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[10]Equivalently, the eigenvalues can be obtained from the zeros of the Fredholm determinant associated with the difference equation (11). This equation can be written in the form

$$a_n = \sum_{l=0}^{\infty} \mathcal{G}_{n,l}\, b_l\, a_l\,,$$

where $\mathcal{G}_{n,l}$ is the Green's function of the difference equation. An exact analytic form for $\mathcal{G}_{n,l}$ has been obtained.
[11]We have also investigated mass renormalization in a one-dimensional model Hamiltonian with nonpolynomial interaction.

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Internal-Symmetry Groups and Their Automorphisms*

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It is proposed that outer automorphisms of degenerate internal-symmetry groups must be symmetry operators themselves. In general, however, they are hidden (spontaneously broken) symmetries. Consequences of this proposition are studied. It is found that internal-symmetry groups are not arbitrary but that their intrinsic properties play an important role. The existence of discrete symmetries (such as charge conjugation) follows naturally from assuming the continuous symmetry groups (such as gauge groups). We also find that the enlargement of the isospin symmetry and parity leads directly to the chiral SU(3)×SU(3), so that the existence of an "exact SU(3) limit" is in principle not allowed.

I. INTRODUCTION

Symmetry has always played an important role in physics. Historically the rotational and translational symmetries were the first to be studied. The theory of relativity brought in the Lorentz invariance. With the advent of quantum mechanics, discrete space-time symmetries (parity and time-reversal) came into existence. Finally, the study of elementary-particle physics brought forth a whole new class of symmetries – the internal symmetries, such as the charge conjugation, isospin, unitary spin, and many more "higher symmetries."

The basic difference between space-time (excluding possibly the discrete symmetries) as compared with the internal symmetries seems to lie in that the space-time coordinates are physically measurable quantities, while the corresponding group space of the internal-symmetry groups are fabrications with no physical significance whatsoever.

Thus, the "isospin space" is only introduced to facilitate the comparison of isospin with ordinary spin. The physically important (and meaningful) quantities are the isospin operators. In fact, we need not talk about the isospin space at all. (Of course it may happen in the future that even the isospin space will acquire some physical meaning. For the moment, at least, this is not the case.)

If the only physically meaningful quantities in the internal-symmetry groups are the group elements, then we may expect that the intrinsic group structure plays an important role. In this work we will discuss the restrictions on the internal-symmetry groups which arise from their automorphisms.

In order to facilitate our discussions, it is convenient to first clarify the origin of symmetry operators in physics. This will be done in Sec. II. We will show that there is a one-to-one correspondence between symmetry operators and conventions. Indeed, in the quantum-mechanical sense, each symmetry operator actually carries out a change of convention.

We are then naturally led to ask the question: Is there any convention in our use of internal-symmetry groups? Section IV is devoted to this problem.

We will find, loosely speaking, that the automorphisms of these groups are indeed inherent conventions associated with their physical applications. A more precise analysis can be done only by a more refined classification of the internal-symmetry groups. This is done in Sec. III, where we emphasize the role of hidden (or spontaneously broken[1,2]) symmetries as compared with the degenerate (or ordinary) symmetries. It turns out that the automorphisms of the internal-symmetry groups are symmetries, but in general they are hidden. In Sec. IV, these considerations are summarized in a proposition which is the main result of this work.

The applications of our results are carried out in Sec. V, in which a number of concrete examples are studied. The sort of intrinsic restrictions on the internal-symmetry groups are evident. We find, for instance, from charge conservation, an additional symmetry identifiable with the charge-conjugation symmetry. The SU(2) and SU(3) symmetries are analyzed. It turns out that considerations of parity invariance lead necessarily to a broken SU(3) symmetry. The natural enlargement of the isospin symmetry (plus parity) is shown to be SU(3)×SU(3), but not SU(3). Therefore, it is perhaps rather unfruitful to consider "the limit in which SU(3) is exact," since only "the limit in which SU(3)×SU(3) is exact" exists. This is actually in line with recent trends, where more emphasis has been put on the study of the SU(3)×SU(3) symmetry. Other detailed questions in connection with the chiral symmetry are studied in Sec. VI.

The difficult task of handling space-time symmetry along the same lines is taken up in Sec. VII. Our considerations seem to unveil more puzzles than answers. Obviously much more has to be done before we understand the problems involved.

A few final remarks are offered in the concluding section, Sec. VIII. Appendix A is devoted to a brief discussion of the mathematical tools necessary in our analysis. Appendix B deals with the considerable controversy generated in the wake of some earlier suggestions[3-5] concerning broken SU(3)×SU(3) symmetry. By relating them to a larger class of physical theories, hopefully the present work will shed some more light on these suggestions.

Summaries of this paper were given elsewhere.[6]

II. CONVENTIONS IN PHYSICS

In our description of a physical system, we are often confronted with different, but equivalent, choices. It has been a time-honored practice to make a particular choice, or a particular convention, and proceed with our description. Thus, in describing a charged system there are *a priori* two possible choices – to say that the system is positively or negatively charged. Similarly, in describing a spherically symmetrical system the orientation of the coordinate system is free for our choice. We are accustomed, in these cases, to pick a particular convention (say, positive charge and any particular orientation, respectively) and disregard the problem. The question that one may ask is: What relations, if any, are there between the different choices? In particular, if we change our convention, does "changing a convention" correspond to any physical operator? In quantum mechanics, coordinate transformations are generally carried out by unitary operators. If the choice of the coordinate is a convention, the unitary operator is furthermore a symmetry operator. Our problem is to study if there is any limitation to this association. We would like to propose, as a general rule, that the operation of changing a convention has a one-to-one correspondence with a symmetry operator.

It was remarked some time ago by Lee[7] that symmetries correspond to "nonmeasurables." It is not surprising that "conventions," which are nothing but nonmeasurables, are intimately related to symmetries. What we wish to emphasize is that a "symmetry operator," in the quantum-mechanical sense, actually carries out the physical process of changing a convention.

In the following, we enumerate a number of familiar examples to illustrate our point.

(1) Rotation symmetry: Let us consider a spherically symmetric system. The orientation of the coordinate system that we may choose to describe our physical system is arbitrary, i.e., it is a convention. To change the orientation, or the convention, is therefore a symmetry operator. It is the rotation operator which, quantum mechanically, either rotates the coordinate or the physical system under consideration.

(2) Parity symmetry: If left-handedness is not distinguishable from right-handedness, then, changing this convention is a symmetry operator. It is carried out by the parity operator.

(3) Lorentz invariance: If, in describing a physical system, it is a convention to use whatever initial frame one likes, then, changing from one initial frame to another is a symmetry operation. It is the Lorentz transformation operator which effects this transition.

(4) Charge conservation: If the state vectors in the physical world may be grouped into sets (each set consists of states with the same "charge"), so that we may attach a common phase factor to the states in the same set, with no measurable consequences, then, the phase factor that one uses

for the state vectors becomes a convention. In particular, one may change this phase factor by the physical operators $e^{i\theta Q}$. These operators $e^{i\theta Q}$ are then symmetry operators of the physical world. They form the symmetry group U(1) – the gauge group for charge conservation.

Before we proceed, it may be worthwhile to analyze the "converse" of our statements. If, in our first example, orientation of the system becomes important, and ceases to be a convention, then, correspondingly, rotation also ceases to be a symmetry operator. Similarly, if left-handedness is distinguishable from right-handedness, then, the parity operator, which effects the transition, is no longer a symmetry operator.

Now, when we describe a system by a Hamiltonian H, an operator S is a symmetry operator if and only if

$$[H, S] = 0.$$

If we may assign S to a physical operation, such as changing the orientation of a system, then, if

$$[H, S] = 0,$$

the orientation is a convention. Conversely, if

$$[H, S] \neq 0,$$

the orientation is not a convention so that we may measure the orientation of our system in an absolute way.

The above are a few familiar examples. It is the purpose of our work to study the conventions associated with the internal-symmetry groups. We will find that these conventions should also generate symmetries. In other words, our consideration suggests that the internal-symmetry groups are not arbitrary, but must satisfy certain restrictions from their intrinsic group structure which, it turns out, is intimately related to these conventions.

III. INTERNAL-SYMMETRY GROUPS

Before we proceed, it is useful to sharpen our definition of internal-symmetry groups. We may classify the internal symmetry into two types: (1) the degenerate (or ordinary) symmetry, (2) the hidden[1,2] (or spontaneously broken) symmetry. Both types of symmetry operators commute with the Hamiltonian, by definition. They differ, however, in their effects when applied to physical states. Alternatively, the possibility of the two types of symmetry is a reflection of two possible types of structure of the Hilbert space for a system described by a given Hamiltonian.

Consider a physical system with the state vectors $|\phi\rangle$, $|\psi\rangle, \ldots$. The existence of a symmetry operator G consists in having[8]

$$G|\phi\rangle = |\phi'\rangle, \ldots \tag{1}$$

with

$$|\langle\phi|\psi\rangle|^2 = |\langle\phi'|\psi'\rangle|^2. \tag{2}$$

It should be emphasized that no *a priori* restrictions are placed on the transformed states $|\phi'\rangle, \ldots$, as far as their relationships with the original states $|\phi\rangle, \ldots$ are concerned. Now, if the ground state of our system, called the "vacuum" $|0\rangle$, is nondegenerate, then the uniqueness of the vacuum forces the relation,

$$|0'\rangle = G|0\rangle = |0\rangle. \tag{3}$$

In this case, G is said to be a degenerate symmetry.[9] On the other hand, the Hilbert space of our system may be more complicated. There may be a number of disjoint,[10] isomorphic subspaces: $(|0\rangle, |\phi\rangle, \ldots)$; $(|0'\rangle, |\phi'\rangle, \ldots)$; . . . . As long as the inner products are preserved, i.e., for all states Eq. (2) holds, the operator G, satisfying

$$G|\phi\rangle = |\phi'\rangle, \ldots,$$

is a (hidden) symmetry operator. Thus, the hidden symmetry originates from the possibility that the Hilbert space is a direct sum of isomorphic subspaces. Each subspace may be used to describe our physical system. (The choice of a particular subspace is a convention.) To switch from one subspace to another is realized by a symmetry operator, in fact, a hidden-symmetry operator.

We may summarize our discussions in the following definition. A physical system with an internal-symmetry group is characterized by a Hamiltonian H, a set of degenerate eigenstates of H $[|\alpha_i\rangle, |\bar{\alpha}_i\rangle, \ldots; |\beta_j\rangle, |\bar{\beta}_j\rangle, \ldots$; with the ground states (vacua) $|0\rangle, |\bar{0}\rangle, \ldots]$, and a group G, which contains a subgroup S and its cosets $\bar{S}$, with the following properties:

(1) For S (the degenerate, or ordinary symmetry),

$$S|0\rangle = |0\rangle, \quad S|\alpha_i\rangle = S_{ij}|\alpha_j\rangle, \ldots, \tag{4}$$

where S_{ij} is a linear representation of S.

