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Motion of Spin-1 Particles in a Homogeneous Magnetic Field – Multispinor Formalism*

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A general method of calculation within the multispinor formalism is proposed. It is then used to calculate the eigenvalues of the theory of particles with any spin moving in a homogeneous magnetic field, without explicitly solving the equation of motion. The spin-1 theory with anomalous-magnetic-moment couplings is examined specifically. The results differ from those obtained by Tsai and Yildiz using the vector theory, and from those obtained by Goldman and Tsai and by Krase, Lu, and Good using the six-component theory. It is found that the discrepancies are due to the fact that different nonminimal couplings are in fact added to various theories. However, when only minimal couplings are considered, all three theories predict the same eigenvalues. In this case, the square of an eigenvalue is a perfect square and is positive definite. In the case where nonminimal couplings are added, the square of an eigenvalue can become negative in all three theories, i.e., the energy eigenvalues can become pure imaginary. Possible physical interpretations of the results are discussed.

I. INTRODUCTION

It is well known that systems with integer spins can be described by tensor fields, while systems with half-integer spins can be described by tensor-spinor fields.[1] It is also known that they can be alternatively described by multispinors with definite symmetry properties.[2] So far, most calculations involving higher-spin fields have used tensor or tensor-spinor theory. The reason is that, while in the former theory the calculation method is well known, in the latter no method exists at the present stage. It is important to find a general method of calculation within the multispinor formalism.

The problem of the motion of charged particles in an external electromagnetic field has been with us for a long time.[3] Until recently, the only cases discussed have been the spin-0 and spin-$\frac{1}{2}$ systems. Recently, the spin-1 system with anomalous-magnetic-moment couplings has been discussed by Tsai and Yildiz[4] in the vector theory, and by Goldman and Tsai[5] and by Krase, Lu, and Good[6] in the six-component theory. It is found that the squares of the energy eigenvalues can become negative and that the results obtained from the two theories are different.[4,5] It is interesting to see whether pure-imaginary eigenvalues also occur in the multispinor theory, and to compare the results obtained from various theories.

The purpose of this paper is to propose a general method of calculation within the multispinor formalism.[7] The method is then applied to calculate the eigenvalues of the theories of spin-$\frac{1}{2}$ and spin-1 particles, with anomalous magnetic moment (a.m.m.), moving in a homogeneous magnetic field. By starting from the eigenequation and by eliminating the dependent variables, we obtain an eigenequation for the independent components of the eigenfunctions. It is at this stage that comparison among various spin-1 theories is made. We found that, when only minimal coupling is considered, all of them have exactly the same form and hence predict the same eigenvalues. However, when a.m.m. couplings are added, they differ by terms proportional to κ, the strength of the a.m.m., and by terms of order H^2 or higher. The discrepancies are due to the fact that different nonminimal couplings are in fact added to various theories. In Sec. II, the general method of approach is presented for the spin-$\frac{1}{2}$ case. It is then extended, in Sec. III, to the spin-1 case and applied to solve the eigenvalue problem. The eigenvalues are solved for explicitly, and comparisons among various spin-1 theories are made. In Sec. IV, the eigenvalues are discussed, as well as possible physical interpretations of the results.

II. METHOD OF APPROACH

In this section, we give a detailed discussion of the method of approach for the spin-$\frac{1}{2}$ system. The techniques learned here can be directly applied to multispinors of higher rank.

To simplify our argument and for later application to the eigenvalue problem, we consider only the case when particles are moving in a homogeneous magnetic field, which is chosen to be in the z direction. The other cases can be similarly discussed. Under this condition, the eigenvalue equation is

$$\left(m-\gamma^0 p^0+\vec{\gamma}\cdot\vec{\pi}-\frac{\kappa}{2m}eq\vec{\sigma}\cdot\vec{\mathrm{H}}\right)\psi(\vec{\mathrm{r}})=0, \qquad (1)$$

with

$$\vec{\pi} = \frac{1}{i}\vec{\nabla} - eq\vec{A}(\vec{r}).$$

Our calculation method is first to decompose the identity into

$$\delta_{\alpha\alpha'} = (\Lambda^{(+)})_{\alpha\alpha'} + (\Lambda^{(-)})_{\alpha\alpha'}, \tag{2}$$

in which

$$\Lambda^{(\pm)} = \tfrac{1}{2}(1 \pm \gamma^0).$$

The latter quantities satisfy the following relations:

$$\Lambda^{(\pm)}\gamma^0 = \pm\Lambda^{(\pm)}, \quad \Lambda^{(\pm)}\sigma_k = \sigma_k\Lambda^{(\pm)},$$

$$\Lambda^{(\pm)}\Lambda^{(\pm)} = \Lambda^{(\pm)}, \quad \Lambda^{(\pm)}\Lambda^{(\mp)} = 0,$$

$$\Lambda^{(\pm)}i\gamma_5 = i\gamma_5\Lambda^{(\mp)}, \quad \Lambda^{(\pm)}\gamma_k = \pm\sigma_k i\gamma_5\Lambda^{(\mp)}.$$

