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Renormalized Scattering Operator in Closed Form. I. ϕ^3 Model

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We present a method of obtaining the results of renormalization which makes no explicit reference either to perturbation theory or to the removal of infinities, but instead is based directly on physical requirements. Applied to scalar ϕ^3 theory, the method yields a closed expression for the renormalized scattering operator S in the Dyson form as an implicitly time-ordered exponential of an interaction Hamiltonian plus quasilocal counterterms. Except for the over-all phase of S, these counterterms are given as explicit functionals of the vacuum expectation value of the bilinear product of operator derivatives of S with respect to the asymptotic in-field $a(x)$, which enables them to be calculated recursively to any given order of perturbation theory from lower orders. S is calculated in a straightforward manner up to third order of perturbation theory, and the two-point function to fourth order, and all are shown to be finite, the infinities canceling automatically. The second- and third-order results are identical with those of conventional renormalized perturbation theory. No comparable calculation of the fourth-order result seems to be available.

I. INTRODUCTION

We know that the renormalized scattering operator S can be written as an implicitly time-ordered exponential of an interaction Hamiltonian H_0 plus quasilocal operator counterterms.[1] However, these counterterms have been defined only in the context of perturbation theory as, for example, canceling divergent integrals, or, strictly speaking, as canceling the cutoff dependence of the cut-off unrenormalized scattering operator. That is, the renormalized S operator could be known completely only in the process of the perturbation calculation itself. On the other hand, many of the renormalized perturbation results in quantum field theory have been obtained by means which do not involve the appearance of divergent integrals.[2] In particular, Scattering Operator Theory (TSO)[3-5] has obtained them without using the interpolating field or the asymptotic condition, basing itself on a single strong equation for the scattering operator S which contains the full content of strong unitarity and strong Bogoliubov causality. But in its present form this theory is also tied to the context of perturbation calculations, because the "interaction term" in it can be specified *a priori* only to first order. The "higher-order interactions" are then determined in the higher-order perturbation calculations by consistency requirements. Even so, these do not give a unique answer, for two finite third-order perturbation solutions were found, one of them the conventional renormalized result.[4]

This paper describes an attempt to find a representation of the renormalized S operator in closed form, by a synthesis of these two approaches. We devise a means of determining the counterterms of the Dyson form for S from conditions derived from the axioms of TSO. Thus the counterterms are defined without explicit reference to perturbation theory or to the removal of divergences. Here we consider only the ϕ^3 model, the neutral scalar field with cubic self-interaction. We take

$$S=(e^{-iH})_+, \tag{1.1}$$

the implicitly positively time-ordered exponential, with

$$H=H_0+\Lambda_0+\int \Lambda_1(x)a(x)d^4x + \tfrac{1}{2}\int \Lambda_2(x, y): a(x)a(y): d^4x\, d^4y \tag{1.2}$$

and

$$H_0=\tfrac{1}{6}g\int : a(x)^3: d^4x\,.$$

$a(x)$ is the "renormalized free-field operator," of physical mass m. g is a constant. Λ_1 and Λ_2 are distributions of point support. We know to write S in this form because renormalized perturbation theory gives us this result,[1] and because this S has been shown to be a formal solution of the fundamental operator equation of TSO.[4] Λ_1 and Λ_2 will be seen to be explicit functionals of a single matrix element of S. Λ_0 is not given explicitly, but this will not hinder the calculation. A vertex counterterm

$$\tfrac{1}{6}\int \Lambda_3(x, y, z): a(x)a(y)a(z): d^4x\, d^4y\, d^4z \tag{1.3}$$

is not included in H, since ϕ^3 theory is superrenormalizable, and so such a term is not needed to obtain finite results. Nevertheless, such a term can be included, and we indicate at the end how it may be determined.

The balance of this paper is organized as follows: In Sec. II we derive from the axioms of TSO and the asymptotic condition on the interpolating field the conditions on S which will determine Λ_1 and Λ_2, and we explain why we have used this mixed approach. In Sec. III we determine Λ_1 and Λ_2. In Sec. IV we calculate S to first and second order of perturbation theory, and in Sec. V to third order. In Sec. VI we give the barest outline and the results of the fourth-order calculation of the two-point function. Section VII is a discussion. Appendix A is a derivation of the results of Sec. II without using the interpolating field and the asymptotic condition. Appendix B contains some of the details of the fourth-order calculation.

II. CONDITIONS ON S

Λ_0 is determined by requiring that

$$\text{(A)}\quad \langle S\rangle = 1, \tag{2.1}$$

where $\langle\cdots\rangle$ signifies the vacuum expectation value. Later, $|0\rangle$ will designate the vacuum state, and $|1\rangle$ the one-particle state.

We suppose that the asymptotic in-field [identified with $a(x)$ in (1.2)] generates a complete set of states from $|0\rangle$, on which S operates. Thus

$$S=\sum_{n=0}^{\infty}\frac{1}{n!}\int S_n(x_1,\ldots,x_n)\times :a(x_1)\cdots a(x_n):\,d^4x_1\cdots d^4x_n\,. \tag{2.2}$$

All other operators will be given in terms of operator derivatives[6] of S with respect to $a(x)$. These are defined by

$$\delta a(x)/\delta a(y)=\delta_4(x-y) \tag{2.3}$$

and

$$S_x\equiv\delta S/\delta a(x) = \sum_{n=0}^{\infty}\frac{1}{n!}\int S_{n+1}(x,x_1,\ldots,x_n)\times :a(x_1)\cdots a(x_n):\,d^4x_1\cdots d^4x_n\,. \tag{2.4}$$

S_n is determined off the mass shell by the axioms of TSO, or by (1.1) and (1.2), regarded as "strong" operator equations.[7] It has the same symmetry off as on the mass shell. A strong equation is one whose operator derivatives (to any order) are valid equations. Since we define operator differentiation as being independent of other mathematical operations, the free-field Klein-Gordon equation $K_x a(x)=0$, where $K_x=\partial_x{}^2-m^2$, must be "weak", since from (2.3)

$$\delta K_x a(x)/\delta a(y)=K_x\delta_4(x-y)\neq 0\,.$$

All operator equations except the free-field equation are strong, so that no special symbol need be used to indicate this. The expansion coefficients in (2.2) can therefore be written in terms of operator derivatives,

$$S_n(x_1,\ldots,x_n)=\langle S_{x_1,\ldots,x_n}\rangle\,. \tag{2.5}$$

We require that S be strongly unitary,

$$S^*S=1\,. \tag{2.6}$$

One operator derivative of (2.6) gives

$$j(x)j(y)=S_x^*S_y\,, \tag{2.7}$$

where

$$j(x)\equiv iS^*S_x\,. \tag{2.8}$$

The second operator derivative of (2.6) gives[3]

$$\mathrm{Re}(S^*S_{xy}+S_x^*S_y)\,, \tag{2.9}$$

where, for any operator A,

$$\mathrm{Re}A\equiv\tfrac{1}{2}(A+A^*),\quad \mathrm{Im}A\equiv\frac{1}{2i}(A-A^*)\,. \tag{2.10}$$

