[36]S. Okubo and V. Mathur, Phys. Rev. D 2, 394 (1970).
[37]Physically, only continuous automorphisms are relevant.
[38]M. Gell-Mann, R. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).
[39]S. Glashow and S. Weinberg, Phys. Rev. Letters 20, 224 (1968).
[40]S. Weinberg, Phys. Rev. Letters 17, 616 (1966).
[41]We may also do the reverse and assign the lighter particle to be the "spinor." This freedom is not to be confused with the statement that there is no "intrinsic" difference between ψ_A and ψ_B. See also Ref. 25.
[42]S. Okubo, Phys. Rev. D 2, 3005 (1970). See also K. Mahanthappa and L. Maiani, Phys. Letters 33B, 499 (1970).
[43]We also thought this to be the case. (See Ref. 5.)
[44]R. Dashen, Phys. Rev. D 3, 1879 (1971).

PHYSICAL REVIEW D VOLUME 4, NUMBER 12 15 DECEMBER 1971

Structure of Internal-Symmetry Groups*

T. K. Kuo
Department of Physics, Purdue University, Lafayette, Indiana 47907
(Received 3 March 1971)

It is shown that, given SU(2) and T invariance, their outer automorphisms, which must themselves be symmetries, form a one-parameter gauge group. Thus, from isospin conservation and time-reversal invariance, one gets hypercharge conservation. Similarly, from angular momentum conservation and T invariance, one has fermion-number conservation. Further, the well-known empirical relations $(-1)^Y=(-1)^{2I}$ and $(-1)^F=(-1)^{2J}$ are derived.

I. INTRODUCTION

Recently[1] it was emphasized that the automorphisms of internal-symmetry groups are inherent conventions associated with their physical applications. Since there is a one-to-one correspondence between conventions and symmetries, this means that the outer automorphisms of internal-symmetry groups play an important role. In fact, according to the discussions in I, if one has an internal-symmetry group G containing a degenerate subgroup S, then outer automorphisms of G which leave S invariant are themselves symmetries. These new symmetries are in general hidden symmetries. Indeed, in the numerous examples investigated in I, we could not decide whether the derived symmetries were degenerate or hidden. The identification can only be done by comparison with reality.

Let us emphasize that there is no fundamental difference between hidden and degenerate symmetries. (In this connection, we may remark that the familiar term "spontaneously broken" symmetry is somewhat misleading.) If G is a symmetry group, and if S is a subgroup of G, then hidden symmetries arise when the physical states form irreducible representations, not of G, but of S. Indeed, the cosets of S in G constitute the hidden symmetries.

The physical significance of outer automorphisms may also be visualized in another way. When we have a (degenerate) symmetry group S (elements labeled as S^a) and some physical states (labeled as $|\alpha_i\rangle$), we are accustomed to thinking that the labeling has already been given. Further, an element S^c is understood to operate on the states and give rise to a reshuffling of them. Thus, $|\alpha_i\rangle \xrightarrow{S^c} |\alpha_{i'}\rangle$ and $S^a \xrightarrow{S^c} S^a$, so that

$$\langle\alpha_i|S^a|\alpha_j\rangle \xrightarrow{S^c} \langle\alpha_{i'}|S^a|\alpha_{j'}\rangle = \langle\alpha_i|S^b|\alpha_j\rangle\,,$$

where $S^b=(S^c)^{-1}S^aS^c$. The symmetry of our system is reflected in the equivalence of the states $|\alpha_i\rangle$, which may be reshuffled. On the other hand, we could have regarded the elements S^c to operate on S, and leave the physical states unchanged: $|\alpha_i\rangle \xrightarrow{S^c} |\alpha_i\rangle$, $S^a \xrightarrow{S^c} S^b$. In other words, the operation of S^c gives rise to a reshuffling of elements in S. The symmetry of our system is reflected, then, in the equivalence of the elements in S. Therefore, we had in the beginning the labeling of S and $|\alpha\rangle$, the symmetry results as a consequence of our freedom in reshuffling the states $|\alpha\rangle$ or the elements S^a. The first view is the usual one, while the second view corresponds to interpreting S^a as elements of the inner automorphism of S. According to this second viewpoint, it is quite clear that outer automorphisms are just as good as the inner ones. Having adjoined these outer automorphisms to our symmetry group, we have still to go back to the first, and "usual," viewpoint. It is then necessary, as was found in I, to interpret them as hidden symmetries, in general. The two viewpoints are actually complementary. If S has neither a center nor an outer automorphism (except the trivial ones), then the two views are the same. If S has a center C, then to regard S as inner automorphisms actually maps C to the identity.[2] On the other hand, if S has some outer automorphisms, the "usual" viewpoint becomes rather cumbersome for visualizing the existence of these symmetries.