With this decomposition and from Eq. (1), we obtain

$$(M - p^0)\psi^{(+)} = -(\vec{\sigma}\cdot\vec{\pi})(i\gamma_5)\psi^{(-)}, \tag{3}$$

$$(M + p^0)\psi^{(-)} = +(\vec{\sigma}\cdot\vec{\pi})(i\gamma_5)\psi^{(+)}, \tag{4}$$

where

$$M = m - \lambda\vec{\sigma}\cdot\vec{H}, \quad \lambda = \frac{\kappa}{2m}eq, \quad \psi^{(\pm)} = \Lambda^{(\pm)}\psi.$$

Then by defining

$$\psi_\pm = \psi^{(+)} \pm \psi^{(-)}, \quad \Psi = \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix},$$

we may rewrite Eqs. (3) and (4) as

$$p^0\psi_+ = M\psi_- + (\vec{\sigma}\cdot\vec{\pi})(i\gamma_5)\psi_+,$$

$$p^0\psi_- = M\psi_+ - (\vec{\sigma}\cdot\vec{\pi})(i\gamma_5)\psi_-,$$

which can be combined to yield

$$p^0\Psi = [M\rho_1 + (\vec{\sigma}\cdot\vec{\pi})(i\gamma_5)\rho_3]\Psi, \tag{5}$$

where ρ_i is the Pauli spin matrix. This is the eigenvalue equation we want to study.

The method to obtain the eigenvalues of Eq. (5) is as follows. By multiplying Eq. (5) from the left by p^0, we obtain

$$(p^0)^2\Psi = [M^2 + \vec{\pi}^2 - eq\vec{\sigma}\cdot\vec{H} + \alpha(i\vec{\sigma}\cdot\vec{H}\times\vec{\pi})]\Psi, \tag{6}$$

which may be rewritten as

$$x\Psi = \alpha(i\vec{\sigma}\cdot\vec{H}\times\vec{\pi})\Psi, \tag{7}$$

with

$$x = (p^0)^2 - (M^2 + \vec{\pi}^2 - eq\vec{\sigma}\cdot\vec{H}),$$
$$\alpha = 2\lambda(i\rho_2)(i\gamma_5), \quad \alpha^2 = -4\lambda^2. \tag{8}$$

The multiplication of Eq. (7) by $\alpha(i\vec{\sigma}\cdot\vec{H}\times\vec{\pi})$ yields

$$Y\Psi = \beta(\vec{\sigma}\cdot\vec{\pi}_\perp)\Psi, \tag{9}$$

where the relations

$$[x, i\vec{\sigma}\cdot\vec{H}\times\vec{\pi}] = 4m\lambda H^2(\vec{\sigma}\cdot\vec{\pi}_\perp),$$

$$(i\vec{\sigma}\cdot\vec{H}\times\vec{\pi})^2 = -H^2\vec{\pi}_\perp{}^2 - eq\vec{\sigma}\cdot\vec{H},$$

$$\beta = 4m\lambda H^2\alpha,$$

$$Y = x^2 + \alpha^2H^2(\vec{\pi}_\perp{}^2 - eq\vec{\sigma}\cdot\vec{H}),$$

are used. However, from the identity

$$H^2(\vec{\sigma}\cdot\vec{\pi}_\perp) = (\vec{\sigma}\cdot\vec{H})(i\vec{\sigma}\cdot\vec{H}\times\vec{\pi}),$$

Eq. (9) becomes

$$[Y - 4m\lambda(\vec{\sigma}\cdot\vec{H})x]\Psi = 0, \tag{10}$$

where we have used Eq. (7).

