\,,$$

or, equivalently,

$$1-\sum_{\text{some } i} x_i>x_n>1-\sum_{\text{some } i} x_i-x_1\,. \tag{A18}$$

Since the intervals of the x_n integration are of $O(x_1)$, so is the integral in (A16). The extension of (A16) to (A17) and then to the general case is straightforward.

The properties (1)–(3) permit us to establish immediately that the fully integrated cluster functions are linear in $\ln s$ and that the partially integrated cluster functions have finite correlation lengths. Actually, the above properties imply some much stronger results. First, they imply that only the first fully integrated cluster function,

$$\overline{G}^{(1)}=\int_{1/s}^1 \frac{dx}{x}\varphi_1(x)=\ln s+O\left(\frac{1}{s}\right), \tag{A19}$$

contains any factors proportional to $\ln s$. All other fully integrated cluster functions, such as

$$\overline{G}^{(2)}=\frac{1}{2!}\int_{1/s}^1 \frac{dx_1}{x_1}\frac{dx_2}{x_2}\varphi_2(x_1,x_2)=-\tfrac{1}{12}\pi^2+O\left(\frac{1}{s}\right)$$

and

$$\overline{G}^{(3)}=\frac{1}{3!}\int_{1/s}^1 \frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{dx_3}{x_3}\varphi_3(x_1,x_2,x_3)=\tfrac{1}{3}\zeta(3)+O\left(\frac{1}{s}\right),$$

approach constants. Second, they imply that the partially integrated cluster functions $\overline{\tau}^n_m$ actually approach zero as any x_i, say x_1, approaches zero,

$$\lim_{x_1\to 0}\overline{\tau}^n_m(x_1,\ldots,x_m)=O(x_1)\,. \tag{A20}$$

To derive (A20), we have, as before, assumed that none of the differences, $1-\sum_{\text{some } i} x_i$, is of $O(x_1)$ and positive. For simplicity, we refer to quantities such as $x_1(\ln x_1)^n$ as $O(x_1)$.

2. Combined Kinematical and Dynamical Correlations

Now we consider the general case. To get some feeling for the results, let us work out the first few correlation functions explicitly. The function $\overline{g}^{(2)}(x_1,x_2)$ is given by

$$\begin{aligned}\overline{g}^{(2)}(x_1,x_2)&=f^{(2)}(x_1,x_2)\theta(1-x_1-x_2)\\&\quad-f^{(1)}(x_1)f^{(1)}(x_2)\theta(1-x_1)\theta(1-x_2)\\&=g^{(2)}(x_1,x_2)\theta(1-x_1-x_2)\\&\quad+g^{(1)}(x_1)g^{(1)}(x_2)\varphi_2(x_1,x_2)\,.\end{aligned} \tag{A21}$$

Equation (A21) reveals that $\overline{g}^{(2)}$ is composed of two terms. The first term, $g^{(2)}(x_1,x_2)\theta(1-x_1-x_2)$, contains all the dynamical correlations, whereas the second term contains only the kinematical correlations. Note that $\varphi_2(x_1,x_2)$ vanishes as any of the x's$\to 0$. Based on the results given in Sec. A1, it is easy to see that the second term in (A21) does not contribute to any $\ln s$ factor in $\overline{G}^{(2)}$. Thus the entire $\ln s$ factor in $\overline{G}^{(2)}$ comes from the first term and hence can at most be linear in $\ln s$.

A similar technique can be applied to $\overline{g}^{(3)}$ and indeed to $\overline{g}^{(n)}$ as well. For simplicity, we denote

$$g^{(3)}(x_1,x_2,x_3) \text{ by } g^{(3)}(1,2,3)\,,$$

$$\varphi_2(x_1,x_2+x_3) \text{ by } \varphi_2(1,2+3)\,, \tag{A22}$$

and

$$\varphi_1(x_1+x_2+x_3) \text{ by } \varphi_1(1+2+3)\,,$$

and similarly for all other functions of the x_i. We then have

$$\begin{aligned}\overline{g}^{(3)}(1,2,3)&=g^{(3)}(1,2,3)\varphi_1(1+2+3)\\&\quad+g^{(1)}(1)g^{(2)}(2,3)\varphi_2(1,2+3)\\&\quad+g^{(1)}(2)g^{(2)}(1,3)\varphi_2(2,1+3)\\&\quad+g^{(1)}(3)g^{(2)}(1,2)\varphi_2(3,1+2)\\&\quad+g^{(1)}(1)g^{(1)}(2)g^{(1)}(3)\varphi_3(1,2,3)\,.\end{aligned} \tag{A23}$$

The first term in (A23) represents the purely dynamical correlations, and the last term represents the purely kinematical correlations. All the remaining terms represent combinations of the two effects.