(2) For $\bar{S}$ (the hidden, or spontaneously broken[1,2] symmetry),

$$\bar{S}|0\rangle \equiv |\bar{0}\rangle \neq |0\rangle, \quad \bar{S}|\alpha_i\rangle \equiv |\bar{\alpha}_i\rangle, \ldots, \tag{5}$$

where $|\bar{\alpha}_i\rangle$ are not linearly related to $|\alpha_i\rangle$.

(3)

$$GHG^{-1} = H. \tag{6}$$

Let us make the following remarks:

(a) We could have used, instead of the state vectors $|\alpha_i\rangle,\ldots,$ the "field operators" $\phi(x),\ldots$ with corresponding transformation laws under G. However, we believe that our considerations are valid for ordinary quantum mechanics, and there is no need to introduce field theory into our framework.

(b) From Eqs. (4) and (5), we can easily deduce that S is a subgroup of G, and that $\bar{S}$ forms cosets with respect to S. Now, the identity element necessarily belongs to S. It follows that $|0\rangle = S^{-1}(S|0\rangle) = S^{-1}|0\rangle$, so that the inverse of any element in S is also degenerate. Also, if S_i and S_j belong to S, then $S_iS_j|0\rangle = S_i|0\rangle = |0\rangle$.

(c) In our definition, S is associated with a certain vacuum state. One may ask: What is the state $S|\bar{0}\rangle$? This is obtained by observing

$$S|\bar{0}\rangle = S\bar{S}|0\rangle = (S\bar{S}S^{-1})S|0\rangle$$
$$= (S\bar{S}S^{-1})|0\rangle .$$

Thus, $S|\bar{0}\rangle$ is determined by the group structure which specifies $S\bar{S}S^{-1}$. In general, $S\bar{S}S^{-1}$ gives another element $\bar{S}'$ and therefore $S|\bar{0}\rangle = |\bar{0}'\rangle$.

(d) The number of degenerate vacua is clearly equal to the number of cosets of S in G. It should be noticed that $\bar{S}$ may be said to be defined only by the cosets of S (actually the right cosets: $\bar{S}S$). Since, given a particular element $\bar{S}$, the complete coset $\bar{S}S$ is already determined. It is then convenient to talk about $\bar{S}$ as a coset. Also, we may choose to pick a particular representative in $\bar{S}S$ by imposing conditions such as $\bar{S}^2 = 1$, if one of the elements in the coset $\bar{S}S$ has order 2.

(e) In the usual discussions of hidden symmetries, much attention has been paid to the case when the number of cosets of S in G is not finite (such as the usual case of exact chiral symmetries). We are then faced with the existence of zero-mass bosons, or Goldstone bosons. Let us emphasize that if the number of cosets of S in G is finite, no zero-mass bosons need exist.

(f) The degenerate symmetries are observed physically through the existence of selection rules. On the other hand, hidden symmetries only lead to relations among matrix elements. Thus, for chiral symmetries we obtain relations between processes involving different number of Goldstone bosons. When $\bar{S}$ has only a finite number of elements, even less can be learned. That its existence is not totally irrelevant will be demonstrated in the subsequent sections (in particular, Sec. VII).

(g) We will propose in Sec. IV that the automorphisms of internal-symmetry groups are hidden symmetries, in general. Since the groups used in physics are often semi-simple Lie groups, according to Appendix A 5, they have only a finite number of outer automorphisms. Thus, although for chiral symmetries the absence of zero-mass bosons implies that they cannot be exact, in our case the hidden (discrete) symmetries are exact. As we will see, hidden, discrete symmetries abound in nature. It seems that much can be learned if more efforts are put into the study of discrete, hidden symmetries.

IV. AUTOMORPHISMS OF INTERNAL-SYMMETRY GROUPS

Consider the physical meaning of an automorphism of G which is also an automorphism of S. [The case when S is mapped to other elements in G (but not in S) will be discussed later.] We shall denote the elements in S by an ordered set $S = \{a, b, c, \ldots\}$. An automorphism of S means that a rearrangement of its elements leaves the group structure unchanged. Thus, if f is an automorphism of S which interchanges the elements a and b, $S(a, b, \ldots) \xrightarrow{f} S(b, a, \ldots)$, then, since S is only defined through the group structure, the existence of f implies that we cannot distinguish a from b. In other words, to choose a over b or vice versa is a convention. Quantum mechanically, changing this convention is realized by the operator f, which must then be a symmetry operator. Thus, by our very assumption that S is an internal-symmetry group, we find that f, which is an automorphism of S, must itself be a symmetry operator. We emphasize that, in drawing our conclusion, it is important that elements of S are specified by the group structure. If there are other external constraints that one might impose on S, then one must investigate the behavior of f upon such constraints.

The above discussion is trivial when f is an inner automorphism of G. But if f is an outer automorphism of G, then we must enlarge G to include f. In other words, if a system has an internal-symmetry group, say G, which possesses some outer automorphisms, then the system actually has as its internal-symmetry group a larger group, namely, the extension of G by its outer automorphisms. (See Appendix A 6.)

This result brings out the important role played by the intrinsic property of an internal-symmetry group. In particular, to serve as a complete internal-symmetry group, the group G must be such that it has no outer automorphism which is also an automorphism of S.

Our discussions, as well as our language, are similar to considerations of group extension.[11,12] The difference lies in that we argue that an internal-symmetry group must be extended by its outer automorphisms, while in group-extension problems we are given some extra symmetry to begin with. Also, in group extensions one studies the question: How many different extensions are possible? In our case, the outer automorphism determines the extension unambiguiously. (It is in general a semi-direct product.)

It is useful to spell out in detail the way the symmetry f is enforced. We start from a system described by a Hamiltonian H, with the eigenstates $|0\rangle$; $|\alpha_i\rangle, \ldots$; $|\beta_j\rangle, \ldots$, satisfying

$$H|0\rangle = 0, \quad H|\alpha_i\rangle = E_\alpha|\alpha_i\rangle, \ldots,$$

$$S|\alpha_i\rangle = S_{ij}|\alpha_j\rangle, \ldots .$$

If we define

$$|\bar{\alpha}_i\rangle \equiv f|\alpha_i\rangle, \ldots,$$

then the physical systems ($|0\rangle, |\alpha_i\rangle, \ldots$) and ($|\bar{0}\rangle, |\bar{\alpha}_i\rangle, \ldots$) have identical behavior if $[H,f]=0$, for both systems obviously have the same energy spectrum. The states $|\bar{\alpha}_i\rangle$ also form a linear representation of S,

$$(fSf^{-1})|\bar{\alpha}_i\rangle = S_{ij}|\bar{\alpha}_j\rangle,$$

which is a linear representation since fSf^{-1} is an element of S. Furthermore, all matrix elements are equal,

$$\langle\alpha_i|\beta_j\rangle = \langle\bar{\alpha}_i|\bar{\beta}_j\rangle.$$

The existence of f also implies that the original Hilbert space must be enlarged by the barred states. This doubling of the Hilbert space generates the symmetry f. Alternatively, we may say that the choice of either of the two identical subspaces in the enlarged Hilbert space is a convention. The change from one subspace to another is carried out by the symmetry operator f. We may thus call f a "duplication symmetry." Let us illustrate this point further by an example. Consider a system in which charge is conserved, but all states have positive charges. We say that this system has the symmetry $U(1) = \{e^{i\theta Q}\}$. Now, when we say $Q>0$, we are making a convention. It is obvious that we may equally well use $Q<0$. That is, we may duplicate the original system by changing $Q>0$ everywhere to $Q<0$. The duplication so created has the identical behavior to the original one. It corresponds to the only outer automorphism of $U(1)$: $e^{i\theta Q} \to e^{-i\theta Q}$. Therefore, given $U(1)$ as a symmetry group, due to the existence of a nontrivial outer automorphism of $U(1)$, there are in general two systems with identical physical behavior. The automorphism of $U(1)$ is realized physically by carrying one system into the other. This example also shows clearly that the duplication symmetry is, in general, hidden. Only when the two systems coalesce do we have a degenerate symmetry.

At the risk of repetition, we wish to discuss the question: What would happen if $[H,f]\neq 0$? In the previous example, this would imply that the two systems characterized by $Q>0$ and $Q<0$ are distinguishable. In other words, to use $Q>0$ or $Q<0$ is no longer a convention. This situation is entirely similar to the case of parity violation.[7] The parity operator changes right-handedness into left-handedness. If $[H,P]=0$, then what we call left-handedness or right-handedness is a convention, and is not physically distinguishable. If $[H,P]\neq 0$, then the "handedness" ceases to be a convention, and is absolutely measurable.

We must now discuss the possibility when G possesses an automorphism, say g, which maps elements in S into $\bar{S}$. Should we also require g to be a symmetry operator? The answer is no, in general. For, as we saw in our definition, a general symmetry group is characterized by both its group structure and its actions on the state vectors. To the extent that S and $\bar{S}$ behave differently on the states, we cannot require g to be a symmetry operator. Alternatively, this follows from a remark in Sec. III, where it was pointed out that $\bar{S}$ really corresponds to cosets of S. Since they refer to different entities, a mapping $S\to\bar{S}$ does not establish a physical equivalence. This does not mean that we should exclude all such automorphisms, though. The inner automorphisms of G, in general, contain mappings $S\to\bar{S}$. They are by assumption symmetry operators. Their existence is due, however, to the particular property of G under consideration.

We end this discussion by summarizing our results in the following proposition. *If an internal-symmetry group G has an outer automorphism f which is also an automorphism of S, then f itself is also a symmetry operator. The symmetry operator f, in general, is hidden.*

Let us add a remark concerning the practical aspects of our proposition. If we start from an internal-symmetry group G with an outer automorphism f, we must decide, by comparing with the real world, whether f is degenerate or hidden. (This question cannot be decided within our framework.) If f is degenerate, we repeat our process. It may appear that when f is hidden, the proposition is rather useless. Actually this is not the case. We note that the symmetry groups in physics usually come in "chains": $G_0 \subset G_1 \subset G_2 \subset \cdots$. We are in fact accustomed to start with a certain symmetry group G_0 and then enlarge it to G_1, etc. Without studying the outer automorphisms of $G_0, G_1, \ldots$ the chain is established by requiring that each G_i is a subgroup of G_{i+1}. It is clear that things are not so simple when outer automorphisms of G_i are introduced into the picture. An example of the severe restrictions so obtained is presented in Sec. VI, where it is found that $SU(3)\times Z_2$ $(P, P^2=1)$ and $U(2)\times Z_2$ cannot fit into a chain.

V. EXAMPLES

In this section we will apply the results of Sec. IV to a number of concrete examples in physics.