Now since Eq. (10) is a function of $\vec{\pi}_\perp{}^2$ and $\vec{\sigma}\cdot\vec{H}$ only, we may choose Ψ as an eigenfunction of $\vec{\pi}_\perp{}^2$ and σ_3, with $\vec{\pi}_\perp{}^2 = (2n+1)eH$ and $\sigma_3' = \pm 1$. The resulting characteristic equation becomes

$$x^2 - 4m\lambda(\vec{\sigma}\cdot\vec{H})x - 4\lambda^2H^2(\vec{\pi}_\perp{}^2 - eq\vec{\sigma}\cdot\vec{H}) = 0. \tag{11}$$

The solution of Eq. (11) which satisfies the weak-field result implied by Eq. (6) is

$$x = 2m\lambda(\vec{\sigma}\cdot\vec{H})\{1 - [1 + (\vec{\pi}_\perp{}^2 - eq\vec{\sigma}\cdot\vec{H})m^{-2}]^{1/2}\}.$$

More explicitly, the eigenvalues are

$$(p^0)^2 = m^2\left((1+\eta)^{1/2} - \frac{b}{2m^2}\right)^{1/2} + p_3{}^2, \tag{12}$$

where

$$b = 2m\lambda\vec{\sigma}\cdot\vec{H}, \quad \xi = eH/m^2, \quad \eta = (2n + 1 - q\sigma_3)\xi.$$

This is just the result obtained by Ternov *et al.*,[3] by using the differential-equation technique and solving for the eigenfunctions and eigenvalues of the system.

We see that even though the second-order form of the eigenvalue equation cannot be diagonalized, it can be diagonalized in the fourth-order form, and the eigenvalues are then easily obtained. The methods of decomposing the eigenfunctions into various parity subspaces and of going to the fourth-order form to diagonalize the eigenvalue equation are quite general and can be applied to any spin cases.

III. SPIN-1 CASE - MULTISPINOR OF SECOND RANK

The method illustrated in Sec. II will be applied in this section to calculate the eigenvalues of the spin-1 theory in the multispinor formalism. The eigenvalue equation, for the spin-1 particle with anomalous magnetic moment, is[2]

$$\left(m - \tfrac{1}{2}(\gamma_1{}^0 + \gamma_2{}^0)p^0 + \tfrac{1}{2}(\vec{\gamma}_1 + \vec{\gamma}_2)\cdot\vec{\pi} - \frac{\kappa}{2m}\tfrac{1}{2}eq(\vec{\sigma}_1 + \vec{\sigma}_2)\cdot\vec{H}\right)\psi = 0. \tag{13}$$

By applying the decomposition method described in

Sec. II to Eq. (13) with

$$\psi_{\alpha\beta}=\delta_{\alpha\alpha'}\delta_{\beta\beta'}\psi_{\alpha'\beta'}$$
$$=(\Lambda^{(+)}+\Lambda^{(-)})_{\alpha\alpha'}(\Lambda^{(+)}+\Lambda^{(-)})_{\beta\beta'}\psi_{\alpha'\beta'}$$
$$=\psi^{(++)}+\psi^{(-+)}+\psi^{(+-)}+\psi^{(--)},$$
$$\psi^{(\pm\pm)}{}_{\alpha\beta}=\Lambda^{(\pm)}{}_{\alpha\alpha'}\Lambda^{(\pm)}{}_{\beta\beta'}\psi_{\alpha'\beta'},$$
$$\psi^{(\pm\mp)}{}_{\alpha\beta}=\Lambda^{(\pm)}{}_{\alpha\alpha'}\Lambda^{(\mp)}{}_{\beta\beta'}\psi_{\alpha'\beta'},$$

we obtain

$$(M-p^0)\psi^{(++)}=-\tfrac{1}{2}[(\;)_1\psi^{(-+)}+(\;)_2\psi^{(+-)}],\qquad(14)$$
$$(M+p^0)\psi^{(--)}=\tfrac{1}{2}[(\;)_1\psi^{(+-)}+(\;)_2\psi^{(-+)}],\qquad(15)$$
$$M\psi^{(+-)}=-\tfrac{1}{2}[(\;)_1\psi^{(--)}+(\;)_2\psi^{(++)}],\qquad(16)$$
$$M\psi^{(-+)}=-\tfrac{1}{2}[-(\;)_1\psi^{(++)}+(\;)_2\psi^{(--)}],\qquad(17)$$

where

$$\vec{S}=\tfrac{1}{2}(\vec{\sigma}_1+\vec{\sigma}_2),\quad M=m-\lambda\vec{S}\cdot\vec{H},$$
$$\lambda=\frac{eq\kappa}{2m},\quad (\;)_i=(\vec{\sigma}\cdot\vec{\pi}_\perp)_i(i\gamma_5),$$
$$(\vec{\sigma}\cdot\vec{\pi}_\perp)_i=(\vec{\sigma}_i\cdot\vec{\pi}_\perp),\quad i=1,2.$$