In this work we wish to extend the previous considerations to include antiunitary symmetry operators – the time-reversal operator T. One of the most interesting properties of T is that phase factors of state vectors become nontrivial under T. Taking advantage of this property, we will show that the outer automorphisms of T and SU(2) (the isospin or rotation group) are realized as phase factors. Further, being just phase factors, these newly obtained symmetries are necessarily degenerate symmetries. By explicit construction, we will in fact find that the outer automorphisms of T and SU(2) generate a gauge group U(1). Moreover, this U(1) group combines with the SU(2) group to form a U(2) group. Thus, given T and isospin, we derive the hypercharge conservation. Similarly, from T and the rotation group SU(2), one obtains the fermion-number conservation. Further, the well-known empirical relations $(-1)^{2I}=(-1)^{Y}$ and $(-1)^{2J}=(-1)^{B+L}=(-1)^{F}$ are direct consequences of our construction.

In Sec. II, we give some general discussions on the questions of the structures of internal-symmetry groups. It seems obvious that these structures are rather nontrivial. Particular emphasis will be laid on the conflicting nature of the various familiar symmetry groups. Section III is devoted to a discussion of some properties of the discrete C, P, and T symmetries when they are adjoined with other internal-symmetry groups. The adjunction of T to an SU(2) group is considered in Sec. IV. Finally, some concluding remarks are offered in Sec. V.

II. GENERAL REMARKS

It was recognized some time ago that the problem of combining several internal-symmetry groups into a larger one is rather nontrivial. In fact, one is naturally led to studying the group extensions.[3-6] Thus, Michel[3] was able to show that, in combining the Poincaré group to the gauge groups

$$[U_Q(1),\ U_B(1),\ U_L(1)],$$

a relation of the form

$$(-1)^{2J}=(-1)^{\epsilon_1 Q+\epsilon_2 B+\epsilon_3 L}$$

must hold, where $\epsilon_i=0$ or 1. The different solutions correspond to different possible extensions of the gauge groups by the Poincaré group. The "physical choice"

$$(-1)^{2J}=(-1)^{B+L}=(-1)^{F}$$

can only be decided by comparing with reality.

Another familiar example has to do with the problem of adjoining the time-reversal operator T to the internal-symmetry groups. For instance, it is well known that T and the isospin operators satisfy

$$\{T, I_2\}=[T, I_{1,3}]=0\,.$$

These nontrivial commutation relations originate from the antiunitary nature of T and the requirement that T does not change the charge of a physical state.

In I, these considerations were carried one step further. It was argued that, by the very nature of the degenerate internal-symmetry groups, their outer automorphisms must themselves be symmetries. This requirement turns out to be very restrictive, as was discussed in I.

As a further example along these lines, let us begin with the chain of symmetries

$$U_Q(1)\subset U(2)\subset SU(3). \tag{1}$$

(A chain $G_0\subset G_1\subset G_2\subset\cdots$ is established if each G_i is a subgroup of G_{i+1}.) What happens if we consider the outer automorphisms of the groups in Eq. (1)? It turns out[1] that they are

$$\begin{aligned}&U_Q(1):\quad Q\overset{C_1}{\to}-Q;\\ &U(2):\quad Y\overset{C_2}{\to}-Y,\quad I_2\overset{C_2}{\to}+I_2,\quad I_{1,3}\overset{C_2}{\to}-I_{1,3};\\ &SU(3):\quad F_{1,3,4,6,8}\overset{C_3}{\to}-F_{1,3,4,6,8},\quad F_{2,5,7}\overset{C_3}{\to}F_{2,5,7}\,.\end{aligned} \tag{2}$$

Since we obviously have the chain

$$C_1\subset C_2\subset C_3, \tag{3}$$

Eqs. (1)–(3) may be combined (with $C_i^2=1$) into

$$U_Q(1)\times_s Z_2\subset U(2)\times_s Z_2\subset SU(3)\times_s Z_2, \tag{4}$$

where $\times_s$ denotes the semi-direct product. Thus, given Eq. (1), it turns out that the outer automorphisms (the charge conjugation) have a natural generalization as we enlarge the symmetry groups, so that Eq. (4) results.

