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Motion of Charged Particles in a Homogeneous Magnetic Field. II*

Terrence Goldman† and Wu-yang Tsai
Jefferson Physical Laboratory, Harvard University, Cambridge, Massachusetts 02138

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The general method proposed by Tsai and Yildiz to obtain the eigenvalues of any spin theory has been used to solve the eigenvalues of the 6-component theory of Shay and Good. This calculation method clearly demonstrates how the vector theory of Proca and Kemmer differs from the 6-component theory. An extra term which contributes to the $S_3=0$ case only is obtained in the 6-component theory, which explains the difference in the results obtained from the two different theories. With the result obtained, it is observed that the 6-component theory is consistent only when there is no anomalous-magnetic-moment coupling. Possible interpretations of the inconsistency are also discussed.

I. INTRODUCTION

Recently a general and simple method to obtain the eigenvalues of any spin theory has been proposed by Tsai and Yildiz.[1] With this method, they easily reproduced the spin-$\frac{1}{2}$ results of Ternov *et al.*[2] Furthermore, they went on to obtain the eigenvalues of the spin-1 theory of Proca and Kemmer[3] and observed that the spin-1 theory is consistent only when there is no anomalous-magnetic-moment coupling (a.m.m.c.). While the vector theory is inconsistent, it is important to see whether the inconsistency also occurs in the other spin-1 theories, namely, the multispinor theory[4] and the 6-component theory.[5]

The eigenvalues of the 6-component theory have been obtained recently by Krase, Lu, and Good[6] by using the quite complicated conventional method of solving the differential equation. However, their results are different from that of Paper I in the $S_3=0$ case. Hence, it is important to see whether the discrepancy comes from the theories themselves or from the calculational methods.

The purpose of this paper is to obtain the exact eigenvalues of the 6-component theory by using the simpler method of Paper I. By starting from the same eigenvalue equation as Ref. 6, we explicitly show that the discrepancy of the results obtained from the two theories is due to the theories themselves, not due to methods of calculation. An extra term which contributes to the $S_3=0$ case only is obtained in the 6-component theory. Furthermore, the eigenvalues are obtained explicitly and from the result obtained, we observe that the same inconsistency as in the vector theory also occurs in the 6-component theory with a.m.m.c. There is even an additional inconsistency in the 6-component theory at $\kappa^2e^2H^2=m^4$. Possible interpretations of the inconsistency common to both theories are discussed.

II. SPIN-1 EIGENVALUE PROBLEM

In this section, we obtain the eigenvalues of the eigenvalue equation of Shay and Good by using the method of Paper I. Before commencing the calculation, we briefly recapitulate the general method of Paper I, which can be summarized as the following three steps:

Step 1. The procedure starts from the eigenvalue equation. The dependent components of the eigenfunctions are eliminated from it by using the constraint equations devolved from the eigenvalue equation. This leaves an eigenvalue equation for the independent components of the eigenfunctions which is rewritten in matrix form by introducing or reducing to the spin matrices for the spin case in question.

Step 2. The eigenvalue equation is rearranged so that only terms which are a function of $\vec{S}\cdot\vec{H}$ and $\vec{\pi}_\perp{}^2$ appear on the left-hand side and all terms which do not commute with these appear on the right-hand side. The iteration-elimination procedure of Paper I is applied to this form of the equation just as in the spin-$\frac{1}{2}$ case described in Paper I to create a new equation in which every term is a function of $\vec{S}\cdot\vec{H}$ and $\vec{\pi}_\perp{}^2$ only. This is the characteristic equation in the usual matrix algebra sense.

Step 3. The eigenfunctions can now be chosen to be simultaneous eigenfunctions of $\vec{\pi}_\perp{}^2$ and $\vec{S}\cdot\vec{H}$ which reduces the characteristic equation to an algebraic one in $(p^0)^2$ of finite order. The roots of this algebraic equation are found and extraneous roots are eliminated by comparison with the result obtained from the original eigenvalue equation in particular simply calculable states, and by comparison with the approximate solution of the original equation in the weak-field case.