Then by multiplying Eq. (14) from the left by M and using Eqs. (16) and (17), we obtain

$$[M(M-p^0)+\tfrac{1}{2}(\vec{\pi}_\perp{}^2-eq\vec{S}\cdot\vec{H})]\psi^{(++)}-\tfrac{1}{4}\{(\;)_1,(\;)_2\}\psi^{(--)}$$
$$=-\tfrac{1}{2}\{[M,(\;)_1]\psi^{(-+)}+[M,(\;)_2]\psi^{(+-)}\}.\qquad(18)$$

Further multiplication by M on the left-hand side of Eq. (18) yields

$$[(M^2-\lambda^2H^2)(M-p^0)+\tfrac{1}{2}M(\vec{\pi}_\perp{}^2-eq\vec{S}\cdot\vec{H})-\tfrac{1}{2}\lambda(\vec{\pi}_\perp{}^2\vec{S}\cdot\vec{H}-eqH^2)]\psi^{(++)}$$
$$=\{M[(\vec{S}\cdot\vec{\pi}_\perp)^2-\tfrac{1}{2}(\vec{\pi}_\perp{}^2-eq\vec{S}\cdot\vec{H})]-\lambda\vec{S}\cdot\vec{H}(\vec{S}\cdot\vec{\pi}_\perp)^2+\lambda eqH^2(1-S_3{}^2)+\tfrac{1}{2}\lambda\vec{\pi}_\perp{}^2\vec{S}\cdot\vec{H}\}(i\gamma_5)_1(i\gamma_5)_2\psi^{(--)},\qquad(19)$$

where we have used Eq. (14) and the relations

$$[M,(\vec{\sigma}\cdot\vec{\pi}_\perp)_i]=-\lambda(i\vec{\sigma}\cdot\vec{H}\times\vec{\pi})_i,$$
$$[M,[M,(\vec{\sigma}\cdot\vec{\pi}_\perp)_i]]=\lambda^2(\vec{\sigma}\cdot\vec{\pi}_\perp)_iH^2,$$
$$\tfrac{1}{4}[(i\vec{\sigma}\cdot\vec{H}\times\vec{\pi})_1(\vec{\sigma}\cdot\vec{\pi}_\perp)_2+(i\vec{\sigma}\cdot\vec{H}\times\vec{\pi})_2(\vec{\sigma}\cdot\vec{\pi}_\perp)_1]=(i\vec{S}\cdot\vec{H}\times\vec{\pi})(\vec{S}\cdot\vec{\pi}_\perp)-\tfrac{1}{2}\vec{\pi}_\perp{}^2\vec{S}\cdot\vec{H},$$
$$\tfrac{1}{4}\{(\vec{\sigma}\cdot\vec{\pi}_\perp)_1,(\vec{\sigma}\cdot\vec{\pi}_\perp)_2\}=(\vec{S}\cdot\vec{\pi}_\perp)^2-\tfrac{1}{2}(\vec{\pi}_\perp{}^2-eq\vec{S}\cdot\vec{H}),$$
$$\tfrac{1}{4}\{[M,(\vec{\sigma}\cdot\vec{\pi}_\perp)_1](\vec{\sigma}\cdot\vec{\pi}_\perp)_1+[M,(\vec{\sigma}\cdot\vec{\pi}_\perp)_2](\vec{\sigma}\cdot\vec{\pi}_\perp)_2\}=-(\lambda/2)(\vec{\pi}_\perp{}^2\vec{S}\cdot\vec{H}-eqH^2),$$

and we have chosen $\pi_3=0$ to simplify the calculation. Similarly we have

$$[(M^2-\lambda^2H^2)(M-p^0)+\tfrac{1}{2}M(\vec{\pi}_\perp{}^2-eq\vec{S}\cdot\vec{H})-(\lambda/2)(\vec{\pi}_\perp{}^2\vec{S}\cdot\vec{H}-eqH^2)]\psi^{(--)}$$
$$=\{M[(\vec{S}\cdot\vec{\pi}_\perp)^2-\tfrac{1}{2}(\vec{\pi}_\perp{}^2-eq\vec{S}\cdot\vec{H})]-\lambda\vec{S}\cdot\vec{H}(\vec{S}\cdot\vec{\pi}_\perp)^2+\lambda eqH^2(1-S_3{}^2)+\tfrac{1}{2}\lambda\vec{\pi}_\perp{}^2\vec{S}\cdot\vec{H}\}(i\gamma_5)_1(i\gamma_5)_2\psi^{(++)}.\qquad(20)$$

Equations (19) and (20) are the two basic eigenequations from which the eigenvalues are to be calculated.