It is straightforward to see that the entire $\ln s$ dependence of $\overline{G}^{(3)}$ comes from the integration of the first term in (A23) and consequently that this dependence can at most be linear in $\ln s$. To understand this result, we consider a typical term in $\overline{G}^{(3)}$, say,

$$\int_{1/s}^{1} \frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{dx_3}{x_3}\, g^{(1)}(1) g^{(2)}(2,3)\varphi_2(1,2+3)\,. \tag{A24}$$

Since $g^{(2)}(2,3)$ has a finite correlation length, the integration over the relative coordinate leads to a well-behaved function even as $1/s \to 0$,

$$\int_0^1 \frac{dx_2}{x_2}\frac{dx_3}{x_3}\, g^{(2)}(2,3)\times\delta(x_2+x_3-x) \equiv h(x)\,. \tag{A25}$$

At small x_1, we obtain from the discussion in Sec. A1

$$\int_0^1 \frac{dx_2}{x_2}\frac{dx_3}{x_3}\, g^{(2)}(2,3)\varphi_2(1,2+3) = \int \frac{dx}{x} h(x)\varphi_2(x_1,x) = O(x_1)\,. \tag{A26}$$

Equation (A26) implies that (A24) does not contribute any $\ln s$ factors.

Similar methods are applicable to the analysis of the partially integrated amplitudes. Consider, for example,

$$\overline{\tau}_2^3(1,2) = \int_{1/s}^{1} \frac{dx_3}{x_3}\, \overline{g}^{(3)}(1,2,3)\,. \tag{A27}$$

As $x_1 \to 0$ and if $1-x_2 \neq O(x_1)$, one can verify that the contributions to $\overline{\tau}_2^3(1,2)$ from the last four terms in (A23) approach zero. Further, the contribution from the first term leads to

$$\int \frac{dx_3}{x_3}\, g^{(3)}(1,2,3)\theta(1-x_1-x_2-x_3) \tag{A28}$$

which has a finite correlation length. Hence, $\overline{\tau}_2^3(1,2)$ has a finite correlation length as well.

The above analysis can be generalized to all orders. In particular, one can show that

$$\overline{g}^{(N)}(1,2,\ldots,N) = \sum_{\text{all partitions}} [g^{(n_1)}(\{n_1\}) g^{(n_2)}(\{n_2\})\cdots g^{(n_r)}(\{n_r\})\varphi_{(r)}(\textstyle\sum\{n_1\},\sum\{n_2\},\ldots,\sum\{n_r\})]\,, \tag{A29}$$

where $\{N\}$ stands for $(1,2,\ldots,N)$, $\{n_1\}$ is a subset of $\{N\}$ with $\{N\}=\{n_1\}\oplus\{n_2\}\oplus\cdots\oplus\{n_r\}$, and $\sum\{n_1\}$ stands for $\sum_{i\in\{n_1\}} x_i$. Equations (A21) and (A23) are special cases of (A29). Using (A29) and the results of Sec. A1, we can show that the only term in (A29) which survives in a partially integrated cluster function as any $x_i \to 0$ is the term

$$g^{(N)}(1,2,\ldots,N)\varphi_1(1+2+\cdots+N) = g^{(N)}(1,2,\ldots,N)\theta(1-x_1-x_2-\cdots-x_N)\,. \tag{A30}$$

This is also the only term which contributes to the $\ln s$ factor in the fully integrated cluster function. Hence the $\ln s$ dependence is linear. Since $g^{(N)}(1,2,\ldots,N)$ has a finite correlation length between any two pairs of x's, we conclude that the $\overline{\tau}_N^m$ also have finite correlation lengths.

3. The Kinematical Boundary

We have thus far restricted our considerations to the inclusive cross sections away from the kinematical boundary: That is, we have considered the case $1-\sum x_i = O(1)$. In this section, we wish to study the properties of the inclusive cross sections near the kinematical boundary, in the region where $1-\sum x_i \ll 1$. We shall examine two separate cases:

(1) $f^{(n)}$ remains finite as $\sum x_i \to 1$

and

(2) $f^{(n)} \sim (1-\sum x_i)^{-a}$ as $\sum x_i \to 1$.

To demonstrate the essential points, let us consider the two-particle inclusive cross sections with $1 \gg 1-x_1-x_2 > 0$.