The situation is completely different when we consider, in addition to $U_Q(1)$, U(2), and SU(3), the parity operator, which forms a Z_2 (P, $P^2=1$) group. Instead of Eq. (1), we now have

$$\mathrm{U}_Q(1)\times Z_2 \subset \mathrm{U}(2)\times Z_2 \subset \mathrm{SU}(3)\times Z_2\,. \quad (5)$$

Besides $C_{1,2,3}$, the introduction of parity gives rise to two new outer automorphisms,[1]

$$\begin{aligned}&\mathrm{U}_Q(1)\times Z_2:\ \ P \xrightarrow{V} (-1)^Q P;\\ &\mathrm{U}(2)\times Z_2:\ \ P \xrightarrow{W} (-1)^{2I} P.\end{aligned} \quad (6)$$

[The automorphism V may be easily obtained with the methods discussed in I (further, $V^2=W^2=1$).] Also, $\mathrm{SU}(3)\times Z_2$ has no outer automorphism except C_3 as was given in Eq. (2). Our argument shows that $\mathrm{U}_Q(1)\times Z_2$ must be enlarged to $[\mathrm{U}_Q(1)\times Z_2]\times_s Z_2$, and $\mathrm{U}(2)\times Z_2$ to $[\mathrm{U}(2)\times Z_2]\times_s Z_2$, where the Z_2's behind the semi-direct product signs are formed out of V and W, respectively. However, Eq. (6) shows that V cannot be included in $\mathrm{U}(2)\times Z_2$ or W, and W not in $\mathrm{SU}(3)\times Z_2$. We have

$$[\mathrm{U}_Q(1)\times Z_2]\times_s Z_2 \not\subset [\mathrm{U}(2)\times Z_2]\times_s Z_2 \not\subset \mathrm{SU}(3)\times Z_2\,. \quad (7)$$

Thus, consideration of parity brings out the incompatibility of the symmetry groups $\mathrm{U}_Q(1)$, U(2), and SU(3). We may in fact call $\mathrm{U}_Q(1)\times Z_2$, $\mathrm{U}(2)\times Z_2$, and $\mathrm{SU}(3)\times Z_2$ "conflicting symmetries." Even though they may be included in a chain as in Eq. (5), their outer automorphisms cannot be so enlarged, as in Eq. (7).

Physically this result is easy to understand. Given $\mathrm{U}_Q(1)\times Z_2$, V corresponds to the ambiguity of the relative parity between states with even and odd charges. If we embed $\mathrm{U}_Q(1)$ in U(2), then V is violated since in $\mathrm{U}(2)\times Z_2$ members in the same isospin multiplet must have the same parity. The same consideration applies to W when we embed U(2) in SU(3), as was discussed in I, where we also showed that, in order to preserve W, we must enlarge U(2) to SU(3)×SU(3).

The conflict between $\mathrm{U}_Q(1)\times Z_2$ and $\mathrm{U}(2)\times Z_2$ may be interpreted in the following way. The strong interaction is known to conserve isospin and parity, as well as the charge. However, one may ask whether the strong interaction only conserves U(2) approximately, in the sense that there may be a genuine isospin-breaking term in H_{st}. Equation (7) says that such a picture is inconsistent, theoretically. Conversely, given that the symmetry group of H_{em} is $\mathrm{U}_Q(1)\times Z_2$, it is also inconsistent to enlarge it to groups with an SU(2) structure.

One has to be a little careful in dealing with the "conflicting symmetries" and the related problem of "broken symmetries." Our analysis was based on the assumption that one "knows" that H_{st} preserves U(2) and H_{em}, $\mathrm{U}_Q(1)$. Suppose one were to insist that $\mathrm{U}(2)\times Z_2$ be only a remnant symmetry of $\mathrm{SU}(3)\times Z_2$; then indeed one might break SU(3) in such a way that the absolute parity of $I=\frac{1}{2}$ states be fixed. (In reality, of course, the parity of $I=\frac{1}{2}$ states is not absolute.) What we are empasizing is that the reverse procedure is more physical and consistent. We should start from $\mathrm{U}(2)\times Z_2$ and enlarge it. In this case the absolute parity of $I=\frac{1}{2}$ states is never measurable. Similarly, the symmetry properties of H_{st} and H_{em} should be regarded as having entirely different characteristics.