Direct manipulation from Eqs. (19) and (20) is quite complicated. However, it becomes much simpler if we separate the eigenvalue equations for the $S_3=0$ and the $S_3=\pm1$ cases. This is accomplished by using the projection operators

$$P(S_3=0)=(1-S_3{}^2),\quad P(S_3=1)=S_3{}^2,$$

which satisfy the relations

$$[S_3{}^2,(\vec{S}\cdot\vec{\pi}_\perp)^2]=0,\quad S_3(1-S_3{}^2)=0,\quad(1-S_3{}^2)(\vec{S}\cdot\vec{\pi}_\perp)^2=\vec{\pi}_\perp{}^2(1-S_3{}^2).$$

In the following, we will discuss these two cases separately.

(a) $S_3=0$ *case*.

By applying $P(S_3=0)$ to the left-hand side of Eqs. (19) and (20), and by defining

$$\phi^{(\pm)}=(1-S_3{}^2)\psi^{(\pm\pm)},\quad(i\gamma_5)_1(i\gamma_5)_2\phi^{(\pm)}=\phi'^{(\pm)},$$

we have

$$[(m^2-\lambda^2H^2)(m-p^0)+\tfrac{1}{2}m\vec{\pi}_\perp{}^2+\tfrac{1}{2}\lambda^2eqH^2]\phi^{(+)}=(\tfrac{1}{2}m\vec{\pi}_\perp{}^2+\lambda eqH^2)\phi'^{(-)},\qquad(19')$$
$$[(m^2-\lambda^2H^2)(m+p^0)+\tfrac{1}{2}m\pi_\perp{}^2+\tfrac{1}{2}\lambda^2eqH^2]\phi^{(-)}=(\tfrac{1}{2}m\vec{\pi}_\perp{}^2+\lambda eqH^2)\phi'^{(+)}.\qquad(20')$$

The combination of Eqs. (19′) and (20′) yields

$$\{[m(m^2-\lambda^2H^2)+\tfrac{1}{2}(m\vec{\pi}_\perp{}^2+\lambda^2 eqH^2)]^2-(p^0)^2(m^2-\lambda^2H^2)^2\}\,\phi^{(\pm)}=(\tfrac{1}{2}m\vec{\pi}_\perp{}^2+\lambda eqH^2)^2\phi^{(\pm)}\,.$$

In the case when $m^2\neq\lambda^2H^2$, we obtain

$$\begin{aligned}(p^0)^2\phi^{(\pm)}&=\frac{m^2}{(m^2-\lambda^2H^2)^2}\left(m^2-\lambda^2H^2-\frac{eq}{2m}\lambda H^2\right)\left(m^2+\vec{\pi}_\perp{}^2-\lambda^2H^2+\frac{3eq\lambda H^2}{2m}\right)\phi^{(\pm)}\\&=\left\{m^2+\vec{\pi}_\perp{}^2+\frac{eq\lambda H^2}{m}+\frac{1}{m^2-\lambda^2H^2}\left[\left(\lambda H^2-\frac{eqH^2}{2m}\right)\left(\vec{\pi}_\perp{}^2+\frac{3eq\lambda H^2}{2m}\right)-\frac{eq\lambda^2H^2}{2m}\right]\right.\\&\quad\left.-\frac{1}{(m^2-\lambda^2H^2)^2}\,\frac{eq\lambda^3H^4}{2m}\left(\vec{\pi}_\perp{}^2+\frac{3eq\lambda H^2}{2m}\right)\right\}\phi^{(\pm)}\,.\end{aligned}\tag{21}$$

This is to be compared with the results obtained from vector theory and from six-component theory. They are

$$(p^0)^2\Phi=(m^2+\vec{\pi}_\perp{}^2)\Phi\quad\text{(vector theory)},\tag{22}$$

$$(p^0)^2\psi^{(\pm)}=[m^2+\vec{\pi}_\perp{}^2+\kappa e^2H^2(m^4-\kappa^2e^2H^2)^{-1}(m^2+\vec{\pi}_\perp{}^2)]\,\psi^{(\pm)}\quad\text{(six-component theory)}.\tag{23}$$

We see that Eqs. (21)–(23) differ from each other by terms proportional to κ and by terms of order H^2 or higher. This may be interpreted as due to the fact that the effect of adding a.m.m.c. in one theory is equivalent to the effect of adding more nonminimal-coupling terms, which are scalar functions constructed from $F_{\mu\nu}$, to the other theories.