We also require that S satisfy strong Bogoliubov causality,[1,3,8]

$$\delta j(x)/\delta a(y)=0\,, \tag{2.11}$$

outside the future light cone of $(x-y)$. [The S operator of (1.1) and (1.2) does in fact satisfy (2.2), (2.6), and (2.11), but for the moment we do not restrict ourselves to that case.] We define an interpolating field operator

$$A(x)\equiv S^*(a(x)S)_+\,, \tag{2.12}$$

where the implicitly time-ordered product is defined by substituting (2.2), and in the integrand of every term taking the explicitly time-ordered product

$$T_+(a(x):a(x_1)\cdots a(x_n):)\,.$$

The implicitly retarded commutator $[a(x),S]_r$ is defined similarly by constructing

$$[a(x),:a(x_1)\cdots a(x_n):]_R = \sum_{i=1}^{n}:a(x_1)\cdots[a(x),a(x_i)]_R\cdots a(x_n):\,.$$

Then

$$[a(x),S]_r=-i\int\Delta_R(x-y)S_y d^4y$$

and

$$(a(x)S)_+=Sa(x)+[a(x),S]_r\,, \tag{2.13}$$

so that

$$A(x)=a(x)-\int\Delta_R(x-y)j(y)d^4y$$

and

$$j(x)=K_x[A(x)-a(x)]. \tag{2.14}$$

We do not delete the term $K_x a(x)$ for reasons already discussed.

One of the two conditions used to determine Λ_1 and Λ_2 is then

$$\text{(B)}\quad \langle j(x)\rangle = 0, \tag{2.15}$$

which follows from (2.14) and the fact that, by (2.12), $A(x)$ satisfies the asymptotic condition, so that[2]

$$\langle A(x)\rangle = \langle a(x)\rangle = 0. \tag{2.16}$$

The other condition (C) for Λ_1 and Λ_2 is obtained from the Källén-Lehmann representation[9]

$$\langle A(x)A(y)\rangle = -i\Delta_+(x-y) -i\int_{4m^2}^{\infty} d\mu^2 \frac{J(\mu^2)}{(\mu^2-m^2)^2}\Delta_+(x-y;\mu^2), \tag{2.17}$$

with $J(\mu^2)$ a real non-negative function, and a similar expression for

$$-i\Delta_c'(x-y)\equiv\langle T_+(A(x)A(y))\rangle, \tag{2.18}$$

with Δ_+ replaced by Δ_c, and[10]

$$\Delta_+(x;\mu^2)=i(2\pi)^{-3}\int d^4p\, e^{ip\cdot x}\theta(p^0)\delta(p^2+\mu^2),$$

$$\Delta_c(x;\mu^2)=(2\pi)^{-4}\int d^4p\, e^{ip\cdot x}(p^2+\mu^2-i\epsilon)^{-1},$$

and

$$\Delta_{()}(x)\equiv\Delta_{()}(x;m^2). \tag{2.19}$$

We have chosen this particular form of the spectral weight function for convenience. It is related to the usual expression $\sigma(\mu^2)$ by

$$J(\mu^2)=(\mu^2-m^2)^2\sigma(\mu^2).$$

Δ_c' is related to S_{xy} by the following two theorems[11]:

(I) If $A(x)$ and a (x) are *any* two operators related by (2.12) (not necessarily satisfying field equations), and S satisfies (2.6) and (2.11), then

$$T_+(A(x)A(y)) = S^*(a(x)a(y)S)_+. \tag{2.20}$$

(II) For any operator S that satisfies (2.2), the implicitly time-ordered product is related to the implicitly retarded commutator by

$$(a(x)a(y)S)_+ = ST_+(a(x)a(y)) + [a(x),S]_r a(y) + [a(y),S]_r a(x) + [a(y),[a(x),S]_r]_r. \tag{2.21}$$

We have given here only the special case of these two theorems required in our derivation. Equation (2.13) is also a special case of Theorem II. Then, from (2.18), (2.20), and (2.21), we have

$$-i\Delta_c'(x-y)=\langle S^*(a(x)a(y)S_+\rangle = -i\Delta_c(x-y) - \int \Delta_R(x-u)\Delta_R(y-v)\langle S^* S_{uv}\rangle d^4u\, d^4v, \tag{2.22}$$

because

$$\langle j(x)a(y)\rangle = \langle 0|j(x)|1\rangle\langle 1|a(y)|0\rangle = 0$$

from the asymptotic condition, which makes[2]

$$\langle 0|A(x)|1\rangle = \langle 0|a(x)|1\rangle$$

and also

$$\langle A(x)a(y)\rangle = \langle a(x)a(y)\rangle, \tag{2.23}$$

and, combined with (2.14), which makes

$$\langle 0|j(x)|1\rangle = 0. \tag{2.24}$$

From (2.22), and the spectral representation for (2.18), we have the second condition for Λ_1 and Λ_2:

$$\text{(C)}\quad \langle S^* S_{xy}\rangle = iK_xK_y\int_{4m^2}^{\infty} d\mu^2\frac{J(\mu^2)}{(\mu^2-m^2)^2}\Delta_c(x-y;\mu^2). \tag{2.25}$$

We eliminate all reference to $A(x)$ by using (2.14), (2.17), and (2.23) to obtain

$$\langle j(x)j(y)\rangle = K_xK_y\langle A(x)A(y)-a(x)a(y)\rangle = -i\int_{4m^2}^{\infty} d\mu^2 J(\mu^2)\Delta_+(x-y;\mu^2), \tag{2.26}$$

which we will use, together with (2.7), to calculate J.

It is interesting to compare the present derivation of (2.25) with one, given in Appendix A, which does not use $A(x)$ and the asymptotic condition, but instead uses the full content of TSO, in order to make explicit the physical content in the former notion. We placed the present derivation in the body of the text, however, because it is shorter, and because we are obliged to concede implicitly the existence of $A(x)$ if we wish to use the Dyson form of S in perturbation theory.