1. U(1) symmetry. The U(1) symmetry corresponds to the "number laws." They are, for instance, the charge, the hypercharge, the baryon-number, and the lepton-number conservation. We may denote these gauge groups by $U_Q(1)$, $U_Y(1)$, $U_B(1)$, and $U_L(1)$. The abstract group U(1) consists of the elements $e^{i\theta}$, $0\le\theta\le 2\pi$. The physical symmetry operator corresponding to $e^{i\theta}$ is $e^{i\theta Q}$, $e^{i\theta Y}$, etc. The condition $e^{i2\pi}=1$ leads to the restriction that Q, Y, B, and L can only take integral values.[14] The group U(1) is Abelian, hence all its inner automorphisms are trivial. It can be shown[12] that the only outer automorphism of U(1) is $e^{i\theta}\to e^{-i\theta}$. For the physical operators, we have then the outer automorphisms,

$$\begin{aligned} &Q \xrightarrow{C_Q} Q' = C_Q Q C_Q^{-1} = -Q,\\ &Y \xrightarrow{C_Y} -Y,\\ &B \xrightarrow{C_B} -B,\\ &L \xrightarrow{C_L} -L. \end{aligned} \tag{7}$$

Thus, the "charge conjugation" C_Q, the "hypercharge conjugation" C_Y, etc., are consequences of the corresponding number laws. It should be emphasized that, separately, our C_Q, C_Y, etc., are not quite the usual charge-conjugation operators which change all these additive quantum numbers. Another important point is that the symmetry operators C_Q etc. are in general hidden, while the usual charge conjugation is supposed to be a degenerate symmetry. We also emphasize that, within our framework, we are unable to judge whether a given automorphism gives rise to a degenerate or a hidden symmetry.

What other algebraic properties can we say about these operators? We note that C_Q, when applied twice, reduces to the identity automorphism,

$$Q \xrightarrow{C_Q^2} Q.$$

This means that C_Q^2 is essentially the identity. Actually, we have only to require that C_Q^2 be in the center of U(1), which is here the entire group,

$$C_Q^2 = e^{2i\theta Q}. \tag{8}$$

However, the associative law may be used[15] to narrow down the choice of θ. Consider

$$C_Q C_Q^2 = C_Q e^{2i\theta Q} = e^{-2i\theta Q} C_Q$$

and

$$C_Q^2 C_Q = e^{2i\theta Q} C_Q.$$

We see immediately

$$e^{2i\theta Q} = e^{-2i\theta Q}.$$

It follows that

$$\theta = 0 \text{ or } \tfrac{1}{2}\pi.$$

Thus

$$C_Q^2 = 1 \text{ or } (-1)^Q \tag{9}$$

are the only possibilities. If C_Q is a hidden symmetry, then either solution in Eq. (9) expresses the fact that there are two cosets of $U_Q(1)$ when enlarged by its outer automorphism. And there is no distinction between these two solutions. On the other hand, if C_Q is a degenerate symmetry, then $C_Q^2=1$ and $C_Q^2=(-1)^Q$ are different. They would have different physical consequences. This situation is analogous to the analysis of Wigner,[15] who found it necessary to introduce different types of representations of the Poincaré group. Just as in the case of the Poincaré group, the analysis is not complete[16] without considering the other internal-symmetry groups. We observe that, in nature, only $C_Q^2=1$ seems to correspond to a degenerate symmetry.[17] From now on, we will only consider the following case:

$$C_Q^2 = 1. \tag{10}$$

Mathematically, we may say that if we have U(1) as an internal-symmetry group, then we must have a larger one: $U(1)\times_s Z_2$, where $\times_s$ denotes the semi-direct product (Appendix A 7).

With the new internal-symmetry group $U(1)\times_s Z_2$, we must study whether it has any further outer automorphisms (when C_Q is degenerate). In Appendix A 13, we shall show that U(1) is a characteristic subgroup of $U(1)\times_s Z_2$. Therefore, the only possible new automorphism is of the form

$$C_Q \to e^{i\theta Q} C_Q,$$

which is actually an inner automorphism,

$$C_Q \to e^{i\theta Q/2} C_Q e^{-i\theta Q/2}.$$

2. U(1) × U(1). We may take it to be $U_Q(1)\times U_B(1)$, for definiteness. The elements in $U_Q(1)\times U_B(1)$ may be denoted as $(e^{i\theta Q}, e^{i\phi B})$. It can be shown that the most general outer automorphisms of $U_Q(1)\times U_B(1)$ are of the form:

$$\begin{pmatrix} Q\\ B\end{pmatrix} \to \begin{pmatrix} Q'\\ B'\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} Q\\ B\end{pmatrix},$$

where a, b, c, and d are integers and the determinant of the transformation satisfies

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm 1.$$

[This prescription admits of an obvious generalization to $U(1)\times U(1)\times \cdots \times U(1)$.]

Physically, if we only have Q and B conservation, both taking integral values, then *a priori* we cannot distinguish Q from B. This symmetry corresponds to $a=d=0$, $b=c=1$. As another example, the case when $a=d=-1$, $b=c=0$ obviously corresponds to the charge conjugation. It is not difficult to convince oneself that similar interpretations holds for the general solution given above.

We should also emphasize that in reality the symmetry groups involving Q and B are more complicated than $U(1)\times U(1)$. Our considerations are not complete without including those larger symmetry groups.

3. $U_Q(1)\times U_B(1)\times U_Y(1)\times U_L(1)$. The usual charge-conjugation operator corresponds to[18]

$$C = C_Q C_B C_Y C_L. \tag{11}$$

Again we are not in a position to claim that C is a degenerate symmetry. On the other hand, our considerations are in line with the proposal[12] that one must define the discrete symmetries with reference to the types of interactions (strong, electromagnetic, and weak). In particular, Eq. (11) suggests a natural reason for the breakdown of C, since $U_Y(1)$ is broken by the weak interactions. The violation of charge conjugation stems from the hypercharge nonconservation. Within the realm of strong and electromagnetic interactions, however, C is conserved (even though it may be hidden).

4. SU(2) (isospin conservation). It turns out[12] that all automorphisms of SU(2) are inner automorphisms. Our considerations do not give anything new for SU(2).

5. U(2) (Y, $\vec{I}$ conservation). We begin by emphasizing that U(2) is not identical to $U(1)\times SU(2)$. (See Appendix A 10.) It is an extension of SU(2) by U(1) satisfying the constraint $(-1)^Y=(-1)^{2I}$.

In Appendix A 12 we will show that SU(2) is a characteristic subgroup of U(2). From examples 1 and 4, we conclude that the only outer automorphism of U(2) is

$$Y \xrightarrow{f} -Y, \quad \vec{I} \xrightarrow{f} \vec{I}. \tag{12}$$

f corresponds clearly to the usual G parity, as far as its effect on the group U(2) is concerned. Just as in example 1, f^2 induces the trivial automorphism. f^2 belongs therefore to the center of U(2), which is U(1) [with $(-1)^Y=(-1)^{2I}$]. The usual convention is to choose[19]

$$f^2=(-1)^Y=(-1)^{2I}.$$

Thus, from U(2) we have the extended group $[U(2)\times_s Z_4]/Z_2$, where $Z_4=(f,f^2,f^3,f^4=1)$ and the factor group is taken since f^2 must be identified with the element $(-1)^Y=(-1)^{2I}$ in U(2).

It is easy to see that U(2) is again characteristic in $[U(2)\times_s Z_4]/Z_2$. Hence the only possibly new automorphism is a transformation on f. Let us write $f\to f'=\chi f$, with $\chi\in U(2)$. In order for f' to commute with $\vec{I}$ [Eq. (12)], χ must be in the center of SU(2); in order for $f'^2=f^2$, χ must be of order 2. It follows that $\chi=(-1)^{2I}$ is a unique solution. Thus, the only possible transformation on f is[20]

$$f \xrightarrow{g} f'=(-1)^{2I}f=(-1)^Y f.$$

However, g does not give anything new since it is actually an inner automorphism,

$$f \xrightarrow{g} f'=e^{i\pi Y/2} f e^{-i\pi Y/2}.$$

6. $U_B(1)\times U(2)$. From examples 1 and 5, the outer automorphisms are C_B and f. [Note that $B\leftrightarrow Y$ is not an automorphism because of the constraint $(-1)^Y=(-1)^{2I}$.] The usual G parity actually corresponds to $fC_B=C_Bf$.

7. SU(3) and $U_B(1)\times SU(3)$. For SU(3) the only outer automorphism[12] is the complex conjugation of the 3×3 unitary, unimodular matrices, which are usually written as $e^{i\theta_i\lambda_i}$. The corresponding physical symmetry operators are $e^{i\theta_i F_i}$, where the F_i's are the familiar generators of SU(3). Since $\lambda_{2,5,7}$ are imaginary, and the other λ_i's real, the automorphism (denoted by C_3)

$$e^{i\theta_i\lambda_i} \xrightarrow{C_3} (e^{i\theta_i\lambda_i})^* = e^{-i\theta_i\lambda_i^*}$$

corresponds to

$$F_{1,3,4,6,8} \xrightarrow{C_3} -F_{1,3,4,6,8}, \quad F_{2,5,7} \xrightarrow{C_3} +F_{2,5,7}. \tag{13}$$

Just as before we may normalize C_3 by

$$C_3^2=1. \tag{14}$$

Thus, from SU(3), we have $SU(3)\times_s Z_2$. That no further outer automorphisms exist is shown in Appendix A 16.

The transition from SU(3) to $U_B(1)\times SU(3)$ is straightforward, since SU(3) is a characteristic subgroup (Appendix A 12). We only note that the "usual" charge-conjugation operator in strong interactions actually corresponds to

$$C=C_BC_3,$$

provided the combination C_BC_3 is degenerate.

It is also important in this connection to distinguish the R reflection from the charge conjugation. While both the R reflection and the charge conjugation induce the same transformations on the SU(3) generators as in Eq. (13), they differ in their effects on the physical states. The R reflection changes the quantum numbers of a particle, but it does not transform a particle into its antiparticle (e.g., it sends a proton into Ξ^-). The usual charge conjugation transforms a particle into its antiparticle, but it still leaves the vacuum invariant. For a hidden symmetry, the transformed state is an independent state and the vacuum is also not invariant. Therefore, the automorphism $C=C_BC_3$ does not necessarily coincide with the usual charge conjugation, or, even less so, with the R reflection.

8. $[SU(3)\times SU(3)]\times_s Z_2$. This is the case of chiral symmetry where the parity operator forms[21] the group Z_2 ($P, P^2=1$). The outer automorphisms are easily seen to be the two charge-conjugation operators in $\mathrm{SU}(3)_+$ and $\mathrm{SU}(3)_-$. They induce the transformations

$$(F_{1,3,4,6,8}) \overset{C^+}{\longleftrightarrow} (-F^5_{1,3,4,6,8}),$$

$$(F_{2,5,7};F^5_{2,5,7}) \overset{C^+}{\longleftrightarrow} (F_{2,5,7};F^5_{2,5,7}),$$

and

$$(F_{1,3,4,6,8}) \overset{C^-}{\longleftrightarrow} (F^5_{1,3,4,6,8}),$$

$$(F_{2,5,7};F^5_{2,5,7}) \overset{C^-}{\longleftrightarrow} (F_{2,5,7};F^5_{2,5,7}).$$

Since we usually assume that F^5_i are hidden and F_i degenerate, both C^+ and C^-, in the notation of Sec. III, map elements in S to elements in $\bar{S}$. According to Sec. IV, we cannot require them to be symmetry operators. However, we may construct a composite operator $C=PC^+C^-$, which induces

$$(F_{1,3,4,6,8};F_{2,5,7}) \xrightarrow{C} (-F_{1,3,4,6,8};F_{2,5,7}),$$

$$(F^5_{1,3,4,6,8};F^5_{2,5,7}) \xrightarrow{C} (F^5_{1,3,4,6,8};-F^5_{2,5,7}).$$

Thus C is an automorphism of the diagonal SU(3) and is thus a symmetry operator. It obviously corresponds to the usual charge conjugation.