From Eq. (21), we obtain the following eigenvalue for the multispinor theory:

$$(p^0)^2=m^2(m^2-\lambda^2H^2)^{-2}\left(m^2-\lambda^2H^2-\frac{eq\lambda}{2m}H^2\right)\left(m^2+(2n+1)eH-\lambda^2H^2+\frac{3eq\lambda H^2}{2m}\right).\tag{21′}$$

In the case when $m^2=\lambda^2H^2$, we have

$$(3\lambda eqH^2+m\vec{\pi}_\perp{}^2)(m\vec{\pi}_\perp{}^2+\lambda eqH^2)\phi^{(\pm)}=0,$$

which in general does not hold unless $\phi^{(\pm)}=0$.

(b) $S_3=\pm1$ *case*.

By applying $P(S_3=\pm1)$ to the left-hand side of Eqs. (19) and (20), we obtain

$$D(A\mp p^0)\phi^{(\pm)}=\frac{1}{m}\{D[(\vec{S}\cdot\vec{\pi}_\perp)^2-\tfrac{1}{2}(\vec{\pi}_\perp{}^2-eq\vec{S}\cdot\vec{H})]+\tfrac{1}{2}\lambda eqH^2\}\,\phi'^{(\mp)}\,,\tag{24}$$

where

$$A=m-\lambda\vec{S}\cdot\vec{H}+\frac{1}{2m}(\vec{\pi}_\perp{}^2-eq\vec{S}\cdot\vec{H}),\quad D=m-2\lambda\vec{S}\cdot\vec{H},$$

$$\phi^{(\pm)}=S_3{}^2\psi^{(\pm\pm)},\quad \phi'^{(\pm)}=(i\gamma_5)_1(i\gamma_5)_2\phi^{(\pm)}\,.$$

If $m^2\neq4\lambda^2H^2$, then D has an inverse. And it is easy to show that the inverse is

$$D^{-1}=\frac{m}{m^2-4\lambda^2H^2}\left(1+\frac{2\lambda}{m}\vec{S}\cdot\vec{H}-\frac{4\lambda^2H^2}{m^2}(1-S_3{}^2)\right).$$

Therefore, when $m^2\neq4\lambda^2H^2$, Eqs. (19) and (20) become

$$(A-p^0)\phi^{(+)}=B\phi^{(-)}\,,\tag{19′′}$$

$$(A+p^0)\phi^{(-)}=B\phi^{(+)}\,,\tag{20′′}$$

with

$$B=\frac{1}{m}\left[(\vec{S}\cdot\vec{\pi}_\perp)^2-\tfrac{1}{2}(\vec{\pi}_\perp{}^2-eq\vec{S}\cdot\vec{H})+a\left(1+\frac{2\lambda}{m}\vec{S}\cdot\vec{H}\right)\right]\times(i\gamma_5)_1(i\gamma_5)_2,$$

$$a=m(m^2-4\lambda^2H^2)^{-1}\tfrac{1}{2}\lambda eqH^2.$$

By defining

$$\psi_\pm=\phi^{(+)}\pm\phi^{(-)},\quad \Psi=\begin{pmatrix}\psi_+\\\psi_-\end{pmatrix},$$

one has

$$p^0\Psi=(A\rho_1+Bi\rho_2)\Psi\,,\tag{25}$$

or in second-order form, one has

$$(p^0)^2\Psi=\{A^2-B^2-\rho_3[A,B]\}\Psi\,.$$

Since

$$[A,B]=\left[-\lambda\vec{S}\cdot\vec{H}+\frac{eq}{2m}\vec{S}\cdot\vec{H},\frac{1}{m}(\vec{S}\cdot\vec{\pi})^2\right]=-\frac{eq(1-\kappa)H}{4m^2}N_-\,,$$

$$N_\pm=S_-{}^2\pi_+{}^2\pm S_+{}^2\pi_-{}^2,$$

we have

$$(p^0)^2\Psi=\Big[m^2+\vec{\pi}_\perp{}^2-eq(1+\kappa)\vec{S}\cdot\vec{H}+\frac{eq(1-\kappa)}{2m^2}\vec{S}\cdot\vec{H}(\vec{\pi}_\perp{}^2-eq\vec{S}\cdot\vec{H})+\lambda^2H^2 - \frac{a^2}{m^2}\Big(1+\frac{2\lambda}{m}\vec{S}\cdot\vec{H}\Big)^2-\frac{a}{2m^2}N_+ +\rho_3\frac{eq(1-\kappa)H}{4m^2}N_-(i\gamma_5)_1(i\gamma_5)_2\Big]\Psi \tag{26}$$

which is the eigenvalue equation to be solved. This is to be compared with the result obtained from both the vector theory and the six-component theory,

$$(p^0)^2\Phi=\Big[m^2+\vec{\pi}_\perp{}^2-eq(1+\kappa)\vec{S}\cdot\vec{H}+\frac{eq(1-\kappa)}{2m^2}\vec{S}\cdot\vec{H}(\vec{\pi}_\perp{}^2-eq\vec{S}\cdot\vec{H})\pm\frac{eqH(1-\kappa)}{4m^2}N_-\Big]\Phi. \tag{27}$$

Again the difference is due to anomalous-moment couplings and is of order H^2 or higher.