We should also emphasize that, even if we were to insist on starting from $\mathrm{SU}(3)\times Z_2$, to break it without preserving W, in a sense, yields a $\mathrm{U}(2)\times Z_2$ which is not "pure." For, what we get is a $\mathrm{U}(2)\times Z_2$ with an extra, exterior, constraint (that the $I=\frac{1}{2}$ states have absolute parities). Finally, in assigning symmetries to H_{st}, we are not saying that SU(3) is in principle not possible. We have merely opted for U(2) as a better choice.

Mathematically, these results seem to hinge on a number of "accidents." We may observe that, in physics, the enlargement of symmetry groups is often "noncentral," in that for $G_i \subset G_{i+1}$, the center of G_i is not in the center of G_{i+1}. The enlargements of $\mathrm{U}_Q(1)$ to U(2) and of SU(2) to SU(3) are all examples of noncentral enlargments. The conflict between SU(2) and SU(3), for instance, has its root in that $(-1)^{2I}$, which is essentially the identity in SU(2) (rotation of 2π), becomes just an undistinguishing member in SU(3). Coupled with these facts are the existence of involutional (reflection-like) symmetries C, P, and T. The adjunction of these operators enhances the importance of elements of order two [such as $(-1)^Q$, $(-1)^{2I}$] in the internal-symmetry groups. In a way SU(3), which has in its center only elements of order three, would be the "natural" symmetry group to consider if we had discrete symmetry operators which show "triality" rather than "duality" properties. All of these seem to suggest a rationale for the fundamental importance of the group SU(2), which happens to posses in its center a nontrivial element of order two. A further illustration will be given in Sec. IV, where we consider the problem of adjoining the time-reversal operator to SU(2).

III. INVOLUTIONAL SYMMETRIES

To facilitate our considerations later, we will now proceed to discuss some general properties of the involutional (reflection-like) symmetries C, P, and T in connection with their adjunction with the continuous internal-symmetry groups. Since their squares C^2, P^2, and T^2 represent the identity operation physically, it seems safe to make the following ansatz:

Ansatz. Given any internal-symmetry group G, the operators C^2, P^2, and T^2 must be in the center of G.

As an immediate consequence, we have the following theorem:

Theorem. If we exclude from our considerations the Poincaré group, then $C^2 = P^2 = T^2 = 1$, the identity element in any internal-symmetry group G.

Several things need clarification.

(1) G shall be understood to stand for any internal-symmetry group, whether it be for the strong, the electromagnetic, or the weak interactions. Here we are assuming that even though the "usual" C and P symmetries are violated by H_{wk}, the violation is known and it is meaningful to talk about C^2 and P^2, which, furthermore, are not violated by H_{wk}.

(2) It is not generally necessary to assume that C^2, P^2, and T^2 be the identity in G. In fact, it is well known that $T^2 = (-1)^{2J}$ so that, for the Poincaré group, T^2 is indeed not the identity.

(3) The ansatz limits, to a certain extent, the freedom which the operators C, P, and T enjoy from the general viewpoints of group extensions. In such considerations, say extending G by P, it is usually argued that gP, where g is any element in G, may be used as the parity operator. Clearly, the ansatz dictates the choice of g to those for which $(gP)^2$ is in the center of G.

(4) The theorem follows immediately since in reality the enlargments of symmetry groups are noncentral. Consider the strong-interaction symmetries. If we believe in the possibility of the chain $SU(2) \subset U(2) \subset SU(3) \times SU(3)$ in the sense that it makes sense to talk about an exact $SU(3) \times SU(3)$ limit, then $C^2 = P^2 = T^2 = 1$, since the identity is the intersection of the centers of SU(2), U(2), and $SU(3) \times SU(3)$. Alternatively, we may consider U(2) for H_{st} and $U_Q(1)$ for H_{em}, the intersection of whose centers again yields the identity uniquely.

(5) Note that in studying, for instance, the adjunction of P to SU(2), it is perfectly all right to have $P^2 = (-1)^{2I}$. However, if P has also to be adjoined to other groups, which do not have $(-1)^{2I}$ in their centers, then we must give up the solution $P^2 = (-1)^{2I}$.

(6) $T^2 = (-1)^{2J}$ is compatible with our discussions only because the rotation group is a symmetry common to all interactions. Note also that $T^2 = (-1)^{2J}$ forbids any noncentral enlargements of the rotation group. (In fact, any "higher-symmetry" schemes which mix half-integral- with integral-spin states are forbidden.)