9. $SU(3)\times Z_2$ [*SU(3) and parity*]. According to Appendix A 15, the only outer automorphism of $\mathrm{SU}(3)\times Z_2$ is the charge conjugation C_3 defined in example 7. The addition of parity invariance to SU(3) does not lead to anything new.

10. $SU(2)\times Z_2$ *(isospin and parity)*. According to Appendix A 14, the only outer automorphism of $\mathrm{SU}(2)\times Z_2$ is

$$P \xrightarrow{W} P' = WPW^{-1}=(-1)^{2I}P,\qquad [W,\vec{I}]=0. \tag{15}$$

Using the proposition of Sec. IV, W is itself a symmetry operator,

$$[W,H]=0. \tag{16}$$

From Eq. (15), we immediately obtain

$$[W^2,P]=[W^2,\vec{I}]=0.$$

Thus W^2 must be in the center of $\mathrm{SU}(2)\times Z_2$, so that

$$W^2=1 \text{ or } (-1)^{2I}. \tag{17}$$

The two solutions are related to each other by

$$W\to W'=e^{i\pi I_3}W=We^{i\pi I_3}.$$

We may therefore choose[22]

$$W^2=1. \tag{18}$$

Hereafter Eq. (18) will always be used.

Finally, then, from $\mathrm{SU}(2)\times Z_2$, we find the enlarged symmetry group $[\mathrm{SU}(2)\times Z_2]\times_s Z_2$.

11. $U(2)\times Z_2$ *(Y, $\vec{I}$ and P)*. According to Appendix A 17, the only outer automorphism (besides the G parity) of $\mathrm{U}(2)\times Z_2$ is

$$P\xrightarrow{W} P' = WPW^{-1}=(-1)^{2I}P=(-1)^{3Y}P,\qquad [W,\vec{I}]=[W,Y]=0. \tag{19}$$

Except for the constraint $(-1)^{3Y}=(-1)^{2I}$, this case is identical with the previous example. We may state, then, that from $\mathrm{U}(2)\times Z_2$ we must have $[\mathrm{U}(2)\times Z_2]\times_s Z_2$.

The physical consequences of these results will be discussed fully in Sec. VI.

VI. CHIRAL SYMMETRIES

Recently much effort has been put into the study of chiral symmetries, be it $\mathrm{SU}(2)\times\mathrm{SU}(2)$ or $\mathrm{SU}(3)\times\mathrm{SU}(3)$. We wish now to discuss the implications of our previous discussion (especially examples 9, 10, and 11) to the chiral symmetries.

Let us start from the strong-interaction symmetry group $\mathrm{U}(2)\times Z_2$. In Sec. V we showed that its outer automorphisms are the G parity and the W operator, given by

$$WPW^{-1}=(-1)^{2I}P=(-1)^{3Y}P,\qquad [W,\vec{I}]=[W,Y]=0,\qquad W^2=1. \tag{20}$$

We emphasize that the same automorphism W results if one considers, instead of $\mathrm{U}(2)\times Z_2$, the symmetry group $\mathrm{SU}(2)\times Z_2$. From now on we will often not make the distinction between $\mathrm{U}(2)\times Z_2$ and $\mathrm{SU}(2)\times Z_2$.

According to the proposition in Sec. IV, W is itself a symmetry operator, satisfying

$$[H,W]=0. \tag{21}$$

Thus, given $\mathrm{U}(2)\times Z_2$, we have a larger symmetry group $[\mathrm{U}(2)\times Z_2]\times_s Z_2$.

It turns out that most existing model Hamiltonians for broken chiral symmetries do not satisfy Eq. (21). In Appendix B we analyze these models in detail, and point out explicitly "what is wrong" if

$$[H, W] \neq 0 .$$

It seems important, at this juncture, to say a few words about the physical origin of the W symmetry. Let us consider, for a moment, the familiar rotation group. It is well known that a spinor ψ is ambiguous up to a sign. Indeed, rotation of 2π, which is the identity, is equal to $(-1)^{2J}$, and transforms ψ into $-\psi$. As Yang and Tiomno[23] pointed out 20 years ago, this ambiguity is carried over when we consider the parity operator (in fact, any discrete symmetry operator). Indeed, they defined four types of spinors:

$$P\psi_{A,B,C,D} = (+1, -1, +i, -i)\psi_{A,B,C,D} . \tag{22}$$

For our purposes we will concentrate on the types A and B only.[24] Equation (22) is merely a reflection of the fact that P or IP, where I is the identity rotation of 2π, are not distinguishable. If, however, ψ_A and ψ_B exist simultaneously, then we "know" that they are different, even though there is no "intrinsic" difference[25] between them. To put it differently, we may define an exchange operator

$$\psi_A \overset{f}{\leftrightarrow} \psi_B . \tag{23}$$

Then, f is a (hidden) symmetry operator.[26] It turns out, however, that only type A or type B seem to exist in nature. The symmetry f, then, is not very restrictive. (More discussions on these problems will be presented in Appendix B.)

We turn now to isospin symmetry. Again an "isospinor," ψ^I, enjoys the ambiguity of a sign change due to rotations of 2π in isospin space. We may then define two types of isospinors $\psi^I_{\alpha,\beta}$:

$$P\psi^I_{\alpha,\beta} = (+1, -1)\psi^I_{\alpha,\beta} . \tag{24}$$

Just as before, as long as isospin symmetry is good, there is no intrinsic difference between types α and β, even though they behave distinctively when they coexist. In particular, the exchange operator effecting $\psi^I_\alpha \leftrightarrow \psi^I_\beta$ is a symmetry operator. According to Eq. (15), this exchange operator is just the W operator:

$$\psi^I_\alpha \overset{W}{\leftrightarrow} \psi^I_\beta . \tag{25}$$

The W symmetry originates from the ambiguity of using for the parity operator either P, or IP, where I is the identity rotation of 2π in isospin space.

So far the isospin symmetry and the rotation symmetry parallel each other exactly. But there is an important difference between them. Whereas only one type of ordinary spinors seems to exist, both types ψ^I_α and ψ^I_β do exist in nature. They are in fact the "vector" and "axial-vector" strangeness-changing current operators in the weak interactions. Physically, the K_{l2} and K_{l3} decays necessitate the existence of two types of $I=\frac{1}{2}$ hadronic currents. On the other hand, to say that the axial-vector current is responsible for K_{l2}, and the vector current for K_{l3}, depends on the conventional assignment of the kaon to be a pseudoscalar particle. To be sure, these currents are different. But the important point is that there is no intrinsic difference between the two. No physics can change if we switch the two currents, since we are only replacing P by IP. It is now clear that Eq. (21) is a consequence of the existence of isospinors. That Eq. (21) is highly restrictive, as we are going to demonstrate in the following, stems from the fact that both types $\psi^I_{\alpha,\beta}$ do exist in nature simultaneously.

After this long digression, let us return to the discussion of Eqs. (20) and (21). Suppose that we wish to enlarge our symmetry group $[U(2)\times Z_2]$ by the addition[27] of the strangeness-changing charge operators $F_{4,5,6,7}$. According to the fundamental postulate of Gell-Mann,[28] even though these operators may be time-dependent, they satisfy, together with $F_{1,2,3,8}$, the exact SU(3) algebra

$$[F_i, F_j] = if_{ijk}F_k . \tag{26}$$

So far we have not discussed the role of the parity operator. Suppose that now $F_{1,2,3,8}$ are even under parity, then we may naturally ask about the behavior of $F_{4,5,6,7}$ under P. It is immediately clear that, since $F_{4,5,6,7}$ are isospinors, they can be either[29] vector or axial vector. More precisely, if one assumes

$$PF_{4,5,6,7}P = +F_{4,5,6,7} , \tag{27}$$

then the operators $WF_{4,5,6,7}W$ are odd under P,

$$P(WF_{4,5,6,7}W)P = -WF_{4,5,6,7}W , \tag{28}$$

and vice versa. In other words, owing to the existence of W, which originates from the spinor representations of SU(2), we see that the operators with $I=\frac{1}{2}$ must appear in "parity pairs."

With this observation it is easy to see that the validity of Eq. (26) actually implies the existence of the chiral SU(3)×SU(3) algebra. Let us choose the convention that $F_{4,5,6,7}$ are even under P [Eq. (27)]. We may define

$$F^5_{4,5,6,7} = WF_{4,5,6,7}W . \tag{29}$$

Equation (28) says that $F^5_{4,5,6,7}$ must be odd under P. Let us further define the (time-dependent) operators $F^5_{1,2,3,8}$ by

$$if_{ijk}F^5_k=[F_i, F^5_j],\quad k=1,2,3,8;\ i,j=4,5,6,7. \tag{30}$$

It is easily verified that, starting from Eq. (26) together with the definitions Eqs. (29) and (30), that F^5_i and F_i, $i=1,\ldots,8$ satisfy the chiral SU(3) $\times$SU(3) current algebra of Gell-Mann,[28]

$$\begin{aligned}[][F_i, F_j]&=if_{ijk}F_k,\\ [F_i, F^5_j]&=if_{ijk}F^5_k,\\ [F^5_i, F^5_j]&=if_{ijk}F_k.\end{aligned} \tag{31}$$

We may summarize our results in the following way. If one assumes (1) exact $U(2)\times Z_2$ symmetry and (2) SU(3) algebra as in Eq. (26), then, owing to the existence of W as implied by assumption (1), there must exist operators F^5_i. Further, F^5_i and F_i necessarily form a closed algebraic system, which is the chiral SU(3)$\times$SU(3) algebra. Alternatively, given SU(3) and W, one may generate[30] a larger group by forming all possible products of the form WSU(3), SU(3)W, WSU(3)W, WSU(3)WSU(3), ... in an obvious notation. This larger group, according to the previous discussion, is precisely the SU(3) $\times$SU(3) group. In this sense, we see that SU(3) and parity generate uniquely the $[SU(3)\times SU(3)]\times_s Z_2$ group.

Several remarks are in order:

(1) The physical interpretation of Eq. (31) remains an assumption, i.e., the identification of F^5_i with the physical weak-interaction currents cannot be obtained from our construction. On the other hand, if one identifies F_i with the vector weak-interaction currents, the symmetry of F_i and F^5_i make it very difficult to do otherwise.

