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Relativistic Wave Equations for Particles with Arbitrary Spin*

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Relativistic wave equations are derived which generalize the recently obtained Galilei-covariant wave equations for massive particles with any integer or half-integer spin. Imposing a minimality condition on the number of components possessed by the relativistic wave function, it is shown that the index transformation properties of the wave function may be either those of the $(s,0)\oplus(s-\frac{1}{2},\frac{1}{2})$ representation of SL(2, C) or of the representation $(0,s)\oplus(\frac{1}{2},s-\frac{1}{2})$. The minimal extension of these representations which accommodates reflection symmetry yields the Dirac equation for $s=\frac{1}{2}$, the Duffin-Kemmer equation for $s=1$, and an equation for particles with $s>1$ whose wave-function indices transform according to the $(s,0)\oplus(s-\frac{1}{2},\frac{1}{2})\oplus(\frac{1}{2},s-\frac{1}{2})\oplus(0,s)$ representation of SL(2,C). The latter theory possesses $4(2s+1)$ independent components, has no subsidiary conditions, and describes a unique mass, $m\neq 0$, and a unique spin. The theory admits a simple Lagrangian and Hamiltonian formulation and yields a conserved current. Finally, it is shown that for any spin the equation remains consistent and causal in the presence of a minimally coupled external-electromagnetic-field interaction.

I. INTRODUCTION

The problem of describing massive $(m>0)$ particles of higher spin $(s>1)$ within the framework of relativistic quantum mechanics has a very long history. Dirac[1] proposed the first higher-spin equations in 1936, but although these equations were satisfactory for describing free particles it was shown by Fierz and Pauli[2] that they led to immediate inconsistencies in the presence of an external electromagnetic field. Fierz and Pauli resolved this immediate difficulty for the cases $s=\frac{3}{2}$ and $s=2$ by introducing, *ad hoc*, subsidiary components which depended upon the field strengths. However, although the immediate inconsistency was thereby avoided, a more subtle difficulty had entered the theory: Spacelike-separated charge densities failed to commute.

This first episode is typical of much of the later work on higher-spin wave equations. There are many formalisms which yield an adequate description of a mass-m, spin-s free particle in that they yield the proper representation of the Poincaré group as classified by Wigner,[3] but when an interaction is introduced the difficulties emerge.

The higher-spin maladies were first studied in terms of the resultant symptoms in the second-quantized theory of such particles.[4] More recently, however, the effects of these diseases have been studied within the framework of the c-number theory.[5] In the present paper we shall consider only the latter. We wish to emphasize, however, that the two are very intimately related.

We shall describe here a new approach to this old problem. Historically the procedure has been as follows: Starting with nonrelativistic quantum mechanics (Schrödinger equation) one first generalized this equation to a relativistic equation (Klein-Gordon) and then further generalized the resultant theory to include the description of spin (Dirac equation and other higher-spin equations). We shall reverse the order of the two generalizations and follow the alternate route: Starting with the Schrödinger equation, we first generalize this equation to describe nonrelativistic particles with arbitrary spin and then generalize this nonrelativistic spin-s theory to a relativistic theory for particles with any spin (see Fig. 1).

We shall see that such an approach leads to a higher-spin wave equation which avoids the difficulties encountered with the usual higher-spin equations in that it admits an external-electromagnetic-field interaction consistently and propagates causally in the presence of such an interaction. The formalism yields the additional feature that in order to have a parity-symmetric theory (and therefore also a Lagrangian formulation) four independent $(2s+1)$-component objects are needed, in contrast with the two $(2s+1)$-component objects which suffice for the first-order equations describing $s=\frac{1}{2}$ and $s=1$ $(m>0)$.

We find it convenient to work mostly with a non-manifestly covariant notation. However, the formalism may be easily cast into manifestly covariant form at any stage. For the sake of concreteness we shall, furthermore, work with particular representations of the appropriate matrices which emerge in the formalism rather than with their abstract algebraic properties. Again, it may be easily verified that the results are invariant under

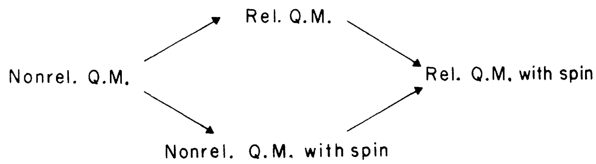

FIG. 1. Two roads to a relativistic higher-spin theory.

any unitary transformation of these matrices.

The first stage of the approach outlined above, i.e., the construction of the nonrelativistic wave equation for particles with arbitrary spin, has been previously described.[6] We shall hereafter refer to this paper as I. In Sec. II we shall briefly review the results of this study. In Sec. III we consider the relativistic generalization of these equations and find that in general for spin s there are two equations which meet the invariance requirements in addition to a minimality condition on the number of components possessed by the wave function. The extension of the theory to allow for parity symmetry is carried out in Sec. IV and a unique spin-s equation results. The spin-s Lagrangian, conserved current, and Hamiltonian equations are displayed in Sec. V and the external-electromagnetic-field interaction is introduced in Sec. VI. The resultant equations are shown to be consistent and causal in the presence of such an interaction. A further discussion of the interpretation of the components of the wave function as well as related issues is deferred until the second-quantized theory is described.

II. GALILEI-COVARIANT WAVE EQUATIONS FOR PARTICLES WITH ARBITRARY SPIN

A $(2s+1)$-component wave function, $\psi_\alpha(\vec{x},t)$, whose indices transform under rotations according to the $(2s+1)$-dimensional representation of SU(2), $D^{(s)}_{\alpha\beta}(R)$, and which satisfies the Schrödinger equation componentwise,

$$-\frac{\nabla^2}{2m}\psi_\alpha(\vec{x},t)=i\partial_t\psi_\alpha(\vec{x},t),\quad \alpha=1,\ldots,2s+1 \tag{2.1}$$

will describe a free, Galilean-invariant, spin-s, mass $m>0$ particle. In I we sought a first-order Lagrangian which would describe such particles, i.e., we assumed a Lagrangian of the form

$$\mathcal{L}(\vec{x},t)=\phi^*_\alpha(\vec{x},t)(iA_{\alpha\beta}\partial_t-i\vec{B}_{\alpha\beta}\cdot\vec{\nabla}+C_{\alpha\beta})\phi_\beta(\vec{x},t), \tag{2.2}$$

where A, $\vec{B}$, and C are numerical matrices and ϕ^*_α denotes the complex conjugate of ϕ_α. We then demanded that $\mathcal{L}$ have the following properties:

(1) $\mathcal{L}(\vec{x},t)$ is a scalar under the transformations of the inhomogeneous Galilei group,

$$\vec{x}'=R\vec{x}+\vec{v}t+\vec{a},$$
$$t'=t+b,$$

where R is a space-rotation matrix, $\vec{v}$ is the boost velocity, and b ($\vec{a}$) is a time (space) translation.

(2) $\phi_\alpha(\vec{x},t)$ has $2s+1$ independent components, $\psi_\alpha(\vec{x},t)=\phi_\alpha(\vec{x},t)$, for $\alpha=1,\ldots,2s+1$, such that variation of $\mathcal{L}(\vec{x},t)$ implies that $\psi_\alpha(\vec{x},t)$ satisfies (2.1) componentwise.

(3) The rest of the components of ϕ_α are dependent upon the ψ_α and there are only as many as are necessary to satisfy (1) and (2).

In I it was shown that the above requirements uniquely determined $\mathcal{L}$ to be

$$\mathcal{L}=\phi^*_\alpha\left[i\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\partial_t-\frac{i}{s}\begin{pmatrix}0&\vec{S}&\vec{K}\\\vec{S}&0&0\\\vec{K}&0&0\end{pmatrix}\cdot\vec{\nabla}+\begin{pmatrix}0&0&0\\0&2m&0\\0&0&2m\end{pmatrix}\right]_{\alpha\beta}\phi_\beta, \tag{2.3}$$

where $\alpha,\beta=1,\ldots,6s+1$ and where the matrix notation is of the form

$$\begin{pmatrix}E&E&F\\E&E&F\\G&G&H\end{pmatrix}, \tag{2.4}$$

where E represents a $(2s+1)$-dimensional square matrix, F has $2s+1$ rows and $2s-1$ columns, G has $2s-1$ rows and $2s+1$ columns, and H is a $(2s-1)$-dimensional square matrix. The $\vec{S}$ represents the three SU(2) generators in the $(2s+1)$-dimensional representation which satisfy ($\alpha,\beta,\gamma,\delta=1,\ldots,2s+1$)

$$R\vec{S}^{\alpha\beta}=D^{(s)}_{\alpha\gamma}(R^{-1})\vec{S}^{\gamma\delta}D^{(s)}_{\delta\beta}(R) \tag{2.5}$$

and the $\vec{K}$ represents three matrices with $2s-1$ rows and $2s+1$ columns which satisfy ($\alpha',\gamma'=1,\ldots,2s-1$; $\beta,\delta=1,\ldots,2s+1$)

$$R\vec{K}^{\alpha'\beta}=D^{(s-1)}_{\alpha'\gamma'}(R^{-1})\vec{K}^{\gamma'\delta}D^{(s)}_{\delta\beta}(R). \tag{2.6}$$