III. DETERMINATION OF Λ_1 AND Λ_2

Take the operator derivative of (1.1) and (1.2). Then from (2.8),

$$j(x)=S^*(H_xS)_+, \tag{3.1}$$

where

$$H_x=\tfrac{1}{2}g{:}a(x)^2{:}+\Lambda_1(x)+\int\Lambda_2(x,y)a(y)d^4y. \tag{3.2}$$

If we substitute (3.1) and (3.2) in (2.15), and use

(2.12) and (2.16), we obtain

$$\Lambda_1(x) = -\tfrac{1}{2}g\langle S^*(:a(x)^2:S)_+\rangle \tag{3.3}$$

and using (2.21) modified for the normal product $:a(x)a(y):$ in place of $a(x)a(y)$, we have

$$\Lambda_1(x) = \tfrac{1}{2}g\int \Delta_R(x-u)\Delta_R(x-v)\langle S^* S_{uv}\rangle d^4u d^4v\,. \tag{3.4}$$

Equation (2.25) then gives us

$$\Lambda_1 = \frac{ig}{2(2\pi)^4}\int d^4p\int_{4m^2}^{\infty} d\mu^2\,\frac{J(\mu^2)}{(\mu^2-m^2)^2(p^2+\mu^2-i\epsilon)}\,. \tag{3.5}$$

Taking the second operator derivative of (1.1) and (1.2), we have

$$S_{xy} = -i(H_{xy}S)_+ - (H_xH_yS)_+\,, \tag{3.6}$$

where

$$H_{xy} = g\delta_4(x-y)a(x) + \Lambda_2(x,y)\,. \tag{3.7}$$

Since $j(x)$ and H_x satisfy the conditions of Theorem I,

$$S^*(H_xH_yS)_+ = T_+(j(x)j(y))\,. \tag{3.8}$$

Using (2.6) and (2.16), we see that

$$\Lambda_2(x,y) = i\langle S^*S_{xy}\rangle - \langle T_+(j(x)j(y))\rangle\,. \tag{3.9}$$

The time-ordered term of (3.9) can be obtained from (2.26) with Δ_+ replaced by Δ_c. Using this and (2.25) we have

$$\Lambda_2(x-y) = (2\pi)^{-4}\int d^4p\,e^{ip\cdot(x-y)}\lambda_2(p)\,,$$

where

$$\lambda_2(p) = \int_{4m^2}^{\infty} d\mu^2\,\frac{J(\mu^2)}{\mu^2-m^2}\left(1-\frac{p^2+m^2}{\mu^2-m^2}\right)\cdot \tag{3.10}$$

As promised, Λ_1 and Λ_2 are given explicitly, although recursively. We note that the unrenormalized theory cannot satisfy (2.15) and (2.25), since if $\Lambda_1=\Lambda_2=0$ we would have $J(\mu^2)=0$, a free-field theory. We also note that, whether or not Λ_2 diverges (it does in perturbation theory), Λ_1 diverges no matter what.

IV. FIRST- AND SECOND-ORDER PERTURBATION

Here we show how we obtain the results of renormalized perturbation theory. This is made possible by the fact that from (2.26) and (2.7), $J(\mu^2)$ is a functional of the bilinear expression $\langle S_x^*S_y\rangle$. Suppose that

$$S = 1+\sum_{n=1}^{\infty} g^nS^{(n)}\,. \tag{4.1}$$

Then S_x and $j(x)$ are at least of first order in g, $J(\mu^2)$ and Λ_2 are at least of second order, and, from (3.4), Λ_1 is at least of third order. In order that the first-order vector $S^{(1)}|0\rangle$ have finite norm, we must modify S by inserting in the integrand of H_0 in (1.2) the real c-number test function $\sigma(x)$.[4] Then $\Lambda_{0,1,2}$ and S become implicit functionals of $\sigma(x)$. At the end of the calculation we take $\sigma(x)=1$. This is a technical requirement, and does not by itself make S finite. It will reduce the divergence of Λ_1 from quadratic to logarithmic, but will not alter the divergence of Λ_2. We will suppress $\sigma(x)$ in the following except in those few integrals that require it, in which cases we can recover it for every variable of integration x by multiplying the integrand by the factor $\sigma(x)$.

The first-order perturbation result, from (1), is

$$\begin{aligned} S^{(1)} &= -iH^{(1)} \\ &= -\tfrac{1}{6}i\int\sigma(x):a(x)^3:d^4x - i\Lambda_0^{(1)}\,, \\ S_x^{(1)} &= -\tfrac{1}{2}i:a(x)^2:\,, \\ S_{xy}^{(1)} &= -i\delta_4(x-y)a(x)\,, \end{aligned} \tag{4.2}$$

and setting $\langle S^{(1)}\rangle=0$ gives $\Lambda_0^{(1)}=0$.

The two-point function to second order is

$$\langle S_{xy}^{(2)}\rangle = \langle S^*S_{xy}\rangle^{(2)} \tag{4.3}$$

because the difference between these two expressions vanishes,

$$\langle S^{*(1)}S_{xy}^{(1)}\rangle = \tfrac{1}{6}\delta_4(x-y)\int\sigma(z)\langle:a(z)^3:a(x)\rangle d^4z = 0\,, \tag{4.4}$$

by the version of Wick's theorem which expresses the product

$$:a(x_1)\cdots a(x_m):\;:a(y_1)\cdots a(y_n): \tag{4.5}$$

as a sum of normally ordered products, with Δ_+ contractions as coefficients.[1,3] As a consequence

$$\langle:a(x_1)\cdots a(x_m):\;:a(y_1)\cdots a(y_n):\rangle = \delta_{nm}\sum_{\text{(combinations)}}\prod_{j,k}[-i\Delta_+(x_j-y_k)]\,, \tag{4.6}$$

which vanishes when $n\neq m$, because the maximum number of Δ_+ contractions will still leave some operators normally ordered. We are obliged to prove (4.3) rather than set

$$S|0\rangle = |0\rangle\,,$$

as ought to be the case, because we have already completely determined S, and therefore may not impose any other conditions. $J^{(2)}(\mu^2)$ is obtained from the spectral representation[12]

$$\Delta_+(x;m^2)\Delta_+(x;\mu^2)=\frac{i}{16\pi^2}\int_{(m+\mu)^2}^{\infty}d\nu^2\rho(\nu^2,\mu^2)\Delta_+(x;\nu^2),\tag{4.7}$$

where

$$\rho(\nu^2,\mu^2)=\left[\left(1-\frac{\mu^2-m^2}{\nu^2}\right)^2-\frac{4m^2}{\nu^2}\right]^{1/2}.$$

From (4.2) and (4.6),

$$\begin{aligned}\langle S_x^* S_y\rangle^{(2)}&=\langle S_x^{*(1)}S_y^{(1)}\rangle\\&=-\tfrac{1}{2}[\Delta_+(x-y;m^2)]^2.\end{aligned}\tag{4.8}$$

Now

$$-2\,\mathrm{Re}[i\Delta_+(x;\mu^2)]=(2\pi)^{-3}\int d^4p\,e^{ip\cdot x}\delta(p^2+\mu^2),\tag{4.9}$$

and, from (2.26) and (2.7),

$$2\,\mathrm{Re}\langle S_x^* S_y\rangle=(2\pi)^{-3}\int d^4p\,e^{ip\cdot(x-y)}J(-p^2).\tag{4.10}$$