IV. ADJOINING THE TIME REVERSAL TO SU(2)

Consider the internal-symmetry group composed of SU(2) (isospin) and T (time reversal), with $T^2 = 1$ (see Sec. III). As we mentioned before, the antiunitary nature of T and the fact that T is not supposed to alter the charge of a physical state imply the following relations:

$$[T, I_1] = [T, I_3] = 0, \quad \{T, I_2\} = 0, \quad T^2 = 1. \tag{8}$$

For our discussions it is convenient to introduce[7] a slight variant of T,

$$\overline{T} \equiv e^{i\pi I_2} T = T e^{i\pi I_2}, \tag{9}$$

which satisfies

$$\{\overline{T}, I_{1,2,3}\} = 0, \quad \overline{T}^2 = (-1)^{2I}, \quad \overline{T}^4 = 1. \tag{10}$$

Since $\overline{T}$ is also antiunitary, we have

$$\overline{T} e^{i\vec{\theta}\cdot\vec{I}} = e^{i\vec{\theta}\cdot\vec{I}}\, \overline{T}. \tag{11}$$

The symmetry $\overline{T}$ and SU(2) now form the direct-product group, $[SU(2) \times Z_4]/Z_2$, where the factor group has to be taken since $\overline{T}^2$ must be identified with the center of SU(2).

From our discussions in I, it follows immediately that the only outer automorphism of the group $[SU(2) \times Z_4]/Z_2$ is

$$\overline{T} \overset{g_0}{\to} g_0 \overline{T} g_0^{-1} = (-1)^{2I}\overline{T} = \overline{T}^{-1}. \tag{12}$$

[g_0 is necessarily an outer automorphism since $\overline{T}$ commutes with SU(2).] Equation (12) is similar to our earlier discussions of the W symmetry,

$$P \overset{W}{\to} (-1)^{2I} P.$$

However, owing to the antiunitary nature of $\overline{T}$, the physical consequences of g_0 are entirely different from those of W.

In order to analyze the properties of g_0, we start from an $I = \frac{1}{2}$ state, say $|\alpha\rangle$, which will be assumed to be "elementary," in the sense that all the states may be regarded as composite states of $|\alpha\rangle$, as far as their isospin properties are concerned. [Of course, on a composite state g_0 is to operate in the usual way: $g_0(|\alpha\rangle|\alpha\rangle\cdots|\alpha\rangle) = (g_0|\alpha\rangle)\cdots(g_0|\alpha\rangle)$.] Without loss of generality, then, we will only consider the state $|\alpha\rangle$.

We now observe that the automorphism g_0 may be realized on $|\alpha\rangle$ by the operation of multiplying by the phase $e^{i\pi/2}$,

$$g_0|\alpha\rangle = e^{i\pi/2}|\alpha\rangle. \tag{13}$$

Algebraically, this solution satisfies

$$g_0^2 = (-1)^{2I}, \quad g_0\overline{T} = \overline{T} g_0^{-1}, \quad [g_0, \vec{I}] = 0, \tag{14}$$

where the second equation follows because $\overline{T}$ is antiunitary. Equation (12) is obviously an immediate consequence of Eq. (14).

The important point about Eq. (13) is that g_0, when applied to $|\alpha\rangle$, does not create a new state. Thus, the fact that $\overline{T}$ is antiunitary has the remarkable consequence that g_0, which induces an outer automorphism on $\overline{T}$, is a degenerate symmetry operator.

Given that g_0 is degenerate, we must now study the enlarged, still degenerate, symmetry group consisting of SU(2), $\overline{T}$, and g_0. There are now two new outer automorphisms,

$$g_0 \xrightarrow{f} g_0^{-1} = (-1)^{2I} g_0 ,\qquad [f, \overline{T}] = [f, \vec{\mathrm{I}}] = 0 , \tag{15}$$

and

$$\overline{T} \xrightarrow{g_1} g_0 \overline{T} ,\qquad [g_1, \vec{\mathrm{I}}] = [g_1, g_0] = 0 . \tag{16}$$

[Note that $(g_0\overline{T})(g_0\overline{T}) = g_0(g_0^{-1}\overline{T})\overline{T} = \overline{T}^2$.] The natures of f and g_1 are completely different. Just as before g_1 must again be a degenerate symmetry, being realized on $|\alpha\rangle$ as a phase factor $e^{i\pi/4}$,

$$g_1 |\alpha\rangle = e^{i\pi/4} |\alpha\rangle . \tag{17}$$

Algebraically, we have

$$g_1^{2} = g_0 ,\qquad g_1\overline{T} = \overline{T} g_1^{-1} ,\qquad [g_1, \vec{\mathrm{I}}] = [g_1, g_0] = 0 . \tag{18}$$