Using the same matrix notation we may write the transformation properties of $\phi_\alpha(\vec{x},t)$ as

$$\phi'_\alpha(\vec{x}',t')=e^{if(\vec{x},t)}\Delta^{(s)}_{\alpha\beta}(\vec{v},R)\phi_\beta(\vec{x},t), \tag{2.7}$$

where $\alpha,\beta=1,\ldots,6s+1$, $f(\vec{x},t)=\frac{1}{2}mv^2t+m\vec{v}\cdot R\vec{x}$, and

$$\Delta^{(s)}_{\alpha\beta}(\vec{v},R)=\begin{pmatrix} D^{(s)}(R) & 0 & 0 \\ -(1/2s)\vec{S}\cdot\vec{v}D^{(s)}(R) & D^{(s)}(R) & 0 \\ -(1/2s)\vec{K}\cdot\vec{v}D^{(s)}(R) & 0 & D^{(s-1)}(R)\end{pmatrix}_{\alpha\beta}, \tag{2.8}$$

which yields a reducible but undecomposable representation of the homogeneous Galilei group whose restriction to the SU(2) subgroup yields a completely reducible representation of SU(2),

$$D^{(s)}(R)\oplus D^{(s)}(R)\oplus D^{(s-1)}(R). \tag{2.9}$$

In I it was shown that the above formalism yields a satisfactory nonrelativistic quantum-mechanical description of spin-s particles and, furthermore, admits the introduction of interactions in a completely consistent fashion for any spin. In the following section we shall derive relativistic equations which reduce to the above equations in the Galilei limit.

III. RELATIVISTIC WAVE EQUATIONS FOR PARTICLES WITH ARBITRARY SPIN

We seek a first-order wave equation of the form[7]

$$(i\beta^{\mu}_{\alpha\beta}\,\partial_{\mu}-m)\phi_{\beta}(x)=0 \tag{3.1}$$

which is form-invariant under the transformations of the proper inhomogeneous Lorentz group and which reduces to the Galilean wave equations of Sec. II in the nonrelativistic limit. The β^{μ} are four numerical square matrices of finite dimension and m is the (nonzero) mass.

We shall first direct our attention to the $\phi_{\beta}(x)$ and the structure of its index transformation, which we assume yields a finite-dimensional representation of the homogeneous Lorentz group homomorphic to SL(2, C). Since we demand that, in the nonrelativistic limit, Eq. (3.1) for a particle of spin s must reduce to the wave equations implied by (2.3), then we must also assume that in this limit the representation of SL(2, C) governing (3.1) must reduce to the representation (2.8) of the homogeneous Galilei group. Two possibilities thus present themselves: Either the SL(2, C) representation has the same dimensionality as (2.8), i.e., $6s+1$; or it is of higher dimension and the added components vanish in the nonrelativistic limit. In keeping with the spirit of the minimality restriction on the number of components of the Galilean wave function which we used in I, we make a similar assumption in the relativistic case and assume that the former of the above two possibilities holds. Thus we require the representation of SL(2, C) which governs (3.1) to have the minimum dimensionality possible consistent with (3.1)'s low-energy limit, i.e., we assume that it is $(6s+1)$-dimensional.

In addition to this restriction on its dimensionality, we may also assume that the rotational properties of the relativistic wave function and those of its nonrelativistic limit are identical.[8] We therefore seek a $(6s+1)$-dimensional representation of SL(2, C) whose SU(2, C) subgroup is represented by (2.9),

$$D^{(s)}(R)\oplus D^{(s)}(R)\oplus D^{(s-1)}(R).$$

Directing our attention to the irreducible representations of SL(2, C) and their direct sums, there are 12 representations which meet the above requirements[9]; eight of the form[10]

$$(s,0)\oplus(s,0)\oplus(s-1,0) \tag{3.2}$$

and four of the form

$$(s,0)\oplus(s-\tfrac{1}{2},\tfrac{1}{2}). \tag{3.3}$$

Having thus limited the index-transformation candidates, we now turn our attention to Eq. (3.1) and ask for which of these 12 representations there exist $(6s+1)$-dimensional matrices, β_{μ}, such that (3.1) will be form-invariant under the transformations of the proper Poincaré group.

Denote the index-transformation matrix as $\Delta_{\alpha\beta}(\Lambda)$, where Λ is a homogeneous Lorentz transformation on the coordinates, $x'=\Lambda x$. The relativistic wave function thus transforms according to

$$\phi'_{\alpha}(\Lambda x)=\Delta_{\alpha\beta}(\Lambda)\phi_{\beta}(x).$$

In terms of the six infinitesimal generators of SL(2, C), $M_{\mu\nu}=-M_{\nu\mu}$, we may express Δ as

$$\Delta(\omega)=e^{(i/2)\omega_{\mu\nu}M^{\mu\nu}},\quad \omega_{\mu\nu}=-\omega_{\nu\mu},$$

where

$$[M_{\mu\nu},M_{\rho\sigma}]=-i(g_{\mu\rho}M_{\nu\sigma}+g_{\nu\sigma}M_{\mu\rho}-g_{\nu\rho}M_{\mu\sigma}-g_{\mu\sigma}M_{\nu\rho}). \tag{3.4}$$

Equation (3.1) will be form-invariant under Lorentz transformations if

$$[M_{\mu\nu},\beta_{\sigma}]=i(g_{\nu\sigma}\beta_{\mu}-g_{\mu\sigma}\beta_{\nu}). \tag{3.5}$$

In terms of $\vec{J}=(M^{23}, M^{31}, M^{12})$ and $\vec{N}=(M^{10},M^{20},M^{30})$ Eq. (3.4) reads

$$[J_i,J_j]=i\epsilon_{ijk}J_k, \tag{3.6a}$$

$$[N_i,N_j]=-i\epsilon_{ijk}J_k, \tag{3.6b}$$

$$[J_i,N_j]=i\epsilon_{ijk}N_k, \tag{3.6c}$$

and Eq. (3.5) becomes

$$[J_i,\beta_0]=0\,, \tag{3.7a}$$

$$[J_i,\beta_j]=i\epsilon_{ijk}\beta_k\,, \tag{3.7b}$$

$$[N_i,\beta_0]=i\beta_i, \tag{3.7c}$$

$$[N_i,\beta_j]=i\delta_{ij}\beta_0\,. \tag{3.7d}$$

We proceed as follows:

(1) Corresponding to each representation, (3.2) and (3.3), construct explicit $\vec{J}$ and $\vec{N}$ such that the Lie algebra, (3.6), is satisfied.

(2) For each representation seek matrices, β_μ, such that (3.7) is satisfied.

We leave the details to Appendixes B, C, and D and present only the following results:

(1) There are no nontrivial β's for the representations of the form (3.2).[11]

(2) There exist β's satisfying (3.7) for two of the representations of the form (3.3) and they are given as

(i) $(s,0)\oplus(s-\frac{1}{2},\frac{1}{2}),\quad \epsilon=+1\,;$

(ii) $(0,s)\oplus(\frac{1}{2},s-\frac{1}{2}),\quad \epsilon=-1\,:$

$$J_i=\begin{pmatrix} S_i & 0 & 0\\ 0 & S_i & 0\\ 0 & 0 & \Sigma_i\end{pmatrix},\quad N_i=-i\epsilon\begin{pmatrix} S_i & 0 & 0\\ 0 & [(s-1)/s]S_i & (1/s)K_i^\dagger\\ 0 & (1/s)K_i & [(s+1)/s]\Sigma_i\end{pmatrix}, \tag{3.8}$$

$$\beta_0=\begin{pmatrix} 0 & \alpha & 0\\ \beta & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad \beta_i=\frac{\epsilon}{s}\begin{pmatrix} 0 & -\alpha S_i & \alpha K_i^\dagger\\ \beta S_i & 0 & 0\\ -\beta K_i & 0 & 0\end{pmatrix}, \tag{3.9}$$

where we have used the $(6s+1)$-dimensional matrix notation, (2.4); α and β are arbitrary complex numbers; S_i (Σ_i) are the three generators of SU(2) in the $(2s+1)$- ($(2s-1)$-) dimensional representation; and K_i are the three rectangular matrices defined by (2.6) (see Appendix B).