If $f^{(n)}(1,2,\ldots,n)$ remains finite (or approaches zero) as $x_1+x_2 \to 1$, we have for $1 \gg 1-x_1-x_2 > \sum_{i=3}^n x_i$ (and hence $x_1+x_2 \gg \sum_{i=3}^n x_i$),

$$f^{(n)}(1,2,\ldots,n) = f^{(2)}(1,2) f^{(n-2)}(3,\ldots,n) + O(1-x_1-x_2)\,. \tag{A31}$$

This is, of course, the basic factorization property of f_n. Thus, we have, to within $O(1-x_1-x_2)$,

$$\frac{d^2\sigma(1,2)}{\sigma_0} = \lambda^2 f^{(2)}(1,2)\frac{dx_1}{x_1}\frac{dx_2}{x_2}\sum \frac{\lambda^{n-2}}{(n-2)!}\int_{1/s}^{1}\frac{dx_3}{x_3}\cdots\frac{dx_n}{x_n} f^{(n-2)}(3,\ldots,n)\,\theta\Big(1-x_1-x_2-\sum_{i=3}^n x_i\Big)\,. \tag{A32}$$

The contribution due to the undetected particles exponentiates into

$$\sum \frac{\lambda^{n-2}}{(n-2)!} \int_{1/s}^{1} \frac{dx_3}{x_3} \cdots \frac{dx_n}{x_n} f^{(n-2)}(3, 4, \ldots, n) \times \theta\left(1 - x_1 - x_2 - \sum_{i=3}^{n} x_i\right) = \gamma(\lambda)[(1 - x_1 - x_2)s]^{\alpha(\lambda)} . \quad \text{(A33)}$$

Recall that the total cross section exponentiates into

$$\sigma_T/\sigma_0 = \beta(\lambda) s^{\alpha(\lambda)} . \quad \text{(A34)}$$

Since the trajectory function $\alpha(\lambda)$ is determined by the contribution from the small-x region (i.e., the pionization region), both expressions (A33) and (A34) have the same $\alpha(\lambda)$ as indicated. Thus, the s dependence disappears in the quotient

$$\frac{d^2\sigma(1,2)}{\sigma_T} = \frac{d^2\sigma(1,2)/\sigma_0}{\sigma_T/\sigma_0} = \lambda^2 \frac{\gamma(\lambda)}{\beta(\lambda)} f^{(2)}(1,2)(1 - x_1 - x_2)^{\alpha(\lambda)} \frac{dx_1}{x_1} \frac{dx_2}{x_2} . \quad \text{(A35)}$$

Note that $\alpha(\lambda) > 0$ and $f^{(2)}(1,2)$ is finite. Hence, the inclusive cross section $d^2\sigma/\sigma_T$ *vanishes* like $O((1 - x_1 - x_2)^{\alpha})$ at the kinematical boundary. Thus, the correlated part of the two-particle inclusive cross section, $\bar{\tau}_2(1,2) = d^2\sigma_c/\sigma_T$, vanishes as $x_1 \gg x_2$, either as $O(x_2/(1-x_1))$ if $1 - x_1 \gg x_2$, or as $O((1 - x_1 - x_2)^{\alpha})$ if $1 \gg (1 - x_1, x_2)$ and $1 - x_1$ is of the same order as x_2. Therefore, $\bar{\tau}_2(1,2)$ has a finite correlation length. Similarly, in this case the correlated parts of all inclusive cross sections have finite correlation lengths even when the sum of the x's is near the kinematical boundary.

The situation is more subtle if $f^{(n)}(1, \ldots, n)$ blows up like $(1 - x_1 - x_2 - \cdots - x_n)^{-a}$, $0 < a < 1$, near the kinematical boundary Then, Eq. (A31) may not be valid because of the singular nature of $f^{(n)}$ near $x_1 + x_2 = 1$. However, one can assert that since

$$h^{(n)}(1, 2, \ldots, n) \equiv (1 - x_1 - \cdots - x_n)^a f^{(n)}(1, 2, \ldots, n) \quad \text{(A36)}$$

is finite near the kinematical boundary, one has for $x_1 + x_2 \approx 1$ and $\sum_{i=3}^{n} x_i \ll 1$

$$h^{(n)}(1, 2, \ldots, n) = h^{(2)}(1,2) h^{(n-2)}(3, \ldots, n) + O(1 - x_1 - x_2) \quad \text{(A37)}$$

or

$$f^{(n)}(1, 2, \ldots, n) = \frac{h^{(2)}(1,2) f^{(n-2)}(3, \ldots, n)}{(1 - x_1 - x_2 - \sum x_i)^a} \times [1 + O(1 - x_1 - x_2)] . \quad \text{(A38)}$$