(2) We showed that SU(3) and W generate SU(3) $\times$SU(3). It is not difficult to identify W explicitly with a finite rotation in SU(3)$\times$SU(3). Since, according to Eqs. (20) and (30), W commutes with $F_{1,2,3,8}$ and $F^5_{1,2,3,8}$, it follows that $W\sim\exp(i\theta Y+i\phi Y_5)$. [$W$ may also be in the center of $SU(2)_\pm$. However, since $(-1)^{2I_\pm}=(-1)^{3Y_\pm}$, they may be expressed in the given form.] Using $W^2=1$ and $WPW=(-1)^{3Y_P}$, θ and ϕ are uniquely determined to give[22]

$$W=e^{i3\pi Y_-}=e^{i2\pi I_3^-}. \tag{32}$$

(3) We have emphasized how $[SU(3)\times SU(3)]\times_s Z_2$ is built up from $U(2)\times Z_2$. This is to be contrasted with the viewpoint of "broken symmetries" according to which $U(2)\times Z_2$ is supposed to be the remnant symmetry when SU(3)$\times$SU(3) is broken. As far as $[SU(3)\times SU(3)]\times_s Z_2$ is only an enlargement from $U(2)\times Z_2$, there may or may not be an exact SU(3) $\times$SU(3) limit. Only Eq. (31), but not the time dependence of F_i and F^5_i, is important. On the other hand, if one does believe in a limit in which SU(3) $\times$SU(3) is exact, then, since W is identified as an element in SU(3)$\times$SU(3) [Eq. (32)], we must require that $[H,W]=0$ [Eq. (21)] when SU(3)$\times$SU(3) is broken. Similarly, if there is an SU(2)$\times$SU(2) limit which is subsequently broken, then we must also have $[H,W]=[H,e^{i2\pi I_3^-}]=0$ (even though the infinitesimal generator I_3^- is time-dependent).

(4) If we compare example 9 with examples 10 and 11, we see that the automorphism W for SU(2) $\times Z_2$ and $U(2)\times Z_2$ is lost when we go over to SU(3) $\times Z_2$. W is restored only if we enlarge SU(3)$\times Z_2$ to $[SU(3)\times SU(3)]\times_s Z_2$. We may then state: If U(2) $\times Z_2$ is exact, then SU(3) must be broken if SU(3) $\times$SU(3) is broken. In other words, there does not exist an "exact SU(3) limit," there is only an "exact SU(3)$\times$SU(3) limit." The construction of SU(3) $\times$SU(3) makes this point amply clear. If one has SU(3) then one must have SU(3)$\times$SU(3). Mathematically, the reason why SU(3) must be broken is that SU(3)$\times Z_2$ does not contain W, even though it does contain $U(2)\times Z_2$ as a subgroup. This result illustrates the internal consistency properties of internal-symmetry groups. It shows that the construction of higher symmetries is not as arbitrary as one might think. Physically, this result is easy to understand. The W symmetry originates from the inherent ambiguity in defining the parity of isospinors. If SU(3) were exact, then this ambiguity would be removed. [For instance, the kaon ($I=\frac{1}{2}$) would have to transform like the pion ($I=1$) under parity, if SU(3) were exact.] Only by going to SU(3)$\times$SU(3) can we recover this ambiguity. (See also Appendix B.)

(5) According to Eq. (29), since $[H,W]=0$, the time dependence of $F_{4,5,6,7}$ and $F^5_{4,5,6,7}$ must be identical. Physically, we may in principle measure the time dependence of these operators. The nonmeasurability of the absolute parity of these operators can only mean that their time dependence must be the same. We may note that it is usually stated that SU(2)$\times$SU(2) is an approximate symmetry, so that $F^5_{1,2,3}$ are "approximately conserved." On the other hand, $F_{4,5,6,7}$ and $F^5_{4,5,6,7}$ are "worse" symmetry operators in that they have "stronger" time dependence. Our result shows that, even though $F_{4,5,6,7}$ and $F^5_{4,5,6,7}$ may have stronger time dependence, their dependence on time must be identical.

(6) So far our discussions are confined to the strong interactions. What if we introduce the electromagnetic and weak interactions? The electromagnetic and weak interactions break the U(2) symmetry. However, it is usually assumed that the symmetry is broken in a definite way. Using the transformation properties of J_μ^{em} and J_μ^{wk}, it was shown[3] earlier that they both commute with W.

If we assume, as usual, the current × current form for $H_{\rm em}$ and $H_{\rm wk}$, we have immediately

$$[W, H_{\rm em}] = [W, H_{\rm wk}] = 0 . \qquad (33)$$

Thus, we find that the W symmetry, owing to the particular transformation properties of $J_\mu^{\rm em}$ and $J_\mu^{\rm wk}$, is actually a symmetry of the strong, the electromagnetic, as well as the weak interactions. It is also interesting to note that, conversely, we may say that the universal $V-A$ form of $J_\mu^{\rm wk}$ is a result of requiring W invariance even in the presence of the weak interactions.

VII. SPACE-TIME SYMMETRIES

So far our considerations have been confined to the internal symmetries (including the parity). We now turn to a discussion of space-time symmetries. As was mentioned in the Introduction, the fundamental difference between internal as compared with space-time symmetries seems to lie in the measurability of the space-time coordinates. Thus, in discussing the isospin, it is adequate to study only the group SU(2), without paying any attention to the "isospin-space coordinates." For space-time symmetries, the (restricted) Poincaré group $\mathcal{P}_0$ is obtained by requiring the invariance of distances in the Minkowski space: $g^{\mu\nu}(x-x')_\mu(x-x')_\nu$. We then go over to the generators of $\mathcal{P}_0$, satisfying

$$\begin{aligned}
&[P_\mu, P_\nu] = 0,\\
&i[M_{\mu\nu}, P_\lambda] = g_{\mu\lambda}P_\nu - g_{\nu\lambda}P_\mu ,\\
&i[M_{\mu\nu}, M_{\rho\sigma}] = g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\rho} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma} .
\end{aligned} \qquad (34)$$

On the one hand, just as for isospin, it seems that one may study $\mathcal{P}_0$ without reference to the coordinates. Thus, one talks about irreducible representations of $\mathcal{P}_0$, the transformation properties of operators under $\mathcal{P}_0$, etc. On the other hand, the coordinates do come in in a subtle way. For instance, we must consider superposition of (nonlocal) eigenstates of momenta to form wave packets which correspond to particle states. In field theory, $\mathcal{P}_0$ and the coordinates are certainly inseparable.

It is not clear whether our considerations on the internal-symmetry groups may or may not be applicable to the space-time symmetries. In the following we assume that it is meaningful to study $\mathcal{P}_0$ as if it behaves like an internal-symmetry group.

It has been shown by Michel[11] that the outer automorphisms of $\mathcal{P}_0$ consist of P (parity), T (time-reversal), and D (dilation). Moreover, there is no further outer automorphism for the extended group. If $\mathcal{P}_0$ is an exact symmetry, as is usually taken to be true, then, applying the proposition of Sec. IV, we are faced with the dilemma that P, T, and D must all be exact symmetries. But experimentally we know that none of the three are exact symmetries. Something must be wrong.

It may be that our considerations simply do not apply to the space-time groups. Yet it is not clear to us why it does not apply. On the other hand, P, T, and D are approximate symmetries, it would be desirable if some rationale can be found for their existence.

Let us observe that P, T, and D violations are actually a very puzzling problem. As was emphasized by Lee and Wick,[12] if P and T are what they are supposed to represent, i.e.,

$$\begin{aligned}
&\vec{x} \xrightarrow{P} -\vec{x},\\
&t \xrightarrow{T} -t,
\end{aligned} \qquad (35)$$

then P and T must commute with the Hamiltonian operator, which is the displacement operator in time,

$$t \xrightarrow{\exp(iH\delta t)} t + \delta t . \qquad (36)$$

The violation of the P and T symmetries implies that the "geometrical interpretation" [Eq. (35)] somehow breaks down. The same remark applies to the dilation operator. If D is really the dilation,

$$x_\mu \xrightarrow{\exp(iD\lambda)} e^\lambda x_\mu ,$$

then we must have

$$[D, H] = -iH, \qquad (37)$$

which leads to dilation invariance,

$$\frac{d}{dt}D = 0.$$

When dilation invariance is broken, Eq. (37) is no longer true,[31] which again implies the breakdown of the geometrical interpretation.

We may summarize the situation in the following way. If P, T, and D are what they are supposed to be, they must be conserved exactly. In reality we seem to be observing the physical operators P', T', and D', which do not coincide with P, T, and D that originate from the geometrical interpretation. What are these operators P', T', and D'? We don't know.

But, if the physical operators P', T', and D' are not conserved, they are also not outer automorphisms of $\mathcal{P}_0$. (Since their commutation relations with P_μ are different from those obtained from P, T, and D.) If P', T', and D' are not outer automorphisms, then they also are not entitled to be symmetry operators.

The whole thing is very puzzling, to say the least, and we do not know how it may be resolved. We wish to offer a few speculative remarks, though. It is noteworthy that the homogeneous

Lorentz group, SL(2, C), is complete (Appendix A 3). All the trouble seems to have come from the displacement operators P_μ. Now, if we are really dealing not with a flat space-time, but rather, say a de Sitter space,[32] then only the commutators $[M_{\mu\nu}, M_{\rho\sigma}]$ are exact. The operators P_μ are limiting cases of rotation operators in the 5-dimensional de Sitter space. The 5-dimensional rotation group,[33] however, has no outer automorphisms. Therefore, we may say that the PTD dilemma actually originates from our assumption of exact Poincaré invariance. If the Poincaré invariance is approximate, we might have approximate P, T, and D invariance. All these remarks are very speculative. It is hoped that something definite can be said about these questions in the future.

VIII. CONCLUSION

In this work we have discussed what may be called the internal consistency properties of internal-symmetry groups. Loosely speaking, the suggestion was made that their automorphisms are themselves symmetries, so that, in the sense of (Appendix A 3), all internal-symmetry groups are complete (if only degenerate symmetries exist). It turns out that things are not so simple. It was found that hidden symmetries emerged naturally. In fact, there seem to be an abundance of hidden symmetries in nature. Although usual discussions of hidden symmetries necessitate the existence of zero-mass bosons, in our case this problem does not arise, since only discrete symmetries are involved.

Through a number of examples, some consequences of our suggestion are demonstrated. Thus, charge conjugation, or at least a close analog of charge conjugation, is found to originate from the gauge groups of charge conservation, baryon-number conservation, etc. For the chiral SU(3)×SU(3) symmetry, it turns out that U(2) and parity conservation imply that there is only an exact SU(3) ×SU(3) limit, but there is no exact SU(3) limit. This perhaps is the root of the discrepancies of our results with many other authors concerning broken SU(3)×SU(3) symmetry. Indeed, it seems that there has always been an assumption, tacit or explicit, that it is in principle meaningful to talk about an exact SU(3) limit, whether it be in the symmetry-breaking terms of a Hamiltonian model, or in the specification of the properties of the vacuum state. We believe that this procedure is not possible, in principle.

We have also applied our considerations to space-time symmetry groups. The results are very perplexing. In short, if a blind application to the Poincaré group is made, then we are faced with (exact) parity, time-reversal, and scale invariance. On the other hand, these invariant operators are geometrical operators which are known to deviate from the physical operators. But what are the differences of these two sets of operators? Or, what are their definitions? We do not know. Some speculations attempting to clarify this very unsatisfactory situation were made in Sec. VII.