Our object is to solve Eq. (26) by using the method outlined in Sec. II. To do this, we rewrite it as

$$y\Psi=(\alpha_4+\alpha_5S_3)N_+\Psi, \tag{28}$$

where

$$y=(p^0)^2-(\alpha_1+\alpha_2\vec{S}\cdot\vec{H}),$$

$$\alpha_1=m^2+\vec{\pi}_\perp{}^2-2eq\vec{S}\cdot\vec{H}+\lambda^2H^2+\frac{e^2H^2}{2m^2}(1-\kappa)-\frac{a^2}{m^2}\Big(1+\frac{4\lambda^2H^2}{m^2}\Big),$$

$$\alpha_2=eq(1-\kappa)\Big(1+\frac{1}{2m^2}(\vec{\pi}_\perp{}^2-2eq\vec{S}\cdot\vec{H})\Big)-\frac{a^2}{m^2}\frac{4\lambda}{m},$$

$$\alpha_3=\frac{eq(1-\kappa)}{2m^2},\quad \alpha_4=-\frac{a}{2m^2},$$

$$\alpha_5=-\rho_3(i\gamma_5)_1(i\gamma_5)_2eq(1-\kappa)H(4m^2)^{-1},$$

and we have used the identity[7]

$$N_\pm=-S_3N_\mp.$$

The iteration procedure described in Secs. II and III (a) can be applied here. By multiplying Eq. (28) on the left by y, one has

$$[y^2-(\alpha_4{}^2-\alpha_5{}^2)N_+{}^2]\Psi=[y,(\alpha_4-\alpha_5S_3)N_+]\Psi, \tag{29}$$

which, by using Eq. (28), becomes

$$[y^2+2\alpha_2\vec{S}\cdot\vec{H}y-(\alpha_4{}^2-\alpha_5{}^2)b]\Psi=0, \tag{30}$$

where

$$N_+{}^2\Psi=4(\vec{\pi}_\perp{}^4+3e^2H^2-4eq\vec{S}\cdot\vec{H}\vec{\pi}_\perp{}^2)\Psi\equiv b\Psi.$$

And the eigenvalues are

$$y=-\alpha_2\vec{S}\cdot\vec{H}\pm[\alpha_2{}^2H^2+(\alpha_4{}^2-\alpha_5{}^2)b]^{1/2},$$

or more explicitly,

$$(p^0)^2=\alpha_1\pm[\alpha_2{}^2H^2+(\alpha_4{}^2-\alpha_5{}^2)b]^{1/2}. \tag{31}$$

The solution of Eq. (30) which satisfies the weak-field limit implied by Eq. (26) is

$$(p^0)^2=m^2\Big\{1+\eta+\tfrac{1}{2}\xi^2(1-\kappa+\tfrac{1}{2}\kappa^2)+qS_3\xi\Big((1-\kappa)(1+\tfrac{1}{2}\eta)-\frac{2\kappa a^2}{m^4}\Big)\times\Big[1-(\eta^2-\xi^2)\Big(\frac{(1-\kappa)^2}{4}-\frac{a^2}{m^4\xi^2}\Big)\Big((1-\kappa)(1+\tfrac{1}{2}\eta)-\frac{2\kappa a^2}{m^4}\Big)^{-2}\Big]^{1/2}\Big\}, \tag{32}$$

where

$$a=\tfrac{1}{4}m^2\kappa\xi^2(1-\kappa^2\xi^2)^{-1},\quad \xi=eH/m,$$

$$\eta=(2n+1-2qS_3)\xi.$$

Equation (32) is our final result. We note that as $\kappa=0$, Eq. (32) becomes

$$(p^0)^2=m^2[(1+\eta+\tfrac{1}{4}\xi^2)^{1/2}+\tfrac{1}{4}qS_3\xi\epsilon]^2, \tag{33}$$

with

$$\epsilon=(1+\tfrac{1}{2}\eta)(|1+\tfrac{1}{2}\eta|)^{-1},$$

which is the same result as obtained in the vector theory and in the six-component theory.

IV. CONCLUSIONS AND DISCUSSION

Equations (21′) and (32) are the eigenvalues of the eigenvalue equation (13). We see that both the method of calculation and the form of the result are quite different from those of the corresponding vector theory and the six-component theory. Some properties of the eigenvalues can be easily obtained by considering the following particular cases:

(i) From Eq. (21′), it is easy to show that $(p^0)^2$ can become negative when the inequality $-1<\kappa<3$ is satisfied.