Now, the inclusive cross section can be written to within a fractional error of $O(1 - x_1 - x_2)$ as

$$\frac{d^2\sigma(1,2)}{\sigma_0} = \lambda^2 h^{(2)}(1,2) \sum \frac{\lambda^{n-2}}{(n-2)!} \int_{1/s}^{1} \frac{dx_3}{x_3} \cdots \frac{dx_n}{x_n} \frac{f^{(n-2)}(3, \ldots, n)}{\left(1 - x_1 - x_2 - \sum_{i=3}^{n} x_i\right)^a} \theta(1 - x_1 - \cdots - x_n)$$
$$= \frac{\lambda^2 h^{(2)}(1,2)}{(1 - x_1 - x_2)^a} \sum \frac{\lambda^{n-2}}{(n-2)!} \int_{[(1-x_1-x_2)s]^{-1}}^{1} \frac{dy_3}{y_3} \cdots \frac{dy_n}{y_n} \frac{f^{(n-2)}(3, \ldots, n)}{(1 - y_3 - \cdots - y_n)^a} \theta(1 - y_3 - y_4 - \cdots - y_n), \quad \text{(A39)}$$

with $y_i = x_i/(1 - x_1 - x_2)$. The kinematic restrictions require $x_3, \ldots, x_n$ to be small. Then the invariance of $f^{(n)}$ under scale transformations in the pionization region implies that

$$f^{(n-2)}(3, \ldots, n) \equiv f^{(n-2)}(x_3, \ldots, x_n) = f^{(n-2)}(y_3, \ldots, y_n) . \quad \text{(A40)}$$

Thus, the sum of the remaining y integrations leads to $\gamma(\lambda)[(1 - x_1 - x_2)s]^{\alpha(\lambda)}$ as before, whereas the total cross section σ_T/σ_0 exponentiates to $\beta(\lambda)s^{\alpha(\lambda)}$. Dividing $d^2\sigma/\sigma_0$ by σ_T/σ_0 we obtain to within a fractional error of $O(1 - x_1 - x_2)$

$$\frac{d^2\sigma(1,2)}{\sigma_T} = \frac{\lambda^2\gamma(\lambda)}{\beta(\lambda)} \frac{h^{(2)}(1,2)}{(1 - x_1 - x_2)^a} (1 - x_1 - x_2)^{\alpha} \frac{dx_1}{x_1} \frac{dx_2}{x_2} = \frac{\lambda^2\gamma(\lambda)}{\beta(\lambda)} h^{(2)}(1,2)(1 - x_1 - x_2)^{\alpha - a} \frac{dx_1}{x_1} \frac{dx_2}{x_2} . \quad \text{(A41)}$$

Since $h^{(2)}(1,2)$ remains finite near the kinematical boundary, the properly normalized inclusive cross section vanishes near the kinematical boundary if $\alpha(\lambda) > a$ but becomes singular if $a > \alpha(\lambda)$. Thus, for $\alpha(\lambda) > a$, the inclusive cross sections have finite correlation lengths as demonstrated earlier. However, for $a > \alpha(\lambda)$, the inclusive cross section diverges at the kinematical boundary. The factorization for the inclusive cross sections then fails at the boundary. Actually, this is not surprising. At small λ [and hence at small $\alpha(\lambda)$] the inclusive spectrum approaches the exclusive spectrum. Thus, the inclusive spectrum blows up if the exclusive spectrum does. The failure of the factorization of the inclusive spectrum at the kinematical boundary is a reflection of the failure of the factorization of the exclusive spectrum near the boundary. We find, however, that the inclusive spectrum is always better behaved than the exclusive spectrum near the kinematical boundary. This is especially so if the coupling is strong.

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*Work supported in part by the National Science Foundation under Grant No. NSF GP 25303.

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[6]A. H. Mueller [Phys. Rev. D 4, 150 (1971)] has recently derived many interesting relations on multiplicity distributions for inclusive cross sections based on the idea of cluster decomposition.

[7]For an explicit verification of the cluster technique in the calculation in the ladder amplitudes in a (3+1)-dimensional φ^3 theory, see S.-J. Chang, T.-M. Yan, and Y. P. Yao, Phys. Rev. D 4, 3012 (1971).

[8]The absorptive part is chosen here to be positive. $A^{(n)}$ introduced here is simply related to the A_{n+1} defined in I.

[9]For a review of the infinite-momentum technique and the usefulness of the light-cone variables $q^{\pm}$, $\vec{q}$, see, e.g., S.-J. Chang and S. Ma, Phys. Rev. 180, 1506 (1969); 188, 2385 (1969).

[10]By $q_1^+ \gg q_2^+$, we mean that either $\lim_{s\to\infty} q_2^+/q_1^+ = 0$, or $\lim_{s\to\infty} q_2^+/q_1^+ = a_2/a_1$, where $a_{1,2}$ are s-independent and $a_1 \gg a_2$.

[11]See, e.g., J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[12]The justifications for writing this integral over the region $1 > x_3 > \mu^2/s$ are discussed briefly in Sec. III of this paper and in more detail in the sequel.

[13]F. Bopp (unpublished) has made a numerical calculation of the inclusive spectra, based on the (3+1)-dimensional φ^3 ladder amplitude. His results agreed qualitatively with our (1+1)-dimensional calculation.

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