On the other hand, for f we may define a state $|\overline{\alpha}\rangle$ by

$$f |\alpha\rangle = |\overline{\alpha}\rangle . \tag{19}$$

Then

$$g_0 |\overline{\alpha}\rangle = e^{-i\pi/2} |\overline{\alpha}\rangle . \tag{20}$$

[Incidentally, Eq. (20) shows that $|\overline{\alpha}\rangle$ must be different from $|\alpha\rangle$.] We have still to find the effect of f on g_1. Since $g_1^2 = g_0$, we have, from Eq. (15),

$$f g_1 f^{-1} = g_1^{-1} . \tag{21}$$

Summarizing, we find, from SU(2), $\overline{T}$, and g_0, an additional degenerate symmetry g_1 (satisfying $g_1^2 = g_0$) as well as an (in general) hidden symmetry f, satisfying $fg_0f^{-1} = g_0^{-1}$ and $fg_1f^{-1} = g_1^{-1}$.

Entirely similar reasoning leads then to the existence of another, degenerate, symmetry operator g_2, with

$$\overline{T} \xrightarrow{g_2} g_1 \overline{T} ,\qquad g_2\overline{T} = \overline{T} g_2^{-1} ,\qquad g_2^{2} = g_1 ,\qquad [g_2, g_0] = [g_2, g_1] = [g_2, \vec{\mathrm{I}}] = 0 , \tag{22}$$
$$g_2 |\alpha\rangle = e^{i\pi/8} |\alpha\rangle ,\qquad f g_2 f^{-1} = g_2^{-1} .$$

Obviously, in this way, starting from $\overline{T}$ and SU(2), we generate the symmetry f and a sequence of degenerate symmetries, $g_0, g_1, g_2, \ldots$, with the properties

$$g_i\overline{T} = \overline{T} g_i^{-1} ,\qquad g_i^{2} = g_{i-1} ,\qquad [g_i, g_j] = [g_i, \vec{\mathrm{I}}] = 0 ,\qquad f g_i f^{-1} = g_i^{-1} ,\qquad g_i |\alpha\rangle = e^{i\pi/2^{i+1}} |\alpha\rangle . \tag{23}$$

Further, it is clear that any product of the g's,

$$g_i g_j \cdots ,$$

is also a degenerate symmetry operator. It follows that the g's actually generate an Abelian U(1) group (since any real number x, $0 \le x \le 1$, may be written as $x = \sum a_n/2^n$, for $a_n = 0$ or 1). Let us define this U(1) group as $\mathrm{U}_Y(1) = \{e^{i\theta Y}\}$. Then

$$Y |\alpha\rangle = |\alpha\rangle ,\qquad g_0 = e^{i(\pi/2)Y} \quad [g_0^{2} = (-1)^{2I},\ g_0^{4} = 1] ,\qquad g_1 = e^{i(\pi/4)Y}, \ldots \tag{24}$$

so that $e^{i\theta Y}$ can always be written as a product of the g_i's.

Therefore, given $\overline{T}$ and SU(2), we have generated a one-parameter degenerate gauge group $\mathrm{U}_Y(1) = \{e^{i\theta Y}\}$ satisfying

$$[Y, \vec{\mathrm{I}}] = 0 ,\qquad [Y, \overline{T}] = 0 \quad (e^{i\theta Y}\overline{T} = \overline{T} e^{-i\theta Y}) , \tag{25}$$

and, by Eqs. (14) and (24),

$$(-1)^{Y} = g_0^{2} = (-1)^{2I} . \tag{26}$$

[In other words, SU(2) and $\mathrm{U}_Y(1)$ generate U(2).] Needless to say, $\mathrm{U}_Y(1)$ is nothing but the hypercharge gauge group. The other symmetry f [Eq. (15)] corresponds obviously to the usual G parity, satisfying [Eqs. (15), (21), etc.],

$$[f, \vec{\mathrm{I}}] = \{f, Y\} = 0 . \tag{27}$$

Let us now pause for a moment, in order to get rid of a number of ambiguities neglected earlier. In Eq. (13), actually another solution exists,