The wave equation, (3.1), thus takes the explicit form

$$\left[i\begin{pmatrix} 0 & \alpha & 0\\ \beta & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\partial_t+i\frac{\epsilon}{s}\begin{pmatrix} 0 & -\alpha\vec{S} & \alpha\vec{K}^\dagger\\ \beta\vec{S} & 0 & 0\\ -\beta\vec{K} & 0 & 0\end{pmatrix}\cdot\vec{\nabla}-m\right]_{\alpha\beta}\begin{pmatrix}\psi_\epsilon\\ \chi_\epsilon\\ \Omega_\epsilon\end{pmatrix}_\beta=0\,, \tag{3.10}$$

where ψ_ϵ and χ_ϵ each have $2s+1$ components, Ω_ϵ has $2s-1$ components, and $\epsilon=+1$ (-1) for the representation $(s,0)\oplus(s-\frac{1}{2},\frac{1}{2})$ $((0,s)\oplus(\frac{1}{2},s-\frac{1}{2}))$.

We may now restrict the α and β by requiring that the theory describe a particle of unique mass. This will be guaranteed if Eq. (3.10) implies that the wave function satisfies the Klein-Gordon equation componentwise. Writing Eq. (3.10) as

$$i\alpha\partial_t\chi_\epsilon-\frac{i\epsilon}{s}\alpha\vec{S}\cdot\vec{\nabla}\chi_\epsilon+\frac{i\epsilon}{s}\alpha\vec{K}^\dagger\cdot\vec{\nabla}\,\Omega_\epsilon-m\psi_\epsilon=0\,, \tag{3.11a}$$

$$i\beta\partial_t\psi_\epsilon+\frac{i\epsilon}{s}\beta\vec{S}\cdot\vec{\nabla}\psi_\epsilon-m\chi_\epsilon=0\,, \tag{3.11b}$$

$$-\frac{i\epsilon}{s}\beta\vec{K}\cdot\vec{\nabla}\,\psi_\epsilon-m\Omega_\epsilon=0\,, \tag{3.11c}$$

and using Eqs. (3.11b) and (3.11c) to substitute for χ_ϵ and Ω_ϵ in Eq. (3.11a), we find that ψ_ϵ satisfies

$$\partial_t{}^2\psi_\epsilon-\nabla^2\psi_\epsilon+\frac{m^2}{\alpha\beta}\psi_\epsilon=0\,, \tag{3.12}$$

where we have used the relation (see Appendix B)

$$S_iS_j+K_i^\dagger K_j=is\epsilon_{ijk}S_k+s^2\delta_{ij}\,. \tag{3.13}$$

Thus ψ_ϵ and hence, by Eqs. (3.11b) and (3.11c), χ_ϵ and Ω_ϵ will satisfy the Klein-Gordon equation for mass m if $\alpha\beta=1$. We take β_0 to be symmetric, and hence choose $\alpha=1=\beta$. The wave equation thus becomes[12]

$$\left[i\begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\partial_t+i\frac{\epsilon}{s}\begin{pmatrix} 0 & -\vec{S} & \vec{K}^\dagger\\ \vec{S} & 0 & 0\\ -\vec{K} & 0 & 0\end{pmatrix}\cdot\vec{\nabla}-m\right]_{\alpha\beta}\begin{pmatrix}\psi_\epsilon\\ \chi_\epsilon\\ \Omega_\epsilon\end{pmatrix}_\beta=0\,, \tag{3.14}$$

where $\alpha, \beta = 1, \ldots, 6s+1$ and $\epsilon = \pm$.

We must now verify that (3.14) meets our primary criterion, i.e., that it reduces to Eqs. (2.3) in the nonrelativistic limit. To show this we first write (3.14) in momentum space and then diagonalize β_0 by means of the unitary transformation

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

to get the form

$$E\tilde{\psi} - \frac{\epsilon}{s}\vec{S}\cdot\vec{p}\,\tilde{\chi} - \frac{\epsilon}{\sqrt{2}s}\vec{K}^\dagger\cdot\vec{p}\,\tilde{\Omega} - m\tilde{\psi} = 0\,, \tag{3.15a}$$

$$-E\tilde{\chi} + \frac{\epsilon}{s}\vec{S}\cdot\vec{p}\tilde{\psi} - \frac{\epsilon}{\sqrt{2}s}\vec{K}^\dagger\cdot\vec{p}\,\tilde{\Omega} - m\tilde{\chi} = 0\,, \tag{3.15b}$$

$$\frac{\epsilon}{\sqrt{2}s}\vec{K}\cdot\vec{p}\,\tilde{\psi} + \frac{\epsilon}{\sqrt{2}s}\vec{K}\cdot\vec{p}\,\tilde{\chi} - m\tilde{\Omega} = 0\,. \tag{3.15c}$$

In the nonrelativistic limit we may write the energy as $E \sim W + m$, where W is the kinetic energy $W = \frac{1}{2}mv^2$. Substituting this form for E and dividing by m implies that (3.15) takes the form

$$\frac{W}{m}\tilde{\psi} - \frac{\epsilon}{s}\vec{S}\cdot\left(\frac{\vec{p}}{m}\right)\tilde{\chi} - \frac{\epsilon}{\sqrt{2}s}\vec{K}^\dagger\cdot\left(\frac{\vec{p}}{m}\right)\tilde{\Omega} = 0\,, \tag{3.16a}$$

$$-\frac{W}{m}\tilde{\chi} + \frac{\epsilon}{s}\vec{S}\cdot\left(\frac{\vec{p}}{m}\right)\tilde{\psi} - \frac{\epsilon}{\sqrt{2}s}\vec{K}^\dagger\cdot\left(\frac{\vec{p}}{m}\right)\tilde{\Omega} - 2\tilde{\chi} = 0\,, \tag{3.16b}$$

$$\frac{\epsilon}{\sqrt{2}s}\vec{K}\cdot\left(\frac{\vec{p}}{m}\right)\tilde{\psi} + \frac{\epsilon}{\sqrt{2}s}\vec{K}\cdot\left(\frac{\vec{p}}{m}\right)\tilde{\chi} - \tilde{\Omega} = 0\,, \tag{3.16c}$$

which exhibits the velocity dependence of the following scheme:

$$O(v^2)\tilde{\psi} + O(v)\tilde{\chi} + O(v)\tilde{\Omega} = 0\,, \tag{3.17a}$$

$$O(v^2)\tilde{\chi} + O(v)\tilde{\psi} + O(v)\tilde{\Omega} + \tilde{\chi} = 0\,, \tag{3.17b}$$

$$O(v)\tilde{\psi} + O(v)\tilde{\chi} + \tilde{\Omega} = 0\,, \tag{3.17c}$$

from which we see that if we neglect terms which go as $O(v^2)$ with respect to those which go as a lower power of v, then in the nonrelativistic limit we may neglect the first and third terms of (3.16b) and the second term of (3.16c). Redefining the components such that $\tilde{\chi}' = -\epsilon\tilde{\chi}$ and $\tilde{\Omega}' = (-\epsilon/\sqrt{2})\tilde{\Omega}$, we get in the nonrelativistic limit

$$W\tilde{\psi} + (1/s)\vec{S}\cdot\vec{p}\,\tilde{\chi}' + (1/s)\vec{K}^\dagger\cdot\vec{p}\tilde{\Omega}' = 0\,, \tag{3.18a}$$

$$(1/s)\vec{S}\cdot\vec{p}\tilde{\psi} + 2m\tilde{\chi}' = 0\,, \tag{3.18b}$$

$$(1/s)\vec{K}\cdot\vec{p}\tilde{\psi} + 2m\tilde{\Omega}' = 0\,, \tag{3.18c}$$

which are the equations implied by the Galilean Lagrangian, (2.3), in momentum space.

Equation (3.14) thus satisfies our basic requirements: It is form-invariant under the proper Poincaré group, it describes particles with a unique mass m and a unique spin s, and it reduces to the proper Galilean equations in the nonrelativistic limit.

IV. THE INCLUSION OF PARITY

Although Eq. (3.14) meets our basic requirements, it has a serious drawback: As is obvious from the SL(2, C) representation itself,[13] the space of solutions to (3.14) does not admit a parity operation. More concretely, if we impose form invariance under parity on an equation of the form

$$(i\beta_0\partial_t + i\vec{\beta}\cdot\vec{\nabla} - m)\psi(x) = 0\,, \tag{4.1}$$

then there must exist a matrix, P, where $\psi'(-\vec{x}, t) = P\psi(\vec{x}, t)$, such that $[\beta_0, P]_- = 0$ and $[\vec{\beta}, P]_+ = 0$. It is easily shown that for $s > \frac{1}{2}$, no such P exists for Eq. (3.14).[14] Indeed, this failing has even more serious consequences. If we hope to be able to derive an equation of the form (4.1) from a Lagrangian, then there must exist a matrix, η, such that the Lagrangian has the form

$$\mathcal{L} = \psi^\dagger\eta(i\beta_0\partial_t + i\vec{\beta}\cdot\vec{\nabla} - m)\psi\,, \tag{4.2}$$

where η is the so-called "Hermitizing" matrix with the property that

$$\eta\beta_\mu = \beta_\mu^\dagger\eta \quad (\eta^\dagger = \eta)\,. \tag{4.3}$$

However, the β_μ in Eq. (3.14) are such that $\beta_0^\dagger = \beta_0$ and $\vec{\beta}^\dagger = -\vec{\beta}$. Consequently, η must have the same properties as P, and the nonexistence of P implies that there is likewise no η for Eq. (3.14).[15] It is therefore imperative that the formalism be extended to allow for reflection symmetry.