It was suggested by Lee[34] that the discrete symmetries should be defined with respect to the different types (strong, electromagnetic, and weak) of interactions. Our discussion is in line with this suggestion. In our framework, the discrete symmetries arise from symmetry groups pertaining to a particular type of interaction. Actually we can say more about the automorphism symmetries if we know how the original symmetry is broken. Thus, as we discussed in Sec. VI, the W symmetry (which arises from the strong interaction) turns out to be a symmetry also of the electromagnetic and weak interactions, if J_μ^{em} and J_μ^{wk} transform in the usual way.

Before we conclude, it seems appropriate to enumerate a number of unsolved problems which, in our opinion, deserve the greatest attention.

(1) Our considerations brought forth a whole class of discrete, hidden symmetries. It is certainly desirable to study these types of symmetries, which have so far been almost completely neglected.

(2) Of particular relevance is the application to chiral symmetries. We have shown that there is a perfect symmetry between the vector and axial-vector strangeness-changing currents. However, we have not been able to exploit its practical consequences. For instance, the significance of partial conservation of axial-vector current (PCAC) for the strangeness-changing current certainly needs reexamination, since PCAC as usually formulated is not symmetrical with respect to interchanging vector and axial-vector currents.

(3) There have been a large number of results pertaining to the exact SU(3) limit. If only an exact SU(3)×SU(3) limit exists, many of these results need reexamination.

(4) As we discussed in Sec. VII, the space-time symmetries should clearly be studied in connection with their automorphism properties.

(5) We have not studied the so-called "higher symmetries," which have been proliferating rapidly. It is conceivable that, by examining their automorphisms, at least some of them will be found to be unsatisfactory symmetry groups.

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APPENDIX A

In this Appendix we collect a number of mathematical definitions and theorems (sometimes with proofs). Further discussions may be found in the literature.[35]

1. Automorphisms of a Group

The automorphisms of a group[11] G form a group $\mathrm{Aut}(G)$. $\mathrm{Aut}(G)$ contains an invariant subgroup $\mathrm{Int}(G)$, which are the inner automorphisms, i.e., those automorphisms which assign to each element g in G the element $g' = \chi g \chi^{-1}$, for a fixed χ in G. The factor group $\mathrm{Aut}(G)/\mathrm{Int}(G) \equiv \mathrm{Out}(G)$ consists of the outer automorphisms of G. If G has a center C, then $\mathrm{Int}(G) \cong G/C$, where $\cong$ denotes isomorphism.

2. Complete Group

Mathematically, a group G is said to be complete[11] if G has only inner automorphisms and if the center of G has no other element except the identity. Thus, for a complete group G, $\mathrm{Aut}(G) \cong \mathrm{Int}(G) \cong G$. For continuous groups, and for applications to physics, it seems convenient to relax this definition, as in Appendix A 3. This will be the definition adopted in this work.

3. Continuous Complete Group

A continuous group G is complete if G has no outer automorphisms and if the center of G contains only a finite number of discrete elements. Under this definition, for a continuous complete group G, we have $\mathrm{Aut}(G) \cong \mathrm{Int}(G) \cong G/C$. Note that the groups SU(2) and SL(2, C) are complete. (See Appendix A 9.)

4. Characteristic Subgroup

An invariant subgroup K of G is characteristic[11] if any automorphism of G is also an automorphism of K (i.e., K cannot be mapped onto other elements of G). The center of G is characteristic. If K is an invariant subgroup of G, and if G does not contain another invariant subgroup K' which is isomorphic to K, then K is a characteristic subgroup of G.

5. Semi-Simple Lie Group

If G is any semi-simple Lie group (see Ref. 13, p. 180), then $\mathrm{Out}(G)$ contains only a finite number of elements.

6. Extension of a Group

The group E is said to be an extension of a group G by a group K if G is an invariant subgroup of E and K is the factor group E/G. (In Ref. 11, however, E is said to be an extension of K by G. To avoid confusion, the extension under consideration is expressed by the exact sequence $1 \to G \to E \to K \to 1$. Our definition is the one used in the modern mathematical literature. See, e.g., MacLane in Ref. 35). Note that since G is invariant in E, inner automorphisms of E induce automorphisms on G. In particular, any element in K induces an automorphism on G. If we write α and a for arbitrary elements in G and K, respectively, we may denote the induced automorphism by $\alpha^a \equiv a\alpha a^{-1}$. In this work we are studying primarily the extension of internal-symmetry groups (G) by their outer automorphisms (K). To be precise, we are just studying the "split extension" of G by $\mathrm{Out}(G)$: $1 \to G \to E \rightleftarrows \mathrm{Out}(G) \to 1$.

7. Semi-Direct Product

E is said to be a semi-direct product[11] of G and K (denoted by $G \times_s K$) if the elements in E consist of pairs (α, a), where $\alpha \in G$ and $a \in K$, with the multiplication law $(\alpha, a)(\beta, b) = (\alpha\beta^a, ab)$. The direct product ($G \times K$) is a particular case of the semi-direct product for which $\beta^a = \beta$, for any β and a.

8. Extension by Outer Automorphism

The extension by outer automorphisms discussed in this work is in general a semi-direct product.

9. The Lie Groups A_n

The Lie groups A_n have either no outer automorphisms (for $n = 1$) or one outer automorphism (for $n \geq 2$). Thus, SU(2) (isospin) and SL(2, C) (homogeneous Lorentz group), both belonging to A_1, do not have outer automorphisms. (Other concrete examples are treated in Sec. V.)

10. The Group U(n)

The group U(n) is[11] an extension of SU(n) by U(1), but is not a direct product of SU(n) and U(1). In particular, in U(n), the intersection of SU(n) and U(1) is nonempty, and consists of the elements $Z_n = e^{i2\pi m/n}$, $m = 0, 1, \ldots, n-1$. This means that we must identify the center of SU(n) with the elements Z_n in U(1). For U(2), this identification reads $(-1)^Y = (-1)^{2I}$. Note also that the group U(n) is a subgroup of SU($n+1$).

11. Useful Identities in the SU(3) Group

It is useful to note the existence of several identities in the SU(3) group. Regardless of whether SU(3) is a symmetry group or not, as long as the SU(3) commutation relations hold, we have

$$e^{i6\pi Y} = e^{i4\pi F_j} = 1, \quad j = 1, 2, \ldots, 7.$$

This follows immediately since the eigenvalues of Y are of the form $\frac{1}{3}n$, and those of F_j, $\frac{1}{2}n$. For the SU(3)×SU(3) group, there are two such sets of equations for $SU(3)_+$ and $SU(3)_-$. Using $F_i^{\pm} = \frac{1}{2}(F_i \pm F_i^5)$, it follows immediately that

$$e^{i2\pi F_j} = e^{\pm i2\pi F_j^5}, \; j = 1, 2, \ldots, 7$$

and

$$e^{i3\pi Y} = e^{\pm i3\pi Y_5}.$$

This last equation was first pointed out explicitly by Okubo and Mathur.[36] Note also that $e^{i3\pi Y_\pm} = e^{i2\pi I_3^\pm}$.

12. Characteristic Subgroups of U(2)

In the group U(2), both SU(2) and U(1) are characteristic subgroups. [Similarly, in U(1)×SU(3), both U(1) and SU(3) are characteristic.] This may be proved directly following a method of Lee and Wick in Ref. 12, Sec. IV. Thus, the generators of U(2) satisfy

$$[I_i, I_j] = i e_{ijk} I_k,$$
$$[I_i, Y] = 0.$$

Since any continuous[37] automorphism f maps the neighborhood of the identity into the same, we have

$$I_i' \equiv f(I_i) f^{-1} = a_{ij} I_j + b_i Y,$$
$$Y' \equiv f(Y) f^{-1} = c_i I_i + dY.$$

Using the condition that I_i' and Y' satisfy the same commutation relations as I_i and Y, the coefficients are immediately determined,

$$b_i = c_i = 0,$$

so that

$$I_i' = a_{ij} I_j,$$
$$Y' = dY,$$

for any automorphism f of U(2). In other words, both U(1) and SU(2) are characteristic subgroups of U(2).

Note that in the foregoing proof we have used the very important property that the automorphism is continuous – it preserves the neighborhood of the identity. It is well known that for Lie groups, a neighborhood of the identity determines the Lie algebra. Conversely, the Lie algebra is almost enough to determine the Lie group. In studying the automorphisms of Lie groups, this property is very useful.

13. Characteristic Subgroups of $U(1)\times Z_2$, $SU(2)\times Z_2$, etc.

Consider an arbitrary automorphism of $SU(2)\times Z_2$. The neighborhood of the identity is $e^{i\epsilon_i I_i} \approx (1 + i\epsilon_i I_i) \to (1 + i\epsilon_i' I_i)$, so that $I_i \to a_{ij} I_j$, $a_{ij} = \partial\epsilon_j'/\partial\epsilon_i$. This means that the generators of SU(2) must transform amongst themselves, or that SU(2) is a characteristic subgroup of $SU(2)\times Z_2$. Entirely the same arguments apply to the groups $U(1)\times Z_2$, $U(2)\times Z_2$, $U(2)\times_s Z_2$, $SU(3)\times Z_2$, and $SU(3)\times_s Z_2$, where U(1), U(2), and SU(3) are characteristic, respectively. Geometrically speaking, the group space of $SU(2)\times Z_2$ $(P, P^2=1)$ consists of two pieces – SU(2) and PSU(2). The identity is in SU(2). The invariance of the neighborhood of the identity carries with it the invariance of the whole piece. Therefore SU(2) is characteristic.

14. Outer Automorphisms of $SU(2)\times Z_2$

Having shown in Appendix A 13 that SU(2) is characteristic in $SU(2)\times Z_2$, and since SU(2) is complete (Appendix A 3), we may easily find all the possible outer automorphisms of $SU(2)\times Z_2$ $(P, P^2=1)$ – they must be transformations on P. Now, the center of $SU(2)\times Z_2$, which is characteristic (Appendix A 4), is $(1, (-1)^{2I})\times(1, P)$. Thus, P can only be mapped into either $(-1)^{2I}$, which is ruled out since SU(2) is characteristic, or

$$P \xrightarrow{W} (-1)^{2I} P = WPW^{-1}.$$

W is the only outer automorphism of $SU(2)\times Z_2$.

15. Outer Automorphisms of $SU(3)\times Z_2$

We turn now to the case of $SU(3)\times Z_2$, in which SU(3) is characteristic (Appendix A 13). Here the center is $(1, e^{i2\pi/3}, e^{i4\pi/3})\times(1, P)$, which, however, admits of no nontrivial automorphism. It follows that the only outer automorphism of $SU(3)\times Z_2$ is that of SU(3) – the charge conjugation.

16. Outer Automorphisms of $SU(3)\times_s Z_2$

We may analogously study the group $SU(3)\times_s Z_2$, where Z_2 consists of the charge conjugation C, with $C^2 = 1$. Since SU(3) is characteristic (Appendix A 13) it is sufficient to study the automorphism on C. Let us assume

$$C \to C' = gC, \quad g \in SU(3).$$

Now, C induces on the SU(3) generators the transformation

$$CF_iC = \eta_i F_i, \quad \eta_{1,3,4,6,8} = -1, \quad \eta_{2,5,7} = +1.$$

It follows from the requirement

$$C'F_iC' = \eta_i F_i$$

that

$$g\eta_i F_i g^{-1} = \eta_i F_i .$$

Therefore, g has to be in the center of SU(3), which consists of the elements $(1, e^{i2\pi Y}, e^{i3\pi Y})$. However, $C \to C'$ is actually an inner automorphism since $e^{i\pi Y}Ce^{-i\pi Y} = e^{i2\pi Y}C$ and $e^{i2\pi Y}Ce^{-i2\pi Y} = e^{i4\pi Y}C$. Thus, $\mathrm{SU}(3)\times_s Z_2$ is complete.