Following these procedures, we start from the eigenvalue equation of Good and Shay,[5,6] which can be written in the form

$$\{\tfrac{1}{2}(1+\rho_1)[\vec{\pi}^2-(p^0)^2]-\rho_1(\vec{S}\cdot\vec{\pi})^2-p^0 i\rho_2(\vec{S}\cdot\vec{\pi}) +m^2-\tfrac{1}{2}(\rho_1+\lambda)eq\vec{S}\cdot\vec{H}\}\psi=0\,, \tag{1}$$

where ρ_i is the Pauli spin matrix. By introducing

$$\psi^{(+)}=\tfrac{1}{2}(1+\rho_1)\psi,\quad \psi^{(-)}=\tfrac{1}{2}(1+\rho_1)i\rho_2\psi,\quad \kappa=\tfrac{1}{2}(\lambda-1)\,,$$

and by multiplying Eq. (1) from the left by $\frac{1}{2}(1+\rho_1)$ and by $\frac{1}{2}(1+\rho_1)i\rho_2$, respectively, we obtain

$$[\ \ ]\psi^{(+)}=p^0(\vec{S}\cdot\vec{\pi})\psi^{(-)}\,, \tag{2}$$

$$M\psi^{(-)}=-p^0(\vec{S}\cdot\vec{\pi})\psi^{(+)}\,, \tag{3}$$

with

$$[\ \ ]=m^2+\vec{\pi}^2-(p^0)^2-(\vec{S}\cdot\vec{\pi})^2-eq(1+\kappa)\vec{S}\cdot\vec{H}\,,$$

$$M=m^2+(\vec{S}\cdot\vec{\pi})^2-eq\kappa\vec{S}\cdot\vec{H}\,.$$

By multiplying Eq. (2) from the left by M and by using Eq. (3), we have

$$\{M[\ \ ]+(p^0)^2(\vec{S}\cdot\vec{\pi})^2\}\psi^{(+)}=-p^0 eq\kappa(i\vec{S}\cdot\vec{H}\times\vec{\pi})\psi^{(-)}\,, \tag{4}$$

where we have used the commutation relation

$$[M,\vec{S}\cdot\vec{\pi}]=-eq\kappa(i\vec{S}\cdot\vec{H}\times\vec{\pi})\,.$$

Further multiplication of Eq. (4) from the left by M yields

$$\{M^2[\ \ ]+(p^0)^2M(\vec{S}\cdot\vec{\pi})^2-(p^0)^2eq\kappa(i\vec{S}\cdot\vec{H}\times\vec{\pi})(\vec{S}\cdot\vec{\pi})\}\psi^{(+)} =-eq\kappa p^0[M,i\vec{S}\cdot\vec{H}\times\vec{\pi}]\psi^{(-)}\,. \tag{4$'$}$$

In the following, to simplify the calculation, we consider only the case when $\pi_3=0$. In this case, with the help of the relations

$$(\vec{S}\cdot\vec{\pi}_\perp)^4=(\vec{\pi}_\perp{}^2-2eq\vec{S}\cdot\vec{H})(\vec{S}\cdot\vec{\pi}_\perp)^2\,,$$

$$(i\vec{S}\cdot\vec{H}\times\vec{\pi})(\vec{S}\cdot\vec{\pi}_\perp)=(\vec{S}\cdot\vec{H})(\vec{S}\cdot\vec{\pi}_\perp)^2-eqH^2(1-S_3{}^2)\,,$$

$$\{(\vec{S}\cdot\vec{\pi}_\perp)^2,\vec{S}\cdot\vec{H}\}=(\vec{\pi}_\perp{}^2-eq\vec{S}\cdot\vec{H})(\vec{S}\cdot\vec{H})\,,$$

$$[\vec{S}\cdot\vec{H},i\vec{S}\cdot\vec{H}\times\vec{\pi}]=(\vec{S}\cdot\vec{\pi}_\perp)H^2\,,$$