Combining (4.7) through (4.11), we obtain

$$\begin{aligned}32\pi^2J^{(2)}(\mu^2)&=\rho(\mu^2,m^2)\theta(\mu^2-4m^2)\\&=(1-4m^2/\mu^2)^{1/2}\theta(\mu^2-4m^2).\end{aligned}\tag{4.11}$$

Substituting (4.11) into (2.25), we see that $\langle S_{xy}^{(2)}\rangle$ is the renormalized result, and is finite. Its Fourier transform is

$$i(p^2+m^2)^2\int_{4m^2}^{\infty}d\mu^2\,\frac{J^{(2)}(\mu^2)}{(\mu^2-m^2)^2(p^2+\mu^2-i\epsilon)}.\tag{4.12}$$

The automatic cancellation involving $\Lambda_2^{(2)}$ is seen in calculating the entire S operator to second order, from (1.1) and (1.2),

$$\begin{aligned}S^{(2)}&=-iH^{(2)}-\tfrac{1}{2}(H^{(1)}H^{(1)})_+\\&=-i\Lambda_0^{(2)}-i\int\Lambda_2^{(2)}(x,y):a(x)a(y):d^4x\,d^4y\\&\quad-\tfrac{1}{72}\int\sigma(x)\sigma(y)T_+(:a(x)^3:\ :a(y)^3:)d^4x\,d^4y.\end{aligned}\tag{4.13}$$

Expanding the last line by Wick's theorem, and combining coefficients of $:a(x)a(y):$, we find the combination

$$\tfrac{1}{2}[\Delta_c(x-y)]^2-i\Lambda_2^{(2)}(x,y).\tag{4.14}$$

From (3.10) and (4.11) we see that $\Lambda_2^{(2)}$ diverges. $\Delta_c{}^2$ diverges for the same reason. Using (4.7) and (4.11), we have

$$\begin{aligned}[\Delta_c(x)]^2&=[\theta(x^0)\Delta_+(x)+\theta(-x^0)\Delta_+(-x)]^2\\&=\theta(x^0)[\Delta_+(x)]^2+\theta(-x^0)[\Delta_+(-x)]^2\\&=2i\int_{4m^2}^{\infty}d\mu^2J^{(2)}(\mu^2)\Delta_c(x;\mu^2).\end{aligned}\tag{4.15}$$

Let us cut off the integrals (3.10) and (4.15) at some large value M of μ^2, do the cancellation in (4.14), and then let $M\to\infty$. Equation (4.14) is then seen to be the Fourier transform of

$$\int_{4m^2}^{\infty}d\mu^2J^{(2)}(\mu^2)\left(\frac{1}{p^2+\mu^2-i\epsilon}-\frac{1}{\mu^2-m^2}+\frac{p^2+m^2}{\mu^2-m^2}\right),\tag{4.16}$$

which is identical with (4.12). Thus

$$\tfrac{1}{2}[\Delta_c(x-y)]^2-i\Lambda_2^{(2)}(x,y)=\langle S_{xy}^{(2)}\rangle\tag{4.17}$$

which is finite, automatically. Putting $\langle S^{(2)}\rangle=0$, we obtain

$$\Lambda_0^{(2)}=-\tfrac{1}{12}\int\sigma(x)\sigma(y)\Delta_c(x-y)^3d^4x\,d^4y.\tag{4.18}$$

Finally

$$\begin{aligned}S^{(2)}=\int d^4xd^4y[&-\tfrac{1}{72}:a(x)^3a(y)^3:\\&+\tfrac{1}{8}i\,\Delta_c(x-y):a(x)^2a(y)^2:\\&+\tfrac{1}{2}\langle S_{xy}^{(2)}\rangle:a(x)a(y):],\end{aligned}\tag{4.19}$$

which is the renormalized result.

V. THIRD-ORDER PERTURBATION

The third-order S operator is, from (1.1) and (1.2),

$$S^{(3)}=-iH^{(3)}-(H^{(2)}H^{(1)})_++\tfrac{1}{6}i(H^{(1)}H^{(1)}H^{(1)})_+.\tag{5.1}$$

We note that $\Lambda_2^{(3)}=0$ because $J^{(3)}(\mu^2)=0$. That is, $J^{(3)}$ is obtained from

$$\langle S_x^* S_y\rangle^{(3)}=\langle S_x^{*(2)}S_y^{(1)}\rangle+\langle S_x^{*(1)}S_y^{(2)}\rangle,\tag{5.2}$$

and if we take one operator derivative of (4.19) combined with (4.2) we see that every term of (5.2) vanishes because of (4.6). Combining (5.1) and (1.2), we have

$$\begin{aligned}S^{(3)}=&-i\Lambda_0^{(3)}-i\int\Lambda_1^{(3)}a(x)d^4x-\tfrac{1}{6}\Lambda_0^{(2)}\int\sigma(z):a(z)^3:d^4z\\&-\tfrac{1}{6}\int d^4xd^4yd^4z\,\sigma(z)[\Lambda_2^{(2)}(x,y)T_+(:a(x)a(y):\ :a(z)^3:)-\tfrac{1}{216}i\,\sigma(x)\sigma(y)T_+(:a(x)^3:\ :a(y)^3:\ :a(z)^3:)].\end{aligned}\tag{5.3}$$

Using Wick's theorem, and combining coefficients of the same normal products, we find that the combination (4.17) occurs three times. We also note that the requirement $\langle S^{(3)}\rangle = 0$ gives $\Lambda_0^{(3)} = 0$. Thus

$$S^{(3)} = \int d^4x d^4y d^4z [6^{-4} i : a(x)^3 a(y)^3 a(z)^3 : + \tfrac{1}{48}\Delta_c(x-y) : a(x)^2 a(y)^2 a(z)^3 :$$
$$- \tfrac{1}{8} i\, \Delta_c(x-y)\Delta_c(y-z) : a(x)^2 a(y) a(z)^2 : - \tfrac{1}{12} i \langle S_{xy}^{(2)}\rangle : a(x)a(y)a(z)^3 :$$
$$- \tfrac{1}{2}\langle S_{xy}^{(2)}\rangle \Delta_c(y-z) : a(x)a(z)^2 : - \tfrac{1}{6}\Delta_c(x-y)\Delta_c(y-z)\Delta_c(z-x) : a(x)a(y)a(z) :]$$
$$+ \int d^4z\, \sigma(z)\left[-\tfrac{1}{6} : a(z)^3 : \left(\Lambda_0^{(2)} + \tfrac{1}{12}\int d^4x d^4y\, \sigma(x)\sigma(y)\Delta_c(x-y)^3\right) - i\Lambda_1^{(3)} + \tfrac{1}{2} i \int d^4x d^4y \langle S_{xy}^{(2)}\rangle \Delta_c(z-x)\Delta_c(z-y)\right]. \tag{5.4}$$

The last two lines of (5.4) vanish, by (4.18), and by (3.4) and (4.12), respectively, giving us the renormalized result.