$$g_0 |\alpha\rangle = e^{-i\pi/2} |\alpha\rangle . \tag{13'}$$

This obviously corresponds to starting with $|\overline{\alpha}\rangle$ instead of $|\alpha\rangle$, as in Eq. (20). Next, in obtaining g_1, $g_2, \ldots$ [Eqs. (17), (22), etc.], one may take the solution $(-1)^{2I}g_i$ instead of g_i. However, since $(-1)^{2I} = e^{i\pi Y}$, all these solutions are recovered when we have generated the $U_Y(1)$ group. Lastly, in extending f from g_0 to operate on g_i, since we can only require $fg_i^2f^{-1} = g_{i-1}^{-1}$, we may, besides $fg_if^{-1} = g_i^{-1}$, get another solution $fg_if^{-1} = (-1)^{2I}g_i^{-1}$. This ambiguity is removed, however, by considering $fg_{i+1}f^{-1}$. For, either solution [g_{i+1}^{-1} or $(-1)^{2I}g_{i+1}^{-1}$] gives $fg_if^{-1} = fg_{i+1}^2f^{-1} = g_i^{-1}$.

In summary, we find that hypercharge conservation follows from time reversal invariance and SU(2) symmetry. Further, an "elementary" isospinor must have $Y = \pm 1$. In general, for composite states, even and odd Y values correspond to integral- and half-integral-isospin states, respectively.

If we start out from T and the rotation group [SU(2)], then, algebraically,

$$\begin{aligned} &\{T, \vec{J}\} = 0, \\ &T^2 = (-1)^{2J}, \\ &Te^{i\vec{\theta}\cdot\vec{J}} = e^{i\vec{\theta}\cdot\vec{J}}T. \end{aligned} \tag{28}$$

It is clear that, by identical arguments as above,[8] we necessarily generate the fermion-number ($F = B + L$) gauge group $U_F(1) = \{e^{i\theta F}\}$ and the fermion-number-conjugation (C_F) operator, satisfying

$$\begin{aligned} &[F, \vec{J}] = [F, T] = 0, \\ &e^{i\theta F}T = Te^{-i\theta F}, \\ &(-1)^F = (-1)^{2J}, \\ &\{C_F, F\} = [C_F, \vec{J}] = 0. \end{aligned} \tag{29}$$

V. CONCLUDING REMARKS

In this work we have found some rather nontrivial consequences in studying the outer automorphisms of symmetry groups. Specifically, we found that the outer automorphisms of the time reversal and an SU(2) group generate unambiguously the degenerate symmetry group U(2). Thus, hypercharge conservation follows from T and isospin conservation. The fermion-number conservation is a consequence of the angular momentum conservation. The well-known empirical relations $(-1)^Y = (-1)^{2I}$ and $(-1)^F = (-1)^{2J}$ were derived in the process.

In retrospect, these results are not as surprising as they might first appear. We may note that the "number laws" are realized as phase factors on the physical states. Since the time-reversal operator is intimately related to phase factors, it is perhaps not too surprising to find a cause and effect relation between them. Another important ingredient in our discussion is the existence of spinors and isospinors, or the empirical fact that the centers of SU(2) [$(-1)^{2J}$ and $(-1)^{2I}$] are nontrivially represented. If there are only integral-spin and -isospin states, or, if $SU(2)/Z_2$ instead of SU(2) were to be our symmetry group, then Eq. (12), which is the starting point of all our discussions, is trivial and nothing would have been learned from our considerations. The empirical relations $(-1)^Y = (-1)^{2I}$ and $(-1)^F = (-1)^{2J}$, in this connection, may be taken as further clues to the intimate relationships between half-integral representations and the "number laws." Indeed, starting from these relations, what we showed in Sec. IV was that we may take "square roots" successively. These "square roots" generate the gauge group U(1).

Our heavy reliance on T may lead to the following objection. Since we know that T is violated (by the "superweak interaction"), but the fermion number is not, is there not something wrong with our derivations? Actually this dilemma is only an apparent one. We may start from the strong, the electromagnetic, and the weak interactions, for which T and SU(2) (rotation symmetry) are exact. We then obtain $U_F(1)$. Having obtained the larger group, we ask: How does the superweak interaction behave under this larger symmetry group? It is not hard to imagine that indeed the superweak interaction behaves so that it violates T, but still conserves U(2). In a similar way, for T and isospin, we have obtained the hypercharge conservation, which is a symmetry for H_{em}, even though H_{em} does not respect the isospin conservation.

Finally, it is not difficult to convince oneself that, had we started from T and U(2) or SU(3)×SU(3), we would only have obtained inner automorphisms. This, again, testifies to the fundamental importance of the SU(2) group.