$$[(\vec{S}\cdot\vec{\pi}_\perp)^2,i\vec{S}\cdot\vec{H}\times\vec{\pi}]=A(\vec{S}\cdot\vec{\pi}_\perp)+B(i\vec{S}\cdot\vec{H}\times\vec{\pi})\,,$$

$$A=-[2(\vec{S}\cdot\vec{H})\vec{\pi}_\perp{}^2-eqH^2+2eq(\vec{S}\cdot\vec{H})^2]\,,$$

$$B=\vec{\pi}_\perp{}^2+4eq\vec{S}\cdot\vec{H}\,,$$

we obtain

$$(p^0)^2L\psi^{(+)}=R\psi^{(+)}\,, \tag{5}$$

where

$$L=(M-B)(m^2-eq\kappa\vec{S}\cdot\vec{H})+eq\kappa A -\kappa^2e^2H^2+eq\kappa(i\vec{S}\cdot\vec{H}\times\vec{\pi})(\vec{S}\cdot\vec{\pi})\,,$$

$$R=[(M-B)M+eq\kappa A-\kappa^2e^2H^2] \times[m^2+\vec{\pi}_\perp{}^2-(\vec{S}\cdot\vec{\pi}_\perp)^2-(1+\kappa)eq(\vec{S}\cdot\vec{H})]\,.$$

If we let

$$R=L[m^2+\vec{\pi}_\perp{}^2-eq(1+\kappa)\vec{S}\cdot\vec{H}]-L(\vec{S}\cdot\vec{\pi}_\perp)^2+R'\,,$$

with

$$\begin{aligned}R'&=[(M-B)(\vec{S}\cdot\vec{\pi}_\perp)^2-eq\kappa(i\vec{S}\cdot\vec{H}\times\vec{\pi})(\vec{S}\cdot\vec{\pi}_\perp)]\\&\quad\times[m^2+\vec{\pi}_\perp{}^2-eq(1+\kappa)\vec{S}\cdot\vec{H}-(\vec{S}\cdot\vec{\pi}_\perp)^2]\\&=[m^2-(6+2\kappa)eq\vec{S}\cdot\vec{H}](\vec{S}\cdot\vec{\pi}_\perp)^2[m^2+eq(1-\kappa)\vec{S}\cdot\vec{H}]\\&\quad+\kappa m^2e^2H^2(1-S_3{}^2)\,,\end{aligned}$$

$$L(\vec{S}\cdot\vec{\pi}_\perp)^2=\{m^2[m^2-(6+2\kappa)eq\vec{S}\cdot\vec{H}] -\kappa^2e^2H^2(1-S_3{}^2)\}(\vec{S}\cdot\vec{\pi}_\perp)^2\,,$$

and use the relations

$$L(1-S_3{}^2)=(m^4-\kappa^2e^2H^2)(1-S_3{}^2)\,,$$

$$(1-S_3{}^2)(\vec{S}\cdot\vec{\pi}_\perp)^2=(1-S_3{}^2)\vec{\pi}_\perp{}^2\,,$$

$$L(\vec{S}\cdot\vec{\pi}_\perp)^2(\vec{S}\cdot\vec{H})=m^2[m^2-(6+2\kappa)eq\vec{S}\cdot\vec{H}] \times(\vec{S}\cdot\vec{\pi}_\perp)^2(\vec{S}\cdot\vec{H})\,,$$

it is straightforward to show that

$$R\psi^{(+)}=\{L[m^2+\vec{\pi}_\perp{}^2-eq(1+\kappa)\vec{S}\cdot\vec{H}+eq(1-\kappa)m^{-2}(\vec{S}\cdot\vec{\pi}_\perp)^2(\vec{S}\cdot\vec{H})]+\kappa e^2H^2(1-S_3{}^2)(m^2+\kappa\vec{\pi}_\perp{}^2)\}\psi^{(+)}$$
$$=L[m^2+\vec{\pi}_\perp{}^2-eq(1+\kappa)\vec{S}\cdot\vec{H}+eq(1-\kappa)m^{-2}(\vec{S}\cdot\vec{\pi}_\perp)^2(\vec{S}\cdot\vec{H})+\kappa e^2H^2(m^2+\kappa\vec{\pi}_\perp{}^2)(m^4-\kappa^2e^2H^2)^{-1}(1-S_3{}^2)]\psi^{(+)}, \quad (6)$$

where we assume that $m^4\neq\kappa^2e^2H^2$. The case when $m^4=\kappa^2e^2H^2$ will be discussed in Sec. III. Note that when $m^4\neq\kappa^2e^2H^2$, L has an inverse; the combination of Eqs. (5) and (6) gives

$$(p^0)^2\psi^{(+)}=[m^2+\vec{\pi}_\perp{}^2-eq(1+\kappa)(\vec{S}\cdot\vec{H})+eq(1-\kappa)m^{-2}(\vec{S}\cdot\vec{\pi}_\perp)^2(\vec{S}\cdot\vec{H})+\kappa e^2H^2(m^2+\kappa\vec{\pi}_\perp{}^2)(m^4-\kappa^2e^2H^2)^{-1}(1-S_3{}^2)]\psi^{(+)}. \quad (7)$$

Further simplification is achieved by the relations[1]

$$\{(\vec{S}\cdot\vec{\pi}_\perp)^2,\vec{S}\cdot\vec{H}\}=(\vec{\pi}_\perp{}^2-eq\vec{S}\cdot\vec{H})(\vec{S}\cdot\vec{H}),$$
$$(\vec{S}\cdot\vec{\pi}_\perp)^2\vec{S}\cdot\vec{H}=\tfrac{1}{2}(\vec{\pi}_\perp{}^2-eq\vec{S}\cdot\vec{H})+\tfrac{1}{4}HN_-,$$

and the final result is

$$(p^0)^2\psi^{(+)}=[m^2+\vec{\pi}_\perp{}^2-eq(1+\kappa)\vec{S}\cdot\vec{H}+eq(1-\kappa)(2m^2)^{-1}(\vec{\pi}_\perp{}^2-eq\vec{S}\cdot\vec{H})(\vec{S}\cdot\vec{H})$$
$$+eqH(1-\kappa)(4m^2)^{-1}N_-+\kappa e^2H^2(1-S_3{}^2)(m^2+\kappa\vec{\pi}_\perp{}^2)(m^4-\kappa^2e^2H^2)^{-1}]\psi^{(+)}, \quad (8)$$

which is equivalent to Eq. (18) (with $\pi_3=0$) in Paper I. We note that the coefficient of N_- differs in sign from that of Paper I. However, it does not affect the final results. The last term in Eq. (8), which contributes to $S_3=0$ only, is completely new in the 6-component theory which explains the different results obtained in Ref. 1 and Ref. 6.

The procedures to solve Eq. (7) are exactly the same as those outlined in Paper I, because of the relation

$$N_\pm(1-S_3{}^2)=0.$$

The solution of Eq. (8), which satisfies conditions (26) and (27) of Paper I, is

$$(p^0)^2=m^2\left\{\{[1+\eta+\tfrac{1}{4}(qS_3\xi)^2]^{1/2}+\tfrac{1}{2}\epsilon qS_3\xi\}^2-\tfrac{1}{2}\kappa(qS_3)^2\xi\left[\xi+qS_3(2+\eta)\left(1-\frac{\eta^2-\xi^2}{(2+\eta)^2}\right)^{1/2}\right]+\kappa\xi^2\frac{1+(2n+1)\kappa\xi}{1-\kappa^2\xi^2}(1-S_3{}^2)\right\}, \quad (9)$$

for $m^4\neq\kappa^2e^2H^2$.

III. CONCLUSIONS AND DISCUSSION

Even though the 6-component theory looks quite different from the vector theory, the decomposition techniques [Eqs. (2) and (3)] and the iteration procedures [Eqs. (4) and (5)] simplify the eigenvalue equation (1) and bring it to the form of Eq. (8) which is similar to Eq. (18) of Paper I. In this way, we can easily compare the two theories. And they differ from each other by a term contributing to the $S_3=0$ case only which explains the two different results obtained by Paper I and Ref. 6.

From Eq. (9), by letting $S_3=0$, we have

$$(p^0)^2=\left(1-\frac{\kappa^2e^2H^2}{m^4}\right)^{-1}\left(m^2+(2n+1)eH+\kappa(1-\kappa)\frac{e^2H^2}{m^2}\right), \quad (10)$$

which is exactly the same result as Eq. (15a) in Ref. 6 while it differs from Eq. (25) in Paper I. As we stated before, the discrepancy is due to the theories themselves, not due to the method of approach. This difference can be removed by adding a nonminimal-coupling term which is bilinear in $F_{\mu\nu}$ to the vector theory.[7]

We comment that the above solution can be obtained from Eq. (5) directly by multiplying Eq. (5) from the left by $\psi_{S_3=0}^{(+)*}$, the eigenfunction for which the eigenvalue of S_3 equals zero. In this way, we obtain

$$(m^4-\kappa^2e^2H^2)[(p^0)^2-m^2-(2n+1)eH]\times\int\psi_{S_3=0}(\vec{r})^*\psi_{S_3=0}(\vec{r})d\vec{r}$$
$$=\kappa e^2H^2[m^2+\kappa(2n+1)eH]\times\int\psi_{S_3=0}(\vec{r})^*\psi_{S_3=0}(\vec{r})d\vec{r}.$$

When $m^4\neq\kappa^2e^2H^2$, we get back to Eq. (10) which checks our calculation. However, in the case when $m^4=\kappa^2e^2H^2$, the left-hand side is zero while the right-hand side is not and we have a contradictory result. Furthermore, inconsistency also occurs in Eq. (10) if we require that $(p^0)^2>0$. This makes the 6-component theory more unpleasant than the vector theory.

In the case when $S_3 = \pm 1$, we have

$$(p^0)^2 = m^2 \left\{ [(1+\eta+\tfrac{1}{4}\xi^2)^{1/2} + \tfrac{1}{2}\epsilon q S_3 \xi]^2 - \tfrac{1}{2}\kappa\xi \left[\xi + qS_3(2+\eta)\left(1 - \frac{\eta^2 - \xi^2}{(2+\eta)^2}\right)^{1/2}\right] \right\}, \tag{11}$$

which is exactly the same result as Eq. (28) in Paper I. We observe that, from Eq. (9), $(p^0)^2$ is positive definite only when $\kappa = 0$. Therefore, we conclude that the 6-component spin-1 theory is consistent only when $\kappa = 0$.

We note that while there are at present three popular spin-1 theories, two of them have now been found to be consistent only when $\kappa = 0$; it remains to be seen whether the inconsistency also occurs in the multispinor theory of second rank.[8] The eigenvalues given in Eq. (8) do not include corrections such as pair creation and radiative corrections. It has been suggested that the omission of these corrections may be the cause of the inconsistency. While there is no existing method which tells us how to calculate these corrections even in the spin-$\frac{1}{2}$ theory with a.m.m.c., it is unclear whether these effects will cancel out the inconsistency of the theory. We would like to point out that these corrections cannot explain the following two points: (1) For sufficiently large values of H, p^0 is *pure imaginary* in the spin-1 theory; (2) the same processes occur for spin $\frac{1}{2}$ and there is no indication of why $(p^0)^2$ is positive definite in spin-$\frac{1}{2}$ theory while it is not positive definite in spin-1 theory.

We argue that when we start from an equation which is inconsistent, it is unlikely that a consistent result will be obtained. The disease is in the formalism itself, not in the incompletion of the processes. A possible interpretation is that the spin-1 particle with anomalous magnetic moment is a composite particle and it simply decomposes into lower-spin particles at sufficiently high magnetic field. We also comment that it is more interesting to find an argument or a method to make the spin-1 theory consistent. One of the possible ways to remove the inconsistency is to start with the assumption that κ is H-dependent. The other is to find a new way to introduce the electromagnetic interactions into the free-particle Lagrangian.

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