VI. OUTLINE OF FOURTH-ORDER PERTURBATION

Since

$$\langle S_{xy}^{(4)}\rangle = \langle S^* S_{xy}\rangle^{(4)}, \tag{6.1}$$

we can use (2.25). $J^{(4)}(\mu^2)$ is obtained from

$$\langle S_x^* S_y\rangle^{(4)} = \sum_{k=1}^{3} \langle S_x^{*(k)} S_y^{(4-k)}\rangle . \tag{6.2}$$

Now

$$\langle S_x^{*(2)} S_y^{(2)}\rangle = -i(4\pi)^{-4}\int_{4m^2}^{\infty} d\mu^2 \int_{(m+\mu)^2}^{\infty} d\nu^2 \times J(\mu^2, \nu^2)\Delta_+(x-y;\nu^2), \tag{6.3}$$

where

$$J(\mu^2,\nu^2) = K(\mu^2,\nu^2) + L(\mu^2,\nu^2),$$
$$K(\mu^2,\nu^2) = \rho(\mu^2, m^2)\rho(\nu^2,\mu^2)/2(\mu^2 - m^2)^2, \tag{6.4}$$

and

$$L(\mu^2,\nu^2) = \frac{1}{(\mu^2 - m^2)\nu^2} \times \ln\left(\frac{\nu^2 - \mu^2 + m^2 + \nu^2\rho(\nu^2,\mu^2)\rho(\mu^2,m^2)}{\nu^2 - \mu^2 + m^2 - \nu^2\rho(\nu^2,\mu^2)\rho(\mu^2,m^2)}\right). \tag{6.5}$$

$\rho(\nu^2,\mu^2)$ is defined in (4.7). Next

$$\langle S_x^{(1)*} S_y^{(3)}\rangle = -\tfrac{1}{2} i(4\pi)^{-4}\int_{4m^2}^{\infty} d\mu^2 M(\mu^2)\Delta_+(x-y;\mu^2), \tag{6.6}$$

where

$$M(\mu^2) = \rho(\mu^2, m^2)[N(\mu^2) + P(\mu^2)], \tag{6.7}$$
$$N(\mu^2) = \tfrac{1}{2}(\mu^2 - m^2)\int_{4m^2}^{\infty} d\nu^2 \frac{\rho(\nu^2, m^2)}{(\nu^2 - m^2)^2(\mu^2 - \nu^2 + i\epsilon)},$$

and

$$P(\mu^2) = \int_{4m^2}^{\infty} d\nu^2 \frac{W(1-\mu^2/\nu^2)}{\nu^2(\nu^2-\mu^2)\rho(\nu^2,m^2)}, \tag{6.8}$$

where

$$W(z) = \ln z + (1-4z)^{-1/2}\ln\left(\frac{1+(1-4z)^{1/2}}{1-(1-4z)^{1/2}}\right). \tag{6.9}$$

The last term in (6.2) is

$$\langle S_x^{(3)*} S_y^{(1)}\rangle = \langle S_y^{(1)*} S_x^{(3)}\rangle^*,$$

and can be obtained from (6.6) using $\Delta_+^*(x) = -\Delta_+(-x)$. Applying (4.9) and (4.10) to the resulting expression in (6.2), we obtain

$$(4\pi)^4 J^{(4)}(\mu^2) = \mathrm{Re}M(\mu^2)\theta(\mu^2 - 4m^2) + \int_{4m^2}^{\infty} d\nu^2 J(\nu^2,\mu^2)\theta(\mu^2 - (m+\nu)^2). \tag{6.10}$$

The second term in (6.10) can be written as

$$\theta(\mu^2 - 9m^2)\int_{4m^2}^{(\mu-m)^2} d\nu^2 J(\nu^2,\mu^2), \tag{6.11}$$

since $\mu > m + \nu > 3m$. From (6.10) and (6.11) it is not difficult to verify that

$$J^{(4)}(\mu^2) \to (4\pi)^{-4}\int_{4m^2}^{\infty} d\nu^2 \frac{\rho(\nu^2, m^2)}{(\nu^2 - m^2)^2} \quad \text{as } \mu^2 \to \infty, \tag{6.12}$$

which is finite, so if we substitute (6.10) in (2.25), the resulting integral for $\langle S_{xy}^{(4)}\rangle$ will converge. The details of obtaining (6.1)–(6.9) are discussed in Appendix B.

VII. DISCUSSION

We admit at once that we have not proven that our method gives the conventional renormalized results to all orders of perturbation theory, but only as far as we have gone. To go further we require Feynman rules, but the recursive definition of Λ_1 and Λ_2 makes it not obvious how these may be obtained.

In terms of conventional renormalization, we may say that $\Lambda_0^{(2)}$ cancels the second-order vacuum diagram, $\Lambda_1^{(3)}$ cancels the third-order tadpole diagram, and $\Lambda_2^{(2)}$ cancels the divergence in the bubble diagram. In addition, as we see from the finiteness of the fourth-order result, the tadpole insertion in $\langle S_{xy}^{(4)}\rangle$ is also canceled. The presence of such a term would violate (6.1) and cause the vacuum state to be unstable. As explained in connection with (4.3), we have not explicitly assumed the full content of $S|0\rangle = |0\rangle$, but in the two cases considered here, our other axioms provide us with more of that content than was apparent.

We emphasize that Λ_1 and Λ_2 are given as explicit functionals of S. Λ_0 is not, but if we had calculated $j(x)$ first, then solved for S subject to (2.1), we could have avoided all explicit mention of Λ_0, as we see from (2.8). However, the perturbation calculation is easier as we have done it. In fact, up to fourth order, it is no more difficult than the conventional renormalized perturbation calculation.

Since we have not included a vertex counterterm like (1.3) we may not be able to identify g with the physical coupling constant in some low-energy limit. We can include such a term in H, and determine it by defining a vertex function Γ by

$$i\langle S^* S_{xyz}\rangle = K_x K_y K_z \int \Delta_c'(x-u)\Delta_c'(y-v)\Delta_c'(z-w) \times \Gamma(u, v, w) d^4u\, d^4v\, d^4w, \tag{7.1}$$

as is suggested by the form of $\langle S_{xyz}^{(1)} + S_{xyz}^{(3)}\rangle$, and by requiring that the Fourier transform of Γ,

$$\gamma(p_1, p_2, p_3)\delta_4(p_1+p_2+p_3),$$

which can be written as $\gamma(-p_3^2)\delta_4(p_1+p_2+p_3)$ when $p_1^2 = p_2^2 = -m^2$, satisfy the following two requirements[13]:

(I) $\gamma(-p^2) \to g$ as $p \to 0$, and

(II) The Lehmann-Symanzik-Zimmermann theorem,[14]

$$(-p^2)^{-1/2}\gamma(-p^2) \to 0 \text{ as } -p^2 \to \infty .$$

The result of this program will be reported on in a sequel.