As noted above, the spin-$\frac{1}{2}$ case is unique in that the representation obtained, $(\frac{1}{2}, 0)\oplus(0, \frac{1}{2})$, is already parity-symmetric. Thus no extension is necessary in order to bring this example into correspondence with the above requirements. The Dirac theory results.

The representation for spin 1 is either $(1, 0)\oplus(\frac{1}{2}, \frac{1}{2})$ or $(\frac{1}{2}, \frac{1}{2})\oplus(0, 1)$. The minimal extension of either representation which allows for parity symmetry is therefore $(1, 0)\oplus(\frac{1}{2}, \frac{1}{2})\oplus(0, 1)$. The determination of the β_μ for this representation results in the Duffin-Kemmer[16] form of the Proca[17] theory. We shall consider the spin-1 case in somewhat more detail in Appendix E.

For all spins greater than one the minimal extension which allows for reflection symmetry results in the direct sum of the two representations (3.8): $(s, 0)\oplus(s-\frac{1}{2}, \frac{1}{2})\oplus(\frac{1}{2}, s-\frac{1}{2})\oplus(0, s)$. The wave equation thus becomes

$$\left[i\begin{pmatrix}\gamma_0 & 0\\ 0 & \gamma_0\end{pmatrix}\partial_t + \frac{i}{s}\begin{pmatrix}\vec{\gamma} & 0\\ 0 & -\vec{\gamma}\end{pmatrix}\cdot\vec{\nabla} - m\right]_{\alpha\beta}\begin{pmatrix}\varphi_+\\ \varphi_-\end{pmatrix} = 0\,, \tag{4.4}$$

where γ_0 and $\vec{\gamma}$ are the $(6s+1)$-dimensional matrices in Eq. (3.14) and $\phi=\binom{\phi_+}{\phi_-}$ is a $(12s+2)$-component wave function. Since Eq. (4.4) is just the direct sum of the two equations (3.14) ($\epsilon=\pm1$), it maintains the properties of those equations (i.e., it exhibits form invariance, implies the Klein-Gordon equation, and reduces to two uncoupled Galilean spin-s equations). Now, however, in addition to these properties, it admits a parity operation whose effect on the indices of the wave function may be given by the $(12s+2)$-dimensional matrix $\begin{pmatrix}0&1\\1&0\end{pmatrix}$. This may also be taken as the "Hermitizing" matrix, η, and a Lagrangian and current may be defined. We shall examine the properties of this equation in both the free case and in the case of the external-field interaction in the following sections.[18]

V. PROPERTIES OF THE FREE-WAVE EQUATION

For $s>1$, the wave equation may be written

$$\left[i\begin{pmatrix}0&1&0&&&\\1&0&0&&&\\0&0&0&&&\\&&&0&1&0\\&&&1&0&0\\&&&0&0&0\end{pmatrix}\partial_t+\frac{i}{s}\begin{pmatrix}0&-\vec{S}&\vec{K}^\dagger&&&\\\vec{S}&0&0&&&\\-\vec{K}&0&0&&&\\&&&0&\vec{S}&-\vec{K}^\dagger\\&&&-\vec{S}&0&0\\&&&\vec{K}&0&0\end{pmatrix}\cdot\vec{\nabla}-m\right]\begin{pmatrix}\psi_+\\\chi_+\\\Omega_+\\\psi_-\\\chi_-\\\Omega_-\end{pmatrix}=0, \tag{5.1}$$

where the $(12s+2)$-component wave function is the direct sum of ϕ_+ and ϕ_- whose indices transform according to the representation $(s,0)\oplus(s-\frac{1}{2},\frac{1}{2})$ and $(0,s)\oplus(\frac{1}{2},s-\frac{1}{2})$, respectively. Note that there are $4(2s+1)$ components entering the time-derivative term.

Using the $(12s+2)$-dimensional η, which may be taken as $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ for the above representation, the Lagrangian density may be written as

$$\mathcal{L}(x)=\overline{\varphi}_\alpha(x)(i\beta^\mu\partial_\mu-m)_{\alpha\beta}\varphi_\beta(x), \tag{5.2}$$

where

$$\overline{\varphi}_\alpha(x)=\varphi^\dagger_\beta(x)\eta_{\beta\alpha},\quad \alpha,\beta=1,\ldots,12s+2.$$

Likewise, the current may be written as

$$j^\mu(x)=\overline{\varphi}_\alpha(x)\beta^\mu_{\alpha\beta}\varphi_\beta(x), \tag{5.3}$$

which, by virtue of the equations of motion, is conserved, $\partial_\mu j^\mu(x)=0$. The fourth component of this current is the "charge" density, $\rho(x)=\overline{\varphi}_\alpha\beta^0_{\alpha\beta}\varphi_\beta(x)$. In the representation in which β_0 is diagonal (denoted by a tilde) as well as η (denoted by the numerical subscripts), $\rho(x)$ becomes

$$\rho(x)=\tilde{\psi}_1^\dagger\tilde{\psi}_1-\tilde{\chi}_1^\dagger\tilde{\chi}_1-\tilde{\psi}_2^\dagger\tilde{\psi}_2+\tilde{\chi}_2^\dagger\tilde{\chi}_2\,. \tag{5.5}$$

Thus the total "charge" receives contributions from four $(2s+1)$-dimensional sets of components, two contributing in a positive definite way and two in a negative definite fashion.

An invariant scalar product may be defined using the conserved current,

$$(\varphi,\psi)_\sigma=\int d\sigma_\mu\,\overline{\varphi}_\alpha\beta^\mu_{\alpha\beta}\psi_\beta, \tag{5.6}$$

where σ is an arbitrary spacelike surface. As mentioned above, however, this product does not produce a positive definite norm.

The Hamiltonian may be found by substituting for the components which do not enter the time-derivative term in (5.1), $\Omega_\pm$. The resultant equation which is now of second order in the spatial derivatives may be cast into Schrödinger form by multiplication with the inverse of the $4(2s+1)$-dimensional β_0 which is now nonsingular. The equation then becomes

$$i\partial_t\begin{pmatrix}\tilde{\psi}_1\\\tilde{\chi}_1\\\tilde{\psi}_2\\\tilde{\chi}_2\end{pmatrix}=\left[-\frac{i}{s}\begin{pmatrix}0&0&0&\vec{S}\\0&0&\vec{S}&0\\0&\vec{S}&0&0\\\vec{S}&0&0&0\end{pmatrix}\cdot\vec{\nabla}+\frac{1}{2ms^2}\begin{pmatrix}-1&-1&0&0\\+1&+1&0&0\\0&0&-1&-1\\0&0&+1&+1\end{pmatrix}G+m\begin{pmatrix}+1&0&0&0\\0&-1&0&0\\0&0&+1&0\\0&0&0&-1\end{pmatrix}\right]\begin{pmatrix}\tilde{\psi}_1\\\tilde{\chi}_1\\\tilde{\psi}_2\\\tilde{\chi}_2\end{pmatrix}, \tag{5.7}$$

where G is a matrix with components $G_{\alpha\beta}=[(\vec{K}^\dagger\cdot\vec{\nabla})(\vec{K}\cdot\vec{\nabla})]_{\alpha\beta}$, $\alpha,\beta=1,\ldots,2s+1$. In terms of the scalar product, (5.6), on a $t=$const spacelike plane we have $(\phi,H\psi)_t=(H\phi,\psi)_t$.

VI. THE EXTERNAL-FIELD INTERACTION

The real testing ground for any relativistic wave equation is the presence of interactions. The inability of the usual formalisms to remain consistent in an external electromagnetic field has long been the principal stumbling block to a higher-spin theory. In this section, therefore, we shall investigate some properties of the present formalism in an external electromagnetic field.

We introduce the external field by means of the usual minimal-coupling replacement,

$$\vec{\nabla} \to \vec{\nabla} - ie\vec{A} \equiv \vec{D} \quad \text{and} \quad \partial_t \to \partial_t + ie\varphi\,, \tag{6.1}$$

where e is the charge and $\vec{A}$ (ϕ) is the vector (scalar) potential. Equation (5.1) may then be written as

$$\left[i\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}(\partial_t + ie\varphi) + i\frac{\epsilon}{s}\begin{pmatrix} 0 & -\vec{S} & \vec{K}^\dagger \\ \vec{S} & 0 & 0 \\ -\vec{K} & 0 & 0 \end{pmatrix}\cdot\vec{D} - m \right]\begin{pmatrix} \psi_\epsilon \\ \chi_\epsilon \\ \Omega_\epsilon \end{pmatrix} = 0\,, \tag{6.2}$$

where $\epsilon = +1$ (-1) corresponds to the representation $(s,0)\oplus(s-\frac{1}{2},\frac{1}{2})$ ($(0,s)\oplus(\frac{1}{2},s-\frac{1}{2})$).