(ii) In the case when $n=0$ and $qS_3=+1$, Eq. (32) becomes

$$(p^0)^2 = m^2\left((1-\tfrac{1}{2}\kappa\xi)^2 - \frac{\kappa^3\xi^5}{8(1-\kappa^2\xi^2)^2}\right), \qquad (34)$$

which is positive definite only when κ is a negative number.

(iii) When case (ii) is excluded, and when $\xi \gg 1$ and $\kappa \neq 0$, we have

$$(p^0)^2 \to \tfrac{1}{4} m^2\xi^2[\kappa^2 + 2(1+qS_3)(1-\kappa)], \qquad (35)$$

which can become negative when $qS_3 = +1$ and $2-2\sqrt{2} < \kappa < 2+2\sqrt{2}$. Therefore, from cases (i)–(iii), we conclude that, outside the region $2-2\sqrt{2} < \kappa < 0$, $(p^0)^2$ is not positive definite.

Up to now, all the three popular spin-1 theories – the vector theory, the six-component theory, and the multispinor theory – have been examined explicitly, and the pure-imaginary eigenvalues are found. We also found that the addition of a.m.m.c. in one theory corresponded in its effect to adding more coupling terms in the other theories – terms of order H^2 or higher. This suggests that the conventional way of adding the a.m.m.c. to the eigenvalue equation is only good for the weak-field case. In the strong magnetic field, more nonminimal-coupling terms, which correspond to the magnetic polarizability effects, must be added to the eigenvalue equation. It is interesting to find a way to add these terms such that $(p^0)^2$ becomes positive definite.

It has been suggested that the inconsistency of the result may be due to the omission of pair creation and radiative corrections. However, we should like to point out that these corrections cannot explain the following two points: (i) For sufficiently large H, p^0 is *pure imaginary* in spin-1 theory; (ii) the same processes occur in the spin-$\frac{1}{2}$ case and there is no indication of why $(p^0)^2$ is positive definite in the spin-$\frac{1}{2}$ theory while it is not positive definite in the spin-1 theory. We argue that the inconsistency is due to the formalism itself, not due to the incompletion of the processes.[8] The usual way of adding a.m.m.c. is good for the weak-field case only. For the strong-field case, other scalar functions of F must be added.

We finally remark that the method of decomposing the eigenfunction into various positive- and negative-parity subspaces is quite general and can be applied to other cases, especially for low-energy processes.[9]

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[1]M. Fierz and W. Pauli, Proc. Roy. Soc. (London) A173, 211 (1939); W. Rarita and J. Schwinger, Phys. Rev. 60, 61 (1941).

[2]V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. 34, 211 (1946); Julian Schwinger, *Particles, Sources and Fields* (Addison-Wesley, Reading, Mass. 1970).

[3]P. A. M. Dirac, Proc. Roy. Soc. (London) A117, 610 (1928); I. I. Rabi, Z. Physik 49, 507 (1928); F. Sauter, *ibid.* 69, 724 (1931); M. H. Johnson and B. A. Lippmann, Phys. Rev. 77, 702 (1948); V. Canuto and C. Chiuderi, Lett. Nuovo Cimento 2, 223 (1969); I. M. Ternov, V. G. Bargrov, and V. Chi. Zhukovskii, Moscow Univ. Phys. Bull. 21, 21 (1966); R. F. O'Connell, Phys. Letters 27A, 391 (1968); V. Canuto and H. Y. Chiu, Phys. Rev. 173, 1210 (1968); 173, 1220 (1968); 173, 1229 (1968).

[4]W. Tsai and A. Yildiz, this issue, Phys. Rev. D 4, 3643 (1971).

[5]T. Goldman and W. Tsai, preceding paper, Phys. Rev. D 4, 3648 (1971).

[6]L. D. Krase, Pao Lu, and R. H. Good, Phys. Rev. D 3, 1275 (1971).

[7]We thank Terrence Goldman for pointing out this identity.

[8]Recently, Goldman and Tsai have succeeded in showing that by adding infinitely many nonminimal-coupling terms to the vector theory, the square of the eigenvalues can become positive definite. See T. Goldman and W. Tsai, Phys. Letters 36B, 467 (1971).

[9]The same method has been used to find the nonrelativistic limit of the eigenvalue equation with all possible nonminimal couplings and to calculate the low-energy Compton scattering processes up to the multispinor of third rank. See Wu-yang Tsai, Harvard University thesis, 1971 (unpublished).