As we saw in Sec. III, an unrenormalized quantum field theory with a positive-metric Hilbert space cannot satisfy the conditions (A), (B), (C) of Sec. II. This is a rather precise expression of the idea that renormalization is done for physical reasons, and would be necessary even for a finite unrenormalized theory.

We began with a synthesis of conventional renormalization and those approaches which do not use the improper exponential form of (1.1), such as TSO, and our result is seen to lie somewhere between theirs. On one hand, our method is far less strenuous to apply than TSO, and does not share its ambiguity. On the other hand, we still have the Dyson exponential form of S, which in our case is seen to be an improper form because Λ_1 is infinite, even without perturbation theory. However, even though the form is improper, the actual values of the matrix elements of S as we have obtained them are finite.

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APPENDIX A

Here we derive (2.25) from a dispersion relation [(A5), below] given by Bogoliubov and Shirkov.[1] First, we summarize the assumptions needed in that derivation. Then we list the additional assumptions we need, and proceed.

$A(x)$ is not used, but we retain (2.6), (2.11), (2.15), and (2.24) as assumptions. The last two equations are required in order that (2.26) should hold. The two expressions

$$Q_c(x-y) \equiv -\langle S^* S_{xy}\rangle \tag{A1}$$

and

$$Q_R(x-y) \equiv i\langle \delta j(x)/\delta a(y)\rangle \tag{A2}$$

are related by the operator identity

$$-S^* S_{xy} = i\,\delta j(x)/\delta a(y) + j(y) j(x), \tag{A3}$$

which follows here from (2.8) and (2.7). If $Q_{c,R}$ are tempered distributions, then their Fourier transforms

$$q_{c,R}(p) = \int d^4x\, e^{-ip\cdot x} Q_{c,R}(x)$$

are analytic continuations of the same function of a complex variable $q(z)$ which is polynomially bounded for large z,

$$q_c(p) = q(-p^2 + i\epsilon), \quad q_R(p) = q(-\vec{p}^{\,2} + (p^0 + i\epsilon)^2) . \tag{A4}$$

If $q(z)$ is bounded by z^n for some n, then

$$q(z) = -i(z-m^2)^{n+1} \times \int_{4m^2}^{\infty} d\mu^2 J(\mu^2)(\mu^2 - m^2)^{-n-1}(\mu^2 - z)^{-1} + \sum_1^n \frac{1}{r!} q^{(r)}(m^2)(z-m^2)^r, \tag{A5}$$

where $q(m^2) = 0$, and $q^{(r)}(m^2)$ is imaginary.

$$q^{(r)}(m^2) = \frac{d^r q(z)}{dz^r} ,$$

with $z = m^2$.

Now we assume that the operators in (A3) lie in the domain of the convolution-integral operators $P_{R,A}$, defined for $f(x, y)$ in that domain by[3,5]

$$P_{R,A}(f(x,y)) \equiv K_x K_y\, \theta(\pm(x^0 - y^0)) \int \Delta_{A,R}(x-u)\Delta_{R,A}(y-v) f(u,v)\, d^4u\, d^4v\,. \tag{A6}$$

This, with the assumptions of strong unitarity and strong causality, implies

$$P_A(\delta j(x)/\delta a(y)) = 0\,. \tag{A7}$$

and

$$-S^* S_{xy} = -iB(\mathrm{Im}(S^* S_{xy})) + B(\mathrm{Re}(S_x^* S_y)) + P_R(S_x^* S_y) + P_A(S_y^* S_x)\,, \tag{A8}$$

where $B \equiv 1 - P_R - P_A$. That is,

$$B(f(x,y)) \equiv K_x K_y \int [\theta(x^0 - y^0)\Delta(x-u)\Delta_R(y-v) + \theta(y^0 - x^0)\Delta(y-v)\Delta_R(x-u)]\, f(u,v)\, d^4u\, d^4v\,. \tag{A9}$$

We take the vacuum expectation value of (A7), first rearranging the terms of (A5) according to the algebraic identity

$$a^{n+1}b^{-1} = \sum_{r=1}^{n+1} a^{n+1-r}(a-b)^{r-1} + (a-b)^{n+1}b^{-1} \tag{A10}$$

with $a = p^2 + m^2$ and $b = p^2 + \mu^2$. Then

$$q(-p^2) = i\int_{4m^2}^{\infty} d\mu^2 J(\mu^2)[(\mu^2 - m^2)^{-1} - (p^2 + \mu^2)^{-1}] + \sum_{r=1}^{n} (-p^2 - m^2)^r \left[\frac{1}{r!} q^{(r)}(m^2) + i\int_{4m^2}^{\infty} d\mu^2 J(\mu^2)(\mu^2 - m^2)^{-r-1}\right]. \tag{A11}$$

(A7) and (A2) then give us $P_A(Q_R(x-y)) = 0$, whose Fourier transform is

$$0 = (p^2 + m^2)^2 \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi i} (\lambda - i\epsilon)^{-1} [\Delta_R(p - \lambda^0)]^2 q_R(p - \lambda^0)\,, \tag{A12}$$

where

$$\Delta_{R,A}(p) = [\vec{p}^{\,2} + m^2 - (p^0 \pm i\epsilon)^2]^{-1}\,,$$

and

$$p - \lambda^0 = (p^0 - \lambda, \vec{p})\,.$$

As we shall see, q_R vanishes sufficiently strongly at the pole of Δ_R for the product $q_R \Delta_R^{\,2}$ to be well defined. Substituting (A4) and (A11) in (A12), we see that the integrand has poles only in the upper half-plane of λ. Therefore, (A12) is equal to (minus) an integral over an infinite semicircle in the lower half-plane of the same integrand. This integral, of the first line and of the term $r = 1$ in the sum over r in (A11), vanishes as the radius becomes infinite, so these terms automatically satisfy (A12). The integrals of the terms $r \geq 3$ diverge, and the degree of divergence increases with r, so the factors in these terms which do not depend on λ must vanish for each $r \geq 3$. Finally, the integral of the term $r = 2$ is finite, but not zero, so its λ-independent factor must vanish. Therefore, we have

$$\frac{1}{r!} q^{(r)}(m^2) = -i\int_{4m^2}^{\infty} d\mu^2 J(\mu^2)(\mu^2 - m^2)^{-r-1} \tag{A13}$$

for $r \geq 2$, and

$$q(-p^2) = -i(p^2 + m^2)^2 \times \int_{4m^2}^{\infty} d\mu^2 J(\mu^2)(\mu^2 - m^2)^{-2}(p^2 + \mu^2)^{-1} - (p^2 + m^2) q^{(1)}(m^2)\,. \tag{A14}$$

Equation (A14) is also correct for $n = 0$, because then there is no sum in (A11), so that

$$q^{(1)}(-p^2) = -i\int d\mu^2 J(\mu^2)(p^2 + \mu^2)^{-2}$$

and setting $-p^2 = m^2$, we have (A13) with $r = 1$.