We shall demonstrate that the solutions to Eq. (6.2) propagate causally, i.e., in or on the light cone. Therefore, they do not suffer from those difficulties which Velo and Zwanziger[19] have studied in other higher-spin theories.

Equation (6.2) may be written more explicitly as

$$(i\partial_t - e\varphi)\chi_\epsilon - \frac{i\epsilon}{s}\vec{S}\cdot\vec{D}\chi_\epsilon + \frac{i\epsilon}{s}\vec{K}^\dagger\cdot\vec{D}\Omega_\epsilon - m\psi_\epsilon = 0\,, \tag{6.3a}$$

$$(i\partial_t - e\varphi)\psi_\epsilon + \frac{i\epsilon}{s}\vec{S}\cdot\vec{D}\psi_\epsilon - m\chi_\epsilon = 0\,, \tag{6.3b}$$

$$-\frac{i\epsilon}{s}\vec{K}\cdot\vec{D}\psi_\epsilon - m\Omega_\epsilon = 0\,. \tag{6.3c}$$

Substitution for χ_ϵ and Ω_ϵ from (6.3b) and (6.3c) into Eq. (6.3a) yields the following equation for ψ_ϵ:

$$(i\partial_t - e\varphi)^2\psi_\epsilon + \frac{i\epsilon}{s}[(i\partial_t - e\varphi)(\vec{S}\cdot\vec{D}) - (\vec{S}\cdot\vec{D})(i\partial_t - e\varphi)]\psi_\epsilon + \frac{1}{s^2}[(\vec{S}\cdot\vec{D})(\vec{S}\cdot\vec{D}) + (\vec{K}^\dagger\cdot\vec{D})(\vec{K}\cdot\vec{D})]\psi_\epsilon - m^2\psi_\epsilon = 0\,. \tag{6.4}$$

Defining the electric field, $\vec{E}$, as $\vec{E} = -\vec{\nabla}\varphi - \partial_t\vec{A}$, we may show that

$$[(i\partial_t - e\varphi)(\vec{S}\cdot\vec{D}) - (\vec{S}\cdot\vec{D})(i\partial_t - e\varphi)]\psi_\epsilon = -e\vec{S}\cdot\vec{E}\psi_\epsilon\,. \tag{6.5}$$

Likewise, if we use relation (3.13) and

$$[D_i, D_j] = -ie\epsilon_{ijk}H_k\,,$$

where

$$\vec{H} = \vec{\nabla}\times\vec{A}\,,$$

then we also have

$$\frac{1}{s^2}[(\vec{S}\cdot\vec{D})(\vec{S}\cdot\vec{D}) + (\vec{K}^\dagger\cdot\vec{D})(\vec{K}\cdot\vec{D})]\psi_\epsilon = \vec{D}^2\psi_\epsilon + \frac{e}{s}\vec{S}\cdot\vec{H}\psi_\epsilon\,. \tag{6.6}$$

We introduce the following four-vector notation[20]: $D_\mu = (\partial_0 + ie\varphi, \vec{\nabla} - ie\vec{A})$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $\Sigma^{(\epsilon)}_{\mu\nu}$, where $\Sigma^{(+)}_{\mu\nu}$ ($\Sigma^{(-)}_{\mu\nu}$) are the six generators of the $(s,0)$ ($(0,s)$) representation of SL(2, C), $\Sigma^{(\epsilon)}_{ij} = \epsilon_{ijk}S_k$ and $\Sigma^{(i0(\epsilon)} = -i\epsilon S_i$, where $i,j,k, = 1,2,3$. Using (6.5), (6.6), and the above notation, we find that ψ_ϵ satisfies the equation

$$[D^\mu D_\mu + m^2 + (e/2s)F^{\mu\nu}\Sigma^{(\epsilon)}_{\mu\nu}]\psi_\epsilon(x) = 0\,, \tag{6.7}$$

where ψ_+ (ψ_-) transforms under the $(s,0)$ ($(0,s)$) representation of SL(2, C). We now see immediately, using the methods described in Ref. 19, that the external field does not affect the causal nature of the solutions. The equation remains hyperbolic and the characteristic surfaces, which determine the maximum propagation velocity, are lightlike.

Since ψ_ϵ propagates causally, so also, by (6.3b) and (6.3c), do χ_ϵ and Ω_ϵ. The present theory is therefore causal in the presence of an external electromagnetic field for any spin.

We note in passing that Eq. (6.7) is the arbitrary-spin generalization of the Feynman–Gell-Mann[21] equation. We also note that since Eq. (5.1) describes particles with a unique mass, we expect no trouble from the Capri-Wightman instability.[22] Equation (5.1), therefore, avoids the usual difficulties associated with higher-spin wave equations in an external electromagnetic field.

As shown above, the present formalism has a matrix η such that $[\beta_0, \eta]_- = 0$ and $[\vec{\beta}, \eta]_+ = 0$, which affords, among other things, the inclusion of parity symmetry. But there also exists a complex-conjugation matrix B such that $\beta_\mu = B\beta^*_\mu B^\dagger$ and a matrix A such that $[\beta_\mu, A]_+ = 0$. The existence of these two matrices permits the formalism to accommodate a time-reversal symmetry, T, and a charge-conjugation symmetry, C, in much the same way as in the Dirac theory.

VII. SUMMARY AND DISCUSSION

Demanding the use of the minimum number of components and the inclusion of reflection symmetry, we have shown that the relativistic generalization of the Galilei-covariant wave equations for massive particles with arbitrary spin has the following properties:

(1) No subsidiary conditions are required.

(2) The wave equation is form-invariant under the full Poincaré group.

(3) The wave function satisfies the Klein-Gordon equation componentwise for a unique mass.

(4) A unique spin is described.

(5) The formalism yields the Dirac equation for $s=\frac{1}{2}$ and the Duffin-Kemmer equation for $s=1$.

(6) For $s>1$, the wave function possesses $12s+2$ components, $4(2s+1)$ of which are independent.

(7) In the nonrelativistic limit, for $s>1$, two uncoupled Galilean particles result, whereas a single particle results for $s=\frac{1}{2}$ and $s=1$.

(8) There exists a simple Lagrangian and Hamiltonian formulation and a conserved current.

(9) The theory is consistent and causal in the presence of a minimally coupled external-electromagnetic-field interaction.

Of course, any consistent interpretation of relativistic quantum mechanics must be given within the framework of a many-particle theory (field theory). It is there that the questions of indefinite metric and negative-energy states, in addition to the physical interpretation of the component doubling for $s>1$, must be treated. These and related field-theoretic issues will be discussed in a subsequent report. It has been the intent of the present paper to describe only the c-number theory which results when the Galilean equations are generalized.

The electromagnetic interaction has been discussed only formally in order to demonstrate the consistency and causality of the higher-spin equations. A more detailed treatment of the electromagnetic properties of arbitrary-spin particles would involve the use of a generalized Foldy-Wouthuysen transformation. Since only the magnetic dipole term persists in the nonrelativistic limit, we have described only this term in the present paper. The addition of "anomalous" electromagnetic-moment interaction terms to the Lagrangian has not been mentioned, even though a prescription for such an addition is essential if these equations are to be useful in describing physical particles of higher spin in an electromagnetic field.

Finally, we have restricted our attention to particles with $m>0$, since only these have a sensible Galilean limit. Massless particles, however, may also be treated by means of the tools developed here for studying relativistic wave equations. A description of these particles will be presented elsewhere.

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APPENDIX A: REPRESENTATIONS OF SL(2, C)

We wish to show that the only representations of SL(2,C) which are $(6s+1)$-dimensional and which have their SU(2,C) subgroups represented by $D^{(s)}\oplus D^{(s)}\oplus D^{(s-1)}$ are given by (3.2) and (3.3). We consider the irreducible representations labeled by (n,m) (where n and m are integer or half integer), and their direct sums.

The SU(2,C) restriction implies that all representations (n,m) where $n+m>s$ or where $|n-m|<s-1$ must be ruled out. Thus we must have (take $n>m$) $n+m\leqslant s$ and $n-m\geqslant s-1$, and hence $n+m\leqslant s\leqslant n-m+1$, which yields $m=0$ with $n=s$, or $s-1$; or $m=\frac{1}{2}$ and $n=s-\frac{1}{2}$. The possible representations are then eight of the form $(s,0)\oplus(s,0)\oplus(s-1,0)$ and four of the form $(s,0)\oplus(s-\frac{1}{2},\frac{1}{2})$, the others being obtained via parity switches, $(n,m)\to(m,n)$.