17. Outer Automorphisms of $\mathrm{U}(2)\times Z_2$

We now consider the case of $\mathrm{U}(2)\times Z_2$, in which U(2) is characteristic (Appendix A 13). The center is $\mathrm{U}(1)\times(1,P)$, with $(-1)^{2I} = (-1)^{3Y}$. (Here, $3Y$ instead of Y is used in order to allow for $Y=\frac{1}{3}n$ states.) Besides the G parity (example 5, Sec. V), any outer automorphism takes the form

$$P \to e^{i\theta Y}P .$$

The condition $P^2=1$ implies that the only nontrivial solution is $\theta = 3\pi$. Therefore, just as in Appendix A 14, the only outer automorphisms of $\mathrm{U}(2)\times Z_2$ are the G parity and W,

$$P \xrightarrow{W} (-1)^{2I}P = (-1)^{3Y}P.$$

APPENDIX B

According to Eqs. (20) and (21), starting from $\mathrm{U}(2)\times Z_2$, an additional symmetry operator W is obtained. If we enlarge $\mathrm{U}(2)\times Z_2$ to $\mathrm{SU}(3)\times\mathrm{SU}(3)$, then Eq. (32) shows that W is actually a finite rotation in $\mathrm{SU}(3)\times\mathrm{SU}(3)$, namely, $W=e^{i3\pi Y_-}=e^{i2\pi I_3^-}$. Suppose that we believe in the existence of a chiral-symmetry limit which is subsequently broken, in the sense that one may write

$$H = H_0 + H', \tag{38}$$

where H_0 conserves $\mathrm{SU}(3)\times\mathrm{SU}(3)$ and H' conserves only U(2), then Eq. (21) becomes highly restrictive, when one assigns definite transformation properties to H'. Let us summarize the consequences of Eq. (21), which was discussed some time ago.[4]

(1) If[38,39] $H' \sim (3,\bar{3})+(\bar{3},3)$, then the only possible form for H' is

$$H' \sim u_0 - \sqrt{2}\,u_8 . \tag{39}$$

(2) Therefore, if H' belongs to $(3,\bar{3})+(\bar{3},3)$, then $\mathrm{SU}(2)\times\mathrm{SU}(2)$ is exact. If we only want U(2) symmetry, H' must contain terms[4] $\sim(6,\bar{6})+(\bar{6},6)$ or (8, 8).

Similar considerations apply to broken SU(2) × SU(2) symmetry, with $W=e^{i2\pi I_3^-}$. Equation (21) implies that the only possible forms of H' are[4,6]

$$H' \sim (1,1), (2,2), \ldots . \tag{40}$$

Thus, the usual assignment[40]

$$H' \sim (\tfrac{1}{2},\tfrac{1}{2}) \tag{41}$$

is ruled out by Eq. (21).

Since most existing models do not satisfy Eq. (21), it is perhaps important to settle explicitly the question: What would happen if $[H,W]\neq 0$?

Let us return to the discussion of Yang and Tiomno[23] which was considered in Sec. VI. If, for the ordinary rotation group, both ψ_A and ψ_B particles exist (all the other quantum numbers are the same for ψ_A and ψ_B), then the mass term in a Hamiltonian model should be written

$$H \sim M(\bar{\psi}_A\psi_A + \bar{\psi}_B\psi_B) \tag{42}$$

so that

$$[H, f] = 0,$$

where f is the exchange operator

$$\psi_A \overset{f}{\longleftrightarrow} \psi_B .$$

If we have

$$H \sim M_A\bar{\psi}_A\psi_A + M_B\bar{\psi}_B\psi_B , \tag{43}$$

with $M_A \neq M_B$, so that $[H,f]\neq 0$, what follows? Since $M_A \neq M_B$, which is measurable, we can always assign the "heavier" particle to be ψ_A, "the spinor," and the "lighter" particle to be ψ_B, the "pseudospinor."[41] Therefore, if $[H,f]\neq 0$, then we would be able to assign a meaning to the term pseudospinor, which is at least inconsistent with our present physical theory.

Why, then, is it that we do not seem to have learned anything from $[H,f]=0$? The answer lies in that, so far, such pairings of states do not seem to exist in nature. If only ψ_A, or ψ_B, exist, we may write

$$H \sim M\bar{\psi}\psi , \tag{44}$$

where ψ may be either ψ_A or ψ_B, and the symmetry f is implicit.

Entirely parallel analysis holds for the isospin group. Consider, as a concrete example, the case of broken $\mathrm{SU}(2)\times\mathrm{SU}(2)$ symmetry where $H'\sim(\frac{1}{2},\frac{1}{2})$. Now the isospinor $(\frac{1}{2},0)$ corresponds to $(\psi_\alpha^I + \psi_\beta^I)$, while $(0,\frac{1}{2}) \sim (\psi_\alpha^I - \psi_\beta^I)$. The even-parity part of $(\frac{1}{2},\frac{1}{2}) \sim (\frac{1}{2},0)\times(0,\frac{1}{2})$, or H', transforms like

$$H' \sim \bar{\psi}_\alpha^I\psi_\alpha^I - \bar{\psi}_\beta^I\psi_\beta^I .$$

Therefore, under W, for which $\psi_\alpha^I \longleftrightarrow \psi_\beta^I$,

$$WH'W = -H' .$$

This corresponds to the case $M_A = -M_B$ in Eq. (43). It is obvious that, if $H'\sim(\frac{1}{2},\frac{1}{2})$, then it would be possible to give absolute meaning, for instance, to an $I=\frac{1}{2}$ vector operator.

The above considerations are easily carried over to $\mathrm{SU}(3)\times\mathrm{SU}(3)$. In the $(3,\bar{3})+(\bar{3},3)$ symmetry-breaking model, we have $(u_0-\sqrt{2}\,u_8)\sim\bar{\psi}_\alpha^I\psi_\alpha^I+\bar{\psi}_\beta^I\psi_\beta^I$ and $(u_0+\frac{1}{2}\sqrt{2}\,u_8)\sim\bar{\psi}_\alpha^I\psi_\alpha^I-\bar{\psi}_\beta^I\psi_\beta^I$. We must then discard terms $\sim(u_0+\frac{1}{2}\sqrt{2}\,u_8)$ to preserve the W symmetry.

Notice that Okubo[42] had shown that if $H' \sim a(u_0+\frac{1}{2}\sqrt{2}\,u_8)$, then in a linear model we have

$$M_K{}^2 \sim M_0{}^2 + a\,,\qquad M_\kappa{}^2 \sim M_0{}^2 - a\,. \tag{45}$$

He concluded that,[43] for the linear model, W symmetry is good since it switches the kaon with the κ meson. Actually this is not the case. If $a\neq 0$, then $M_K \neq M_\kappa$. We may then define, for instance, the heavier particle to be the kaon, which means, in turn, that an absolute meaning can be given to a pseudoscalar, $I=\frac{1}{2}$ particle.

Another point which we wish to stress concerns the limit of exact SU(3)×SU(3), in particular, the behavior of the vacuum state in this limit. Let us consider the model

$$H = H_0 + \epsilon H'\,, \tag{46}$$

with $[W, H]=0$. Regardless of the value ϵ, the vacuum is doubly degenerate. For $\epsilon = 0$, the vacuum is actually infinitely degenerate. Since SU(3) is a (degenerate) subgroup of SU(3)×SU(3), out of this infinity of vacua we can pick out one which satisfies

$$\mathrm{SU}(3)|0\rangle = |0\rangle\,. \tag{47}$$

However, the existence of $|0\rangle$ actually gives rise to another vacuum, which we define as

$$|\bar{0}\rangle \equiv W|0\rangle\,, \tag{48}$$

then

$$\tilde{\mathrm{S}}\mathrm{U}(3)|\bar{0}\rangle = |\bar{0}\rangle\,, \tag{49}$$

where $\tilde{\mathrm{S}}\mathrm{U}(3)$ is the hybrid SU(3) defined earlier.[3] As $\epsilon\to 0$, then, the doubly degenerate vacua of H go over into $|0\rangle$ and $|\bar{0}\rangle$. The symmetry W is preserved in the interchangeability of $|0\rangle$ and $|\bar{0}\rangle$, whether ϵ vanishes or not. It was argued[44] that one could pick out a unique SU(3) and a unique vacuum by $\mathrm{SU}(3)|0\rangle = |0\rangle$. However, to say that we have chosen SU(3), and not $\tilde{\mathrm{S}}\mathrm{U}(3)$, already implies that we can assign absolute parity to the strangeness-changing $I=\frac{1}{2}$ generators. We have seen earlier that this is not possible.

We have emphasized in Sec. VI that there is no such thing as an exact SU(3) limit, there is only an exact SU(3)×SU(3) limit. Let us show explicitly how this happens. If we construct a model

$$H = H_0 + H'\,,$$

so that H' breaks SU(3)×SU(3) but preserves SU(3), then

$$H' \sim (n, \bar{n})\,,$$

where n is an arbitrary irreducible representation in $\mathrm{SU}(3)_+$, and $\bar{n}$ the corresponding conjugated representation in $\mathrm{SU}(3)_-$. We wish to show that, if

$$[H', W] = 0\,,$$

then

$$n = \bar{n} = 1\,,$$

or, H' is a singlet in SU(3)×SU(3) and $\sim H_0$. Let us decompose the product $(n, \bar{n})$ into sums of terms with definite isospin,

$$(n,\bar{n}) \sim \sum_{I_+} (I_+, I_-)\,,$$

with $I_+ = -I_-$. Since

$$W(I_+, I_-)W = (-1)^{2I_-}(I_+, I_-)\,,$$

it is obvious that $(n, \bar{n})$ can be invariant under W only if the representation n does not contain any half-integral isospin multiplets. For the SU(3) group, unless $n=1$, there are no such representations. This proves our statement.

Finally, let us summarize our discussions of the W symmetry in broken and exact SU(3)×SU(3) symmetry.

(1) Linear model [SU(3)×SU(3)-degenerate]: Here the vacuum is unique. If SU(3)×SU(3) is exact, there will be 16 degenerate scalar and pseudoscalar mesons. If SU(3)×SU(3) is broken, they split. The (degenerate) W symmetry is reflected in that the kaon and κ meson are still exactly degenerate.