We take the vacuum expectation value of (A8) using (2.26) and (2.7), and in the same way we obtained (A12), we obtain the Fourier transform of $P_R\langle S_x^* S_y\rangle$,

$$-i(p^2 + m^2)^2 \int_{-\infty}^{\infty} d\lambda\, (\lambda - i\epsilon)^{-1} [\Delta_A(p + \lambda^0)]^2 \times \theta(p^0 + \lambda) J(-(p + \lambda^0)^2)\,. \tag{A15}$$

This integral is well defined because we may take $J(\mu^2)=0$ for all $\mu^2<4m^2$, and therefore J vanishes uniformly in the neighborhood of the pole of Δ_A. If we change the variable of integration to μ^2 according to

$$p^0+\lambda=(\vec{p}^{\,2}+\mu^2)^{1/2}$$

and treat $P_A\langle S_y^* S_x\rangle$ similarly, we find that the Fourier transform of $P_R\langle S_x^* S_y\rangle+P_A\langle S_y^* S_x\rangle$ is

$$-i(p^2+m^2)^2\int_{4m^2}^{\infty} d\mu^2 J(\mu^2)(\mu^2-m^2)^{-2} \times(p^2+\mu^2-i\epsilon)^{-1}. \qquad \text{(A16)}$$

Similarly, the Fourier transform of $B(\mathrm{Im}Q_c(x-y))$ is

$$(p^2+m^2)^2\int_{-\infty}^{\infty} d\lambda\,\epsilon(p^0+\lambda)\delta((p+\lambda^0)^2+m^2) \times[\Delta_A(p+\lambda^0)(\lambda-i\epsilon)^{-1} +\Delta_R(p+\lambda^0)(\lambda+i\epsilon)^{-1}]\mathrm{Im}q_c(p+\lambda^0), \qquad \text{(A17)}$$

because $q_c(p)$ is an even function of p. As in (A12) the fact that q_c vanishes at the pole of $\Delta_{A,R}$ removes the conflict with the δ function. Substituting (A14) and (A4) in (A17), and remembering that $q^{(1)}(m^2)$ is imaginary, we obtain

$$\text{(A17)}=2i(p^2+m^2)q^{(1)}(m^2). \qquad \text{(A18)}$$

Similarly,

$$B(\mathrm{Re}\langle S_x^* S_y\rangle)=0, \qquad \text{(A19)}$$

for the reason given following (A15). Combining (A8), (A1), (A16), (A18), and (A19), we have

$$q(-p^2+i\epsilon)=-2(p^2+m^2)q^{(1)}(m^2) -i(p^2+m^2)^2\int_{4m^2}^{\infty} d\mu^2 J(\mu^2)(\mu^2-m^2)^{-2} \times(p^2+\mu^2-i\epsilon)^{-1}. \qquad \text{(A20)}$$

Comparing (A20) with (A14) we see that $q^{(1)}(m^2)=0$, and (2.25) follows.

Equation (2.25) can be written in the usual form of a dispersion relation by combining (4.10) and (2.9). This gives

$$\pi J(\mu^2)=\mathrm{Re}\,q(\mu^2+i\epsilon),$$

and

$$q(-p^2+i\epsilon)=-\frac{i}{\pi}(p^2+m^2)^2 \times\int_{4m^2}^{\infty} d\mu^2\,\frac{\mathrm{Re}\,q(\mu^2+i\epsilon)}{(\mu^2-m^2)^2(p^2+\mu^2-i\epsilon)}, \qquad \text{(A21)}$$

a twice-subtracted dispersion relation, whose subtraction terms have been determined.

APPENDIX B

(1) *Proof of (6.1)*. The difference between the two sides of (6.1) is

$$\sum_{k=1}^{3}\langle S^{*(k)}S_{xy}^{(4-k)}\rangle, \qquad \text{(B1)}$$

each term of which will be seen to vanish separately. Using (4.2), (4.19), and the second operator derivatives of (4.19) and (5.4), we construct the products in (B1), and delete from the result all terms of the form (4.6) with $m\neq n$. This wipes out the term $k=3$ in (B1). We perform the contractions using (4.6) with $n=m$. We will find several terms with the factor

$$\int d^4u\langle :a(u)^3:\ :a(x)a(y)a(z):\rangle = 6i\int d^4u\,\Delta_+(u-x)\Delta_+(u-y)\Delta_+(u-z) = (2\pi)^{-5}\int d^4p\,d^4q\,e^{ip\cdot(z-x)}e^{iq\cdot(z-y)} \times\delta_m^+(p)\delta_m^+(q)\delta_m^+(-p-q), \qquad \text{(B2)}$$

where

$$\delta_m^+(p)=\theta(p^0)\delta(p^2+m^2).$$

Now $\theta(p^0)\theta(q^0)\theta(-p^0-q^0)=0$ unless $p^0=q^0=0$, in which case $p^2=\vec{p}^{\,2}\geq 0$, and therefore $\delta(p^2+m^2)=0$, and (B2) vanishes. This conclusion also holds when $x=y$, by using (4.7). For the case $x=y=z$, we use the limit

$$\int d^4u\,d^4z\,\sigma(u)\sigma(z)\Delta_+(u-z)^3\to 0 \quad\text{as } \sigma(x)\to 1. \qquad \text{(B3)}$$

The surviving terms are contained in $\langle S^{*(2)}S_{xy}^{(2)}\rangle$. Of these, two will have the factor

$$\int d^4u\,d^4v\langle S_{uv}^{(2)}\rangle^*\,\Delta_+(u-x)\Delta_+(v-y)=0, \qquad \text{(B4)}$$

because of (4.12). Equation (B4) also holds when $x=y$. What survives is

$$\tfrac{1}{4}i\int d^4u\,d^4v\,\Delta_c^*(u-v) \times[\Delta_+(u-x)^2\Delta_+(v-y)^2 +2\Delta_+(u-x)\Delta_+(u-y)\Delta_+(v-x)\Delta_+(v-y)]. \qquad \text{(B5)}$$

The first term of (B5) vanishes, because after applying (4.7) to it, we find in the Fourier transform $\delta_\mu^+(p)\delta_\nu^+(-p)$, which vanishes by an argument similar to the one following (B2). The second term in (B5) vanishes by a similar but longer argument.