APPENDIX B: THE $\vec{K}$ MATRICES

The $\vec{K}$ matrices are closely related to the angular momentum coupling coefficients and have been discussed previously.[6] In the present appendix we list and prove some of their properties. $\vec{S}$ ($\vec{\Sigma}$) represents the three generators of SU(2) in the $(2s+1)$- ($(2s-1)$-) dimensional representation. The $\vec{K}$ represents three rectangular matrices with $2s-1$ rows and $2s+1$ columns. In the following, $i,j,k,=1,2,3$; ϵ_{ijk} is the Levi-Civita symbol, and summation over repeated indices is implied.

The $\vec{K}$ matrices are defined to a factor by the relation

$$K_iS_j-\Sigma_jK_i=i\epsilon_{ijk}K_k\ . \tag{B1}$$

This factor is reduced to a phase by the relation

$$S_iS_j+K_i^\dagger K_j=is\epsilon_{ijk}S_k+s^2\delta_{ij}\ . \tag{B2}$$

K_i satisfying (B1) and (B2) exist for any s and, for example, in the representation where S_z and Σ_z are diagonal they are given as

$$(K_1)_{ij}=\begin{cases}-\omega_i\,, & i=j\\ \omega_{2s-i}\,, & i=j-2\\ 0\,, & \text{otherwise,}\end{cases}$$

$$(K_2)_{ij} = -i \begin{cases} \omega_i\,, & i=j \\ \omega_{2s-i}\,, & i=j-2 \\ 0, & \text{otherwise}, \end{cases}$$

$$(K_3)_{ij} = \begin{cases} z_i\,, & i=j-1 \\ 0, & \text{otherwise}, \end{cases}$$

where $i=1,\ldots,2s-1$, $j=1,\ldots,2s+1$, and

$$\omega_i = \tfrac{1}{2}(2s-i)^{1/2}(2s-i+1)^{1/2}\,,$$

$$z_i = [i(2s-i)]^{1/2}\,.$$

We present some examples: For $s=\frac{1}{2}$, $\vec{K}=0$. For $s=1$, we have

$$K_1 = (-(\tfrac{1}{2})^{1/2}, 0, (\tfrac{1}{2})^{1/2})\,,$$

$$K_2 = -i((\tfrac{1}{2})^{1/2}, 0, (\tfrac{1}{2})^{1/2})\,,$$

$$K_3 = (0, 1, 0)$$

in the representation where S_z is diagonal, and for $S_j^{\alpha\beta} = -i\epsilon_{j\alpha\beta}$ we have

$$K_1 = (1,0,0), \quad K_2 = (0,1,0), \quad K_3 = (0,0,1).$$

For $s=\frac{3}{2}$, we have[23]

$$K_1 = \begin{pmatrix} -(\frac{3}{2})^{1/2} & 0 & (\frac{1}{2})^{1/2} & 0 \\ 0 & -(\frac{1}{2})^{1/2} & 0 & (\frac{3}{2})^{1/2} \end{pmatrix},$$

$$K_2 = -i\begin{pmatrix} (\frac{3}{2})^{1/2} & 0 & (\frac{1}{2})^{1/2} & 0 \\ 0 & (\frac{1}{2})^{1/2} & 0 & (\frac{3}{2})^{1/2} \end{pmatrix},$$

$$K_3 = \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \end{pmatrix}.$$

One may explicitly verify that for any s the following relation holds:

$$K_i K_j^\dagger + \Sigma_i \Sigma_j = -is\epsilon_{ijk}\Sigma_k + s^2\delta_{ij}\,. \tag{B3}$$

Using the above properties we may show

$$K_i^\dagger K_j - K_j^\dagger K_i = (2s-1)i\epsilon_{ijk}S_k\,, \tag{B4}$$

$$K_i S_j - K_j S_i = (s+1)i\epsilon_{ijk}K_k\,, \tag{B5}$$

$$\Sigma_i K_j - \Sigma_j K_i = (1-s)i\epsilon_{ijk}K_k\,, \tag{B6}$$

$$K_i K_j^\dagger - K_j K_i^\dagger = -(2s+1)i\epsilon_{ijk}\Sigma_k\,, \tag{B7}$$

$$K_i S_j - \Sigma_i K_j = si\epsilon_{ijk}K_k\,. \tag{B8}$$

For A a $(2s+1)$-by-$(2s-1)$ rectangular matrix,

$$S_i A - A\Sigma_i = 0, \quad i=1,2,3$$

$$\Rightarrow A = 0. \tag{B9}$$

For B (C) a $(2s+1)$- ($(2s-1)$-) dimensional square matrix,

$$K_i B - CK_i = 0, \quad i=1,2,3$$

$\Rightarrow B$ and C are multiples of the identity,

$$B=\lambda\,, \quad C=\lambda\,. \tag{B10}$$

Note that to each of the above properties corresponds an "adjoint" property. For example, in addition to (B5) we have

$$S_i K_j^\dagger - S_j K_i^\dagger = (s+1)i\epsilon_{ijk}K_k^\dagger\,. \tag{B5†}$$

We indicate some proofs.

Proof of (B4): Use (B2) to write

$$K_i^\dagger K_j - K_j^\dagger K_i = -S_i S_j + S_j S_i + 2is\epsilon_{ijk}S_k$$

and then $[S_i, S_j] = i\epsilon_{ijk}S_k$ to get the desired result. (B7) is proved similarly using (B3).

Proof of (B5) and (B6): Since the result is trivially true for $i=j$, we assume $i\neq j$. Take, e.g., $i=1$ and $j=2$. Using (B1) we have

$$K_1 = i(K_3 S_2 - \Sigma_2 K_3) \quad\text{and}\quad K_2 = -i(K_3 S_1 - \Sigma_1 K_3). \tag{B11}$$

Thus

$$K_1 S_2 - K_2 S_1 = i(K_3 S_2 - \Sigma_2 K_3)S_2 + i(K_3 S_1 - \Sigma_1 K_3)S_1$$
$$= iK_3(S_2 S_2 + S_1 S_1) - i(\Sigma_2 K_3 S_2 + \Sigma_1 K_3 S_1).$$

Again using (B11), $S^2 = s(s+1)$, and $\Sigma^2 = s(s-1)$, we find

$$K_1 S_2 - K_2 S_1 = iK_3[s(s+1) - S_3 S_3]$$
$$-i[s(s-1) - \Sigma_3\Sigma_3]K_3 + \Sigma_1 K_2 - \Sigma_2 K_1\,,$$

and using $K_3 S_3 = \Sigma_3 K_3$ we have

$$(K_1 S_2 - K_2 S_1) - (\Sigma_1 K_2 - \Sigma_2 K_1) = 2isK_3\,. \tag{B12}$$

Direct application of (B1) implies that

$$(K_1 S_2 - K_2 S_1) + (\Sigma_1 K_2 - \Sigma_2 K_1) = 2iK_3\,. \tag{B13}$$

First adding and then subtracting the last two equations yields the desired results.

Proof of (B8): Use (B1) and (B5).

Proof of (B9):

$$S_i A - A\Sigma_i = 0$$
$$\Rightarrow S_{(i)}S_{(i)}A = S_{(i)}A\Sigma_{(i)}$$
$$\Rightarrow S_{(i)}S_{(i)}A = A\Sigma_{(i)}\Sigma_{(i)} \quad \text{(no summation)},$$

and summing over all i implies that

$$s(s+1)A = As(s-1) \Rightarrow A = -A = 0.$$

Proof of (B10): Use the matrices $K_\pm = K_1 \pm iK_2$ to demonstrate the result explicitly.

The above relations are helpful in demonstrating the results of Appendixes C and D. However, they by no means exhaust the list of interesting relations obeyed by the $\vec{K}$ matrices.

APPENDIX C: SOME REPRESENTATIONS OF THE LIE ALGEBRA OF SL(2, C)

In this appendix we shall construct explicitly the Lie algebra of SL(2,C), (3.6), in the $(s,0)$ and $(s-\frac{1}{2},\frac{1}{2})$ representations.