(2) Nonlinear model [only a subgroup SU(3)-degenerate]: The vacuum is nonunique. If SU(3)×SU(3) is exact, there is an infinity of vacua out of which we can pick two, $|0\rangle$ and $|\bar{0}\rangle$. One is invariant under SU(3), the other under $\tilde{\mathrm{S}}\mathrm{U}(3)$. There are eight zero-mass mesons. The (hidden) W symmetry is reflected in our inability to distinguish SU(3) from $\tilde{\mathrm{S}}\mathrm{U}(3)$ and $|0\rangle$ from $|\bar{0}\rangle$. As a consequence, four of the zero-mass mesons may be either pseudoscalar or scalar. For broken SU(3)×SU(3), there are only two degenerate vacua. Both are invariant under U(2). Again four of the mesons may be either scalar or pseudoscalar, due to the doubling of the vacua. In the limit when the symmetry-breaking term tends to zero, these two vacua go over into $|0\rangle$ and $|\bar{0}\rangle$ discussed above.

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[1]J. Goldstone, Nuovo Cimento 19, 154 (1961).

[2]Y. Nambu, in *Group Theoretical Concepts and Methods in Elementary Particles*, edited by F. Gürsey (Gordon and Breach, New York, 1964). In this paper the term "hidden symmetry," due to Nambu, will always be used instead of "spontaneously broken symmetry," since the symmetry is not "broken" in that it still commutes with the Hamiltonian operator.

[3]T. K. Kuo, Lett. Nuovo Cimento 3, 803 (1970); Phys. Rev. D 2, 342 (1970).

[4]T. K. Kuo, Phys. Rev. D 2, 2439 (1970).

[5]T. K. Kuo, Phys. Rev. D 2, 1761 (1970).

[6]T. K. Kuo, reports, 1970 (unpublished).

[7]T. D. Lee, in *Lectures at the Second Hawaii Topical Conference on Particle Physics*, edited by S. Pakvasa and S. F. Tuan (Univ. of Hawaii Press, Honolulu, Hawaii, 1967).

[8]E. P. Wigner, *Group Theory* (Academic, New York, 1959).

[9]For antiunitary operators, the transformed ket vectors become bra vectors. However, Eq. (2) still holds.

[10]Disjoint means that there does not exist any nonvanishing matrix elements between different subspaces.

[11]L. Michel, in Ref. 2. Also L. Michel, in *Particle Symmetries and Axiomatic Field Theory*, 1965 Brandeis Summer Institute in Theoretical Physics, edited by M. Chrétien and S. Deser (Gordon and Breach, New York, 1966).

[12]T. D. Lee and G. C. Wick, Phys. Rev. 148, 1385 (1966). For the mathematical proofs, see e.g., Ref. 13.

[13]M. Hansner and J. T. Schwartz, *Lie Groups; Lie Algebras* (Gordon and Breach, New York, 1968).

[14]When we discuss the SU(3) symmetry, or the U(2) symmetry as a sub-group of SU(3), it is convenient to allow for $Y=\frac{1}{3}n$ states. In that case, corresponding to the element $e^{i\theta}$ in $U_Y(1)$, we shall assign the symmetry operator $e^{i3\theta Y}$.

[15]This method was used in discussing the extended Poincaré group by Wigner in Ref. 2.

[16]See Ref. 12 and B. Zumino and D. Zwanziger, Phys. Rev. 164, 1959 (1967).

[17]The cases $C_Q{}^2=1$ and $C_Q{}^2=(-1)^Q$ are not related in any simple way. The case $C_Q{}^2=(-1)^Q$ is analogous to the "unusual theories" discussed by Lee and Wick in Ref. 12. It is interesting to compare the above with similar cases for $U_B(1)$. Here, owing to the existence of the rotation group, the two types are actually the same. For, using $[C_B, \vec{J}]=0$ and $(-1)^B=(-1)^{2J}$, if $C_B{}^2=(-1)^B$, then the operator $C_B'=e^{i\pi J_z}C_B=C_Be^{i\pi J_z}$ satisfies $C_B'^2=1$. Entirely similar arguments apply to the groups $U_L(1)$ and $U_Y(1)$ because of the relations $(-1)^L=(-1)^{2J}$ and $(-1)^Y=(-1)^{2I}$. It is a curious conspiracy of nature that no such equations seem to connect $(-1)^Q$ to the center of another SU(2) group.

[18]Here we are ignoring the existence of approximate symmetries such as isospin conservation.

[19]Actually, from our point of view, if U(2) were the only symmetry under consideration, a simpler (but nonequivalent) choice would be $f^2=1$. However, the usual choice comes from $f=Ce^{i\pi I_2}$, with $C^2=1$. Had we used $f^2=1$, we would have $C^2=(-1)^{2I}$. Thus the choice $f^2=1$ would lead to an "unusual theory." (See Ref. 17.)

[20]It is important to consider g only because f is usually a degenerate symmetry. If f is hidden, then f and $(-1)^{2I}f$ are merely two representatives in the same coset of U(2).

[21]From now on we will regard the parity operator as an internal-symmetry operator.

[22]It should be noted that we had earlier used the formula $W^2=(-1)^{2I}$ in Refs. 3–5. Anticipating the final enlargement of SU(2)× Z_2 to SU(3)× SU(3), we may state that $W=e^{i3\pi Y_5/2}$ and $W=e^{i3\pi Y_-}$ satisfy $W^2=(-1)^{2I}$ and $W^2=1$, respectively. In obtaining Eq. (15), note also that the identity $e^{i3\pi Y}=e^{i3\pi Y_5}$ is used. (See Appendix A 11.)

[23]C. N. Yang and J. Tiomno, Phys. Rev. 79, 495 (1950).

[24]The types C and D originate from the 4-dimensionality of space-time, and was first discussed by G. Racah, Nuovo Cimento 14, 322 (1937). In our subsequent discussions on isospin, it is clear that the types C and D do not arise.

[25]We may compare this situation with the charge-conjugation symmetry. If both particle and antiparticle exist in the same system, they certainly behave differently. However, there is no "intrinsic" difference between them if charge conjugation is a good symmetry. In other words, if we exchange each particle with its antiparticle everywhere, the transformed system behaves identically as the original system.

[26]The operator f should not be confused with the usual "mass-reversal" operator, $\psi\to\gamma_5\psi$. The mass-reversal symmetry is broken since it is, by definition, a degenerate symmetry.

[27]It is interesting to note the role of G parity in the enlargement. According to example 11 of Sec. V, from U(2) symmetry we must have G-parity invariance. Now, when we add, say, F_{4+i5} to U(2), the G parity forces the addition of F_{4-i5}. Isospin invariance brings in further the operators $F_{6\pm i7}$. In other words, if one adds to U(2) an operator with $I=\frac{1}{2}$, $Y=1$, then we necessarily generate four. However, that the algebra closes, i.e., that the commutators of the newly added operators do not generate new elements [as in Eq. (26)], remains to be an assumption.

[28]M. Gell-Mann, Phys. Rev. 125, 1067 (1962); Physics 1, 63 (1964).

[29]It is well known that the structure of Eq. (26) admits of the classification of "even" operators ($F_{1,2,3,8}$) and "odd" operators ($F_{4,5,6,7}$). Thus as far as Eq. (26) is concerned, it has nothing to say about the parity of $F_{4,5,6,7}$, when we assume that $F_{1,2,3,8}$ are even under $\boldsymbol{P}$. [Eq. (26) also implies that $F_{1,2,3,8}$ cannot be odd under $\boldsymbol{P}$.]

[30]Note that, mathematically, SU(3)× SU(3) is not a group extension of SU(3), since SU(3) is not an invariant subgroup of SU(3)× SU(3).

[31]See, e.g., M. Gell-Mann, in *Particle Physics*, edited by W. A. Simmons and S. F. Tuan (Western Periodicals, Los Angeles, 1970).

[32]For a discussion of de Sitter groups, see, for instance, the lectures by Gürsey in Ref. 2.

[33]It is classified as the Lie group B_2. According to Ref. 13, the groups B_n do not have any outer automorphisms.

[34]T. D. Lee, Phys. Rev. 140, B959 (1965).

[35]The lectures by Michel, Ref. 11, are particularly relevant for the physicists. Reference 13 is a useful reference on Lie groups. For the subject of group extensions, it is also useful to consult the book by S. MacLane, *Homology* (Academic, New York, 1963).

[36]S. Okubo and V. Mathur, Phys. Rev. D 2, 394 (1970).
[37]Physically, only continuous automorphisms are relevant.
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Structure of Internal-Symmetry Groups*

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It is shown that, given SU(2) and T invariance, their outer automorphisms, which must themselves be symmetries, form a one-parameter gauge group. Thus, from isospin conservation and time-reversal invariance, one gets hypercharge conservation. Similarly, from angular momentum conservation and T invariance, one has fermion-number conservation. Further, the well-known empirical relations $(-1)^Y=(-1)^{2I}$ and $(-1)^F=(-1)^{2J}$ are derived.

I. INTRODUCTION

Recently[1] it was emphasized that the automorphisms of internal-symmetry groups are inherent conventions associated with their physical applications. Since there is a one-to-one correspondence between conventions and symmetries, this means that the outer automorphisms of internal-symmetry groups play an important role. In fact, according to the discussions in I, if one has an internal-symmetry group G containing a degenerate subgroup S, then outer automorphisms of G which leave S invariant are themselves symmetries. These new symmetries are in general hidden symmetries. Indeed, in the numerous examples investigated in I, we could not decide whether the derived symmetries were degenerate or hidden. The identification can only be done by comparison with reality.

Let us emphasize that there is no fundamental difference between hidden and degenerate symmetries. (In this connection, we may remark that the familiar term "spontaneously broken" symmetry is somewhat misleading.) If G is a symmetry group, and if S is a subgroup of G, then hidden symmetries arise when the physical states form irreducible representations, not of G, but of S. Indeed, the cosets of S in G constitute the hidden symmetries.

The physical significance of outer automorphisms may also be visualized in another way. When we have a (degenerate) symmetry group S (elements labeled as S^a) and some physical states (labeled as $|\alpha_i\rangle$), we are accustomed to thinking that the labeling has already been given. Further, an element S^c is understood to operate on the states and give rise to a reshuffling of them. Thus, $|\alpha_i\rangle \xrightarrow{S^c} |\alpha_{i'}\rangle$ and $S^a \xrightarrow{S^c} S^a$, so that

$$\langle\alpha_i|S^a|\alpha_j\rangle \xrightarrow{S^c} \langle\alpha_{i'}|S^a|\alpha_{j'}\rangle = \langle\alpha_i|S^b|\alpha_j\rangle\,,$$

where $S^b=(S^c)^{-1}S^aS^c$. The symmetry of our system is reflected in the equivalence of the states $|\alpha_i\rangle$, which may be reshuffled. On the other hand, we could have regarded the elements S^c to operate on S, and leave the physical states unchanged: $|\alpha_i\rangle \xrightarrow{S^c} |\alpha_i\rangle$, $S^a \xrightarrow{S^c} S^b$. In other words, the operation of S^c gives rise to a reshuffling of elements in S. The symmetry of our system is reflected, then, in the equivalence of the elements in S. Therefore, we had in the beginning the labeling of S and $|\alpha\rangle$, the symmetry results as a consequence of our freedom in reshuffling the states $|\alpha\rangle$ or the elements S^a. The first view is the usual one, while the second view corresponds to interpreting S^a as elements of the inner automorphism of S. According to this second viewpoint, it is quite clear that outer automorphisms are just as good as the inner ones. Having adjoined these outer automorphisms to our