(2) *Proof of (6.3)–(6.5)*. Take the operator derivative of (4.19) and construct $\langle S_x^{*(2)}S_y^{(2)}\rangle$. Apply (4.6) and delete all terms which vanish for any of

the reasons discussed so far. This leaves

$$\langle S_x^{*(2)} S_y^{(2)}\rangle = \tfrac{1}{2}i \int d^4u\, d^4v\, \Delta_c^*(x-u)\Delta_c(y-v) \times [\Delta_+(u-v)^2\Delta_+(x-y) + 2\Delta_+(x-v)\Delta_+(u-y)\Delta_+(u-v)]\,. \tag{B6}$$

We apply (4.7) to the first term of (B6), and obtain

$$-\frac{1}{32\pi^2}\int_{4m^2}^{\infty} d\mu^2\, \frac{\rho(\mu^2, m^2)}{(\mu^2-m^2)^2}\Delta_+(x-y;\mu^2)\Delta_+(x-y;m^2)\,.$$

A second application of (4.7) gives us the term (6.4) in (6.3). The second term in (B6) is related to (4.7), but is rather more complicated. Here is an outline of its derivation. That term equals

$$(2\pi)^{-9}\int d^4p\, d^4q\, e^{ip\cdot(x-y)}\frac{\delta_m^+(p-q)}{q^2+m^2-i\epsilon}I(p,q)\,, \tag{B7}$$

where

$$I(p,q) = \int d^4r\, \delta_m^+(q-r)\delta_m^+(r)[(p-r)^2+m^2+i\epsilon]^{-1}\,.$$

In the reference frame in which $\vec{p}=0$, and setting $w=(\vec{r}^{\,2}+m^2)^{1/2}$,

$$I(p,q) = \frac{\pi}{4p_0}\int_0^\infty \frac{r^2dr}{w(w-\frac{1}{2}p_0)}\int_{-1}^{1}\frac{dz}{x}\delta(q_0-w-x)\,, \tag{B8}$$

where $x=[(\vec{q}-\vec{r})^2+m^2]^{1/2}$, and $\vec{q}\cdot\vec{r}=qrz$. Changing the variables of integration from r to w, and from z to x, we find that w is limited by $m\le w\le q_0$ and

$$[(q-r)^2+m^2]^{1/2}\le q-w\le[(q+r)^2+m^2]^{1/2}\,. \tag{B9}$$

After considerable manipulation, we obtain

$$m\le \tfrac{1}{2}q_0-\tfrac{1}{2}q\rho(\mu^2,m^2)\le w\le \tfrac{1}{2}q_0+\tfrac{1}{2}q\rho(\mu^2,m^2)$$

and

$$\mu^2 = q_0^{\,2}-\vec{q}^{\,2}\ge 4m^2\,,$$

where

$$\rho(\mu^2,m^2) = (1-4m^2/\mu^2)^{1/2}\,. \tag{B10}$$

Equation (B8) may then be written in spectral form as

$$\frac{\pi}{4qp^0}\int_{4m^2}^{\infty} d\mu^2\,\delta_\mu^+(q)\ln\left(\frac{q_0-p_0+q\rho(\mu^2,m^2)}{q_0-p_0-q\rho(\mu^2,m^2)}\right)\,. \tag{B11}$$

We now integrate over q, remembering to keep $\vec{p}=0$:

$$\int d^4q\,\delta_m^+(p-q)(q^2+m^2-i\epsilon)I(p,q) = \frac{\pi^2}{4p_0}\int_{4m^2}^{\infty}\frac{d\mu^2}{\mu^2-m^2}\int_m^{p_0}\frac{dE}{E'}\delta(p_0-E-E')\ln\left(\frac{E+q\rho(\mu^2,m^2)}{E-q\rho(\mu^2,m^2)}\right)\,, \tag{B12}$$

where $E=(\vec{q}^{\,2}+m^2)^{1/2}$ and $E'=(\vec{q}^{\,2}+\mu^2)^{1/2}$, and we have used the Fourier transform of (4.7). The limits on E come from $\delta_m^+(p-q)\delta_\mu^+(q)$. Now

$$(1/2E')\delta(p_0-E-E') = \theta(p_0-E)\delta(p_0^{\,2}+m^2-\mu^2-2p_0E)\,,$$

so the integral over E vanishes unless

$$p_0\ge(p_0^{\,2}+m^2-\mu^2)/2p_0\ge m>0$$

or, after some manipulation, $p^0\ge m+\mu>0$. Transforming back to $\vec{p}\ne 0$, (B12) may be written in spectral form as

$$\frac{\pi^2}{4}\int_{4m^2}^{\infty}d\mu^2\int_{(m+\mu)^2}^{\infty}d\nu^2 L(\mu^2,\nu^2)\delta_\nu^+(p)\,,$$

which is the contribution of (6.5) to (6.3).

(3) *Proof of (6.6)–(6.9)* Take (4.2) and the operator derivative of (5.4), and construct $\langle S_x^{*(1)}S_y^{(3)}\rangle$, deleting all terms which vanish for any reason mentioned so far. Then

$$\langle S_x^{*(1)}S_y^{(3)}\rangle = \tfrac{1}{2}i\int d^4u\,d^4v \times[\langle S_{yu}^{(2)}\rangle\Delta_c(u-v)\Delta_+(x-v)^2 + T(y,u,v)\Delta_+(x-u)\Delta_+(x-v) + 2\langle S_{uv}^{(2)}\rangle\Delta_c(y-u)\Delta_+(x-v)\Delta_+(x-y)]\,, \tag{B13}$$

where

$$T(y,u,v) = \Delta_c(y-u)\Delta_c(u-v)\Delta_c(v-y)\,.$$

Now the last term of (B13) vanishes, as can be seen from the Fourier transform, which will have the factor

$$(p^2+m^2-i\epsilon)^{-1}(p^2+m^2)^2\delta(p^2+m^2)=0\,.$$

Using (2.25) and (4.7), we see that the first term of (B13) is

$$-\tfrac{1}{4}i(4\pi)^{-4}\int_{4m^2}^{\infty}d\mu^2\,\rho(\mu^2,m^2)N(\mu^2)\Delta_+(x-y;\mu^2)\,,$$

which gives (6.7). The second term of (B13) involves the triangle graph, which is known to be finite in the ϕ^3 model,

$$T(y, u, v) = (2\pi)^{-12} \int d^4q\, d^4r\, e^{iq\cdot(u-y)}\, e^{ir\cdot(v-y)}\, t(q, -r),$$

where

$$t(q, r) = \int d^4p\, \delta_c(p)\delta_c(p-q)\delta_c(p-r) \tag{B14}$$

and

$$\delta_c(p) = (p^2 + m^2 - i\epsilon)^{-1}.$$

Then

$$\int d^4u d^4v\, T(y, u, v)\Delta_+(x-u)\Delta_+(x-v)$$
$$= -(2\pi)^{-10}\int d^4p d^4q\, e^{ip\cdot(x-y)}\, \delta_m^+(p-q)\delta_m^+(q)t(p, q). \tag{B15}$$

Since $q^2 = (p-q)^2 = -m^2$, t, being invariant, can depend only on p^2. That is, $t(