The $(s,0)$ case may be written as

$$J_i = S_i \quad\text{and}\quad N_i = -iS_i\ . \tag{C1}$$

These matrices satisfy the Lie algebra, (3.6); they are irreducible and yield the following representations of the SU(2) algebras:

$$X_i \equiv \tfrac{1}{2}(J_i + iN_i) = S_i\ ,\quad Y_i \equiv \tfrac{1}{2}(J_i - iN_i) = 0.$$

They therefore yield the desired representation, $(s,0)$. Similarly the $(0,s)$ case is realized by the matrices

$$J_i = S_i \quad\text{and}\quad N_i = iS_i\ . \tag{C2}$$

For the $(s-\frac{1}{2},\frac{1}{2})$ representation $(s>\frac{1}{2})$, we take

$$J_i = \begin{pmatrix} S_i & 0 \\ 0 & \Sigma_i \end{pmatrix}$$

and (C3)

$$N_i = -i\begin{pmatrix} [(s-1)/s]S_i & (1/s)K_i^\dagger \\ (1/s)K_i & [(s+1)/s]\Sigma_i \end{pmatrix},$$

where, as before, Σ_i represents the three generators of SU(2) in the $(2s-1)$-dimensional representation. To verify that (C3) satisfies (3.6) we take the commutators explicitly:

(a) $[J_i, J_j] = i\epsilon_{ijk}J_k$,

(b)
$$[N_i,N_j] = -\frac{1}{s^2}\begin{pmatrix} (s-1)^2[S_i,S_j] + (K_i^\dagger K_j - K_j^\dagger K_i) & (s-1)(S_iK_j^\dagger - S_jK_i^\dagger) + (s+1)(K_i^\dagger\Sigma_j - K_j^\dagger\Sigma_i) \\ (s-1)(K_iS_j - K_jS_i) + (s+1)(\Sigma_iK_j - \Sigma_jK_i) & (K_iK_j^\dagger - K_jK_i^\dagger) + (s+1)^2[\Sigma_i,\Sigma_j] \end{pmatrix}$$

$$= -\begin{pmatrix} i\epsilon_{ijk}S_k & 0 \\ 0 & i\epsilon_{ijk}\Sigma_k \end{pmatrix} = -i\epsilon_{ijk}J_k\ ,$$

where we have used (B4), (B5), (B6), and (B7).

(c)
$$[J_i,N_j] = -i\frac{1}{s}\begin{pmatrix} (s-1)[S_i,S_j] & (S_iK_j^\dagger - K_j^\dagger\Sigma_i) \\ (\Sigma_iK_j - K_jS_i & (s+1)[\Sigma_i,\Sigma_j] \end{pmatrix}$$
$$= -i\frac{1}{s}\begin{pmatrix} (s-1)i\epsilon_{ijk}S_k & i\epsilon_{ijk}K_k^\dagger \\ i\epsilon_{ijk}K_k & (s+1)i\epsilon_{ijk}\Sigma_k \end{pmatrix}$$
$$= i\epsilon_{ijk}N_k\ ,$$

where we have used (B1) and its adjoint. Thus we have shown that (C3) satisfies the SL(2,C) Lie algebra.

The set is irreducible since, if an arbitrary $4s$-dimensional matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

commutes with J_i and N_i, then we must have

$$[J_i,A] = \begin{pmatrix} S_ia_{11} - a_{11}S_i & S_ia_{12} - a_{12}\Sigma_i \\ \Sigma_ia_{21} - a_{21}S_i & \Sigma_ia_{22} - a_{22}\Sigma_i \end{pmatrix} = 0\ .$$

The irreducibility of the S_i and Σ_i implies that $a_{11} = \lambda_1 I$ [$(2s+1)$-dimensional] and $a_{22} = \lambda_2 I$ [$(2s-1)$-dimensional]. Also by (B9) we have $a_{12} = 0 = a_{21}$. If we now commute

$$[N_i,A] = -i\frac{1}{s}\begin{pmatrix} 0 & K_i^\dagger\lambda_2 - \lambda_1K_i^\dagger \\ K_i\lambda_1 - \lambda_2K_i & 0 \end{pmatrix} = 0\ ,$$

we have that $\lambda_1 = \lambda_2 = \lambda$ and $A = \lambda I$. Thus J_i and N_i form an irreducible set.

Finally, the matrices $Y_i \equiv (\frac{1}{2})(J_i - iN_i)$, in addition to satisfying the SU(2) Lie algebra, may be easily shown via (B1), (B2), and (B3) to satisfy

$$Y_iY_j + Y_jY_i = \tfrac{1}{2}\delta_{ij}$$

and thus to have eigenvalues $\pm\frac{1}{2}$ and $Y\cdot Y = (\frac{1}{2})(\frac{1}{2}+1)$. We therefore have that J_i and N_i form the $4s$-dimensional irreducible representation of SL(2,C), $(n,\frac{1}{2})$. Since $(2n+1)\times 2 = 4s$, we have $n = s - \frac{1}{2}$ which completes the verification.

In view of the above results we may immediately write down the $(\frac{1}{2}, s-\frac{1}{2})$ representation as

$$J_i = \begin{pmatrix} S_i & 0 \\ 0 & \Sigma_i \end{pmatrix} \quad\text{and}\quad N_i = i\frac{1}{s}\begin{pmatrix} (s-1)S_i & K_i^\dagger \\ K_i & (s+1)\Sigma_i \end{pmatrix}. \tag{C4}$$

APPENDIX D: CONSTRUCTION OF β_μ

In this appendix we shall seek the β_μ matrices satisfying (3.7) for each of the 12 representations, (3.2), (3.3). The case of (3.2) will be carried out explicitly to illustrate the procedure.

The $(6s+1)$-dimensional matrix generators corresponding to the eight representations, (3.2), may be written using (C1) and (C2),

$$J_i=\begin{pmatrix} S_i & 0 & 0\\ 0 & S_i & 0\\ 0 & 0 & \Sigma_i\end{pmatrix} \quad\text{and}\quad N_i=i\begin{pmatrix}\epsilon_1 S_i & 0 & 0\\ 0 & \epsilon_2 S_i & 0\\ 0 & 0 & \epsilon_3 S_i\end{pmatrix}, \tag{D1}$$

where $\epsilon_1, \epsilon_2, \epsilon_3 = \pm 1$ yield the eight possibilities. We now seek $(6s+1)$-dimensional matrices, β_μ, such that (3.7) will be satisfied.

Equation (3.7a) implies that

$$\beta_0=\begin{pmatrix}\gamma_{11} & \gamma_{12} & 0\\ \gamma_{21} & \gamma_{22} & 0\\ 0 & 0 & \gamma_{33}\end{pmatrix}, \tag{D2}$$

where the irreducibility of the S_i and Σ_i has been used in addition to property (B9). The γ_{ij} are multiples of the identity in their respective matrix blocks.

Likewise, (3.7b) implies that

$$\beta_j=\begin{bmatrix}\beta_{11}S_j & \beta_{12}S_j & \beta_{13}K_j^\dagger\\ \beta_{21}S_j & \beta_{22}S_j & \beta_{23}K_j^\dagger\\ \beta_{31}K_j & \beta_{32}K_j & \beta_{33}\Sigma_j\end{bmatrix}, \tag{D3}$$

where the β_{ij} are complex numbers.

Putting the forms (D1), (D2), and (D3) into (3.7d), $[N_i, \beta_j]=i\delta_{ij}\beta_0$, we find the following conditions on the nine matrix blocks:

$$(1,1):\quad \epsilon_1\beta_{11}(S_iS_j-S_jS_i)=\gamma_{11}\delta_{ij}\Rightarrow\gamma_{11}=0=\beta_{11},$$

$$(1,2):\quad \epsilon_1\beta_{12}S_iS_j-\beta_{12}\epsilon_2S_jS_i=\gamma_{12}\delta_{ij}\Rightarrow\begin{cases}\text{either} & \beta_{12}(S_iS_j-S_jS_i)=\gamma_{12}\delta_{ij}\Rightarrow\gamma_{12}=0=\beta_{12}\\ \text{or} & \beta_{12}(S_iS_j+S_jS_i)=\gamma_{12}\delta_{ij},\end{cases}$$

which can be satisfied for $\beta_{12}, \gamma_{12}\neq 0$ only for the case $s=\frac{1}{2}$. This is the expected exception, so we assume in the following that $s>\frac{1}{2}$:

$$(1,3):\quad \epsilon_1\beta_{13}S_iK_j^\dagger-\beta_{13}\epsilon_3K_j^\dagger\Sigma_i=0\Rightarrow\begin{cases}\text{either} & \beta_{13}(S_iK_j^\dagger-K_j^\dagger\Sigma_i)=0\Rightarrow\beta_{13}=0\\ \text{or} & \beta_{13}(S_iK_j^\dagger+K_j^\dagger\Sigma_i)=0\Rightarrow\beta_{13}=0.\end{cases}$$

Proceeding in a similar fashion for the remaining six matrix blocks we find that $\beta_\mu=0$ except for the case $s=\frac{1}{2}$ which leads us to the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation, i.e., the Dirac theory. Thus we must abandon the higher-spin candidates corresponding to the SL(2, C) representations, (3.2).