ACKNOWLEDGMENT

We would like to thank our colleagues for discussions.

*Supported in part by the U. S. Atomic Energy Commission.

[1]T. K. Kuo, preceding paper, Phys. Rev. D 4, 3620 (1971), hereafter referred to as I.

[2]Compare with the analogous situation in angular momentum conservation. If we have only SO(3) (the pure rotation group) instead of SU(2), then the center of SU(2) – $(-1)^{2J}$ – becomes trivial. The half-integral representations are obtained only if we enlarge our "geometrical picture" of rotation to include the extra element $(-1)^{2J}$.

[3]L. Michel, in *Group Theoretical Concepts and Methods in Elementary Particle Physics*, edited by F. Gürsey (Gordon and Breach, New York, 1964); and in *Particle Symmetries and Axiomatic Field Theory*, 1965 Brandeis Summer Institute in Theoretical Physics, edited by M. Chrétien and S. Deser (Gordon and Breach, New York, 1966).

[4]Von F. Kamber and N. Straumann, Helv. Phys. Acta. 37, 563 (1964).

[5]T. D. Lee and G. C. Wick, Phys. Rev. 148, 1385 (1966).

[6]B. Zumino and D. Zwanziger, Phys. Rev. 164, 1959 (1967).

[7]The existence of $\overline{T}$ for SU(2) is well known. We may note that, if we adjoin T to an internal-symmetry group S which commutes with P, then the operator $CT \equiv \overline{T}$ anticommutes with all the generators of S. For SU(2), $C = e^{i\pi I_2}$; and hence we have Eqs. (9) and (10).

[8]To the extent that hadrons and leptons can be separated unambiguously, our arguments actually lead to separate baryon-number and lepton-number conservation laws.

PHYSICAL REVIEW D VOLUME 4, NUMBER 12 15 DECEMBER 1971

Motion of Charged Particles in a Homogeneous Magnetic Field*

Wu-yang Tsai† and Asim Yildiz
Jefferson Physical Laboratory, Harvard University, Cambridge, Massachusetts 02138
(Received 18 March 1971)

A general and simple method is presented to calculate the eigenvalues of the eigenequation for the theory of a particle with any spin with anomalous-magnetic-moment coupling moving in a homogeneous magnetic field. The eigenvalues of the spin-$\frac{1}{2}$ and spin-1 systems are obtained specifically. This method of approach does not require an explicit solution of the eigenfunction equation. Different spin-1 theories (vector theory, multispinor theory, and 6-component theory) are discussed, and in the case of no anomalous-magnetic-moment coupling terms, all these theories predict the same eigenvalues. Furthermore, by requiring that the energy eigenvalues be positive definite, we show that the vector spin-1 theory is consistent only when no anomalous-magnetic-moment coupling terms exist.

I. INTRODUCTION

The problem of describing the motion of charged particles in an external electromagnetic field has been with us for a long time.[1,2] Until now the only cases discussed have been the spin-0 and spin-$\frac{1}{2}$ systems, and the generally accepted method of approach is to define the Lagrangian equation of motion and to use the differential-equation technique to solve for the eigenfunctions and eigenvalues of the system. This approach becomes increasingly complicated when anomalous-magnetic-moment coupling terms (a.m.m.c.t.) are introduced[2] and when we discuss the higher-spin cases.

The purpose of this paper is to present a simple, general method for calculating the eigenvalues of particles of any spin with anomalous magnetic moment moving in a homogeneous magnetic field. The method is introduced in Sec. II, and the results of Ternov *et al.*[2] are easily reproduced. In Secs. III and IV, the eigenvalues of a spin-1 theory with a.m.m.c.t. are calculated explicitly within the vector theory of the Kemmer-Proca equation.[3] By observing the physical requirement that the squares of the energy eigenvalues be positive definite, we show that the spin-1 theory is consistent only when the a.m.m.c.t. are not present. The predictions from three different spin-1 theories – vector theory,[3] multispinor theory,[4] and the 6-component theory[5] – are discussed in Sec. V. We find that for the case where the a.m.m.c.t. are absent, these theories all predict the same results.

II. METHOD OF APPROACH

In this section we illustrate the method of approach in the spin-$\frac{1}{2}$ theory with a.m.m. couplings. Our starting point is the eigenvalue equation[6]

$$\gamma^0 p^0 \psi = \left(m + \vec{\gamma}\cdot\vec{\pi} - \frac{eq\kappa}{2m}\sigma_3 H\right)\psi\,, \tag{1}$$