For the four representations of the form (3.3), $(s,0)\oplus(s-\frac{1}{2},\frac{1}{2})$, we may write the generators as

$$J_i=\begin{pmatrix}S_i & 0 & 0\\ 0 & S_i & 0\\ 0 & 0 & S_i\end{pmatrix}$$

and

$$N_i=-i\begin{pmatrix}\epsilon_1 S_i & 0 & 0\\ 0 & \epsilon_2[(s-1)/s]S_i & \epsilon_2(1/s)K_i^\dagger\\ 0 & \epsilon_2(1/s)K_i & \epsilon_2[(s+1)/s]\Sigma_i\end{pmatrix},$$

where again $\epsilon_1, \epsilon_2=\pm 1$.

The rotational properties again imply that β_μ has the form (D2) and (D3). The imposition of the boost commutators (3.7c) and (3.7d) again results in nine algebraic conditions linking the coefficients of the submatrix blocks. We shall omit the straightforward algebra for this case, which is somewhat more tedious than the previous example. Application of the $\vec{K}$-matrix properties of Appendix B yields the following result.

There exist nontrivial β_μ only for $\epsilon_1=\epsilon_2$, i.e., only for the representations $(s,0)\oplus(s-\frac{1}{2},\frac{1}{2})$ and $(0,s)\oplus(\frac{1}{2},s-\frac{1}{2})$. These are the matrices (3.9).

APPENDIX E: SOME COMMENTS ON $s=\frac{1}{2}$ AND $s=1$

In the extension to the parity-symmetric theory we have seen that there are two exceptions to the general formalism: The spin-$\frac{1}{2}$ case, which is already parity-symmetric and therefore needs no extension, and the spin-1 case, whose minimal parity extension is of lower dimensionality than the $(s,0)\oplus(s-\frac{1}{2},\frac{1}{2})\oplus(\frac{1}{2},s-\frac{1}{2})\oplus(0,s)$ representation due to the parity symmetry of the $(\frac{1}{2},\frac{1}{2})$ representation, i.e., the representation $(1,0)\oplus(\frac{1}{2},\frac{1}{2})\oplus(0,1)$ will suffice. In this appendix we wish to point out that the $(1,0)\oplus(\frac{1}{2},\frac{1}{2})\oplus(0,1)$ theory does indeed fit into our general requirements and, second, that although the minimality requirement favors the representation $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ for spin-$\frac{1}{2}$ and $(1,0)\oplus(\frac{1}{2},\frac{1}{2})\oplus(0,1)$ for spin 1 the larger representations $(s,0)\oplus(s-\frac{1}{2},\frac{1}{2})\oplus(\frac{1}{2},s-\frac{1}{2})\oplus(0,s)$ for $s=\frac{1}{2}$ or 1 are consistent with invariance requirements and, therefore, might also be considered as wave-equation candidates.

In order to elaborate the first point, we may construct the $(1,0)\oplus(\frac{1}{2},\frac{1}{2})\oplus(0,1)$ representation using the results of Appendixes B, C, and D. If we then seek the appropriate β_μ matrices we will find that they are determined up to two parameters. Setting one or the other of them equal to zero yields the $(1,0)\oplus(\frac{1}{2},\frac{1}{2})$ or the $(\frac{1}{2},\frac{1}{2})\oplus(0,1)$ representation again. If we demand parity symmetry, however, the parameters must be equal and the $(1,0)\oplus(\frac{1}{2},\frac{1}{2})\oplus(0,1)$ representation emerges. This ten-dimensional theory can be shown to be equivalent to the Duffin-Kemmer form of the Proca theory. Furthermore, taking the nonrelativistic limit of this theory, as in Sec. III, yields the desired seven-component Galilean theory.[24] Three of the relativistic components vanish in the limit. The Duffin-Kemmer equation is, therefore, consistent with our general requirements and is in fact implied by them.

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[1]P. A. M. Dirac, Proc. Roy. Soc. (London) A155, 447 (1936).

[2]M. Fierz and W. Pauli, Proc. Roy. Soc. (London) A173, 211 (1939).

[3]E. P. Wigner, Ann. Math. 40, 149 (1939).

[4]See J. Weinberg, Ph.D. thesis, University of California, 1943 (unpublished); J. Weinberg and S. Kusaka (unpublished); K. Johnson and E. C. G. Sudarshan, Ann. Phys. (N.Y.) 13, 126 (1961).

[5]For a description of the Capri-Wightman instability see A. S. Wightman, in *Proceedings of the Fifth Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1968,* edited by A. Perlmutter, C. Angas Hurst, and B. Kurşunoğlu (Benjamin, New York, 1968). Noncausalities and other inconsistencies are described in G. Velo and D. Zwanziger, Phys. Rev. 186, 1337 (1969); 188, 2218 (1969). Further discussion may be found in B. Schroer, R. Seiler, and A. Swieca, Phys. Rev. D 2, 2927 (1970), and in the review talks given at the 1971 Coral Gables Conference (to be published).

[6]W. J. Hurley, Phys. Rev. D 3, 2339 (1971).

[7]We use the metric $g_{\mu\nu}=\text{diag}(1,-1,-1,-1)$ and units such that $\hbar=1=c$.

[8]More technically, we are "contracting" the SL(2,C) representation with respect to its SU(2,C) subgroup and "saving" the SU(2,C) representation. See E. Inönü and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. 39, 510 (1953), and E. J. Saletan, J. Math. Phys. 2, 1 (1961).

[9]See Appendix A.

[10]The other seven are obtained by switching the order of the pairs, e.g., $(0,s)\oplus(s,0)\oplus(s-1,0)$ etc.

[11]Except for the case $s=\frac{1}{2}$, which also is included among the representations (3.3).

[12]β_0 with its $(2s-1)$-dimensional null space exhibits the structure obtained by Umezawa and Visconti for unique mass equations. See H. Umezawa and A. Visconti, Nucl. Phys. 1, 348 (1956).

[13]We remind the reader that under the parity operation the (m,n) representation of SL(2,C) becomes (n,m).

[14]The nonrelativistic limit of (3.14) does, however, admit a parity operation.

[15]Except, as before, for $s=\frac{1}{2}$.

[16]R. J. Duffin, Phys. Rev. 54, 1114 (1938).

[17]A. Proca, Compt. Rend. 202, 1420 (1936).

[18]At the completion of the present investigation the author learned that equivalent equations have been proposed by several others. To his present knowledge the earliest study is that of F. Cap [Z. Naturforsch. 8a, 740 (1953); 8a, 748 (1953)] and H. Donnert [Z. Naturforsch. 8a, 745 (1953); Acta Phys. Austriaca 11, 321 (1957)]. See also J. S. Dowker, Proc. Roy. Soc. (London) A297, 351 (1967). The author is grateful to Dr. J. S. Dowker for a correspondence concerning his work and these earlier references.

The parity doubling seems to have been first explicitly studied by J. D. Harris [Ph.D. thesis, Purdue University, 1955 (unpublished)]. The author is grateful to Dr. A. Z. Capri for bringing this work to his attention.

K. Johnson and E. C. G. Sudarshan (see Ref. 4) considered the components for $s=\frac{3}{2}$, $(\frac{3}{2},0)\oplus(1,\frac{1}{2})\oplus(\frac{1}{2},1)\oplus(0,\frac{3}{2})$, but rejected this possibility because it led to an indefinite metric. They then showed, however, that within their framework the indefinite metric was unavoidable in any case.

W. K. Tung [Phys. Rev. Letters 16, 763 (1966); Phys. Rev. 156, 1385 (1967)] found that the component transformation $(s,0)\oplus(s-\frac{1}{2},\frac{1}{2})\oplus(\frac{1}{2},s-\frac{1}{2})\oplus(0,s)$ was the simplest of a class of representations which led to equations without subsidiary conditions. The resultant parity doubling and indefinite energy [see also S. J. Chang, Phys. Rev. Letters 17, 1024 (1966)] were considered grounds for rejecting these equations. The author is indebted to Dr. W. K. Tung for a correspondence on these points.

The present author feels that the firm theoretical foundations for these equations as demonstrated by their present derivation and by their properties in an external electromagnetic field (see below) are sufficient to motivate a careful re-examination of their less orthodox features.

[19]G. Velo and D. Zwanziger, Phys. Rev. 186, 1337 (1969); 188, 2218 (1969).

[20]Take $A^\mu=(\phi,\vec{A})$, $\partial_\mu=(\partial_t,\vec{\nabla})$, $g_{\mu\nu}=\text{diag}(1,-1,-1,-1)$.

[21]R. P. Feynman and M. Gell-Mann, Phys. Rev. 109, 193 (1958). See also L. M. Brown, *ibid.* 111, 957 (1958).

[22]A. S. Wightman, see Ref. 5.

[23]Similar matrices for the spin-$\frac{3}{2}$ case were introduced by F. J. Belinfante [Phys. Rev. 92, 997 (1953)]. The present matrices are generalizations of these matrices to any spin.

[24]C. R. Hagen, Commun. Math. Phys. 18, 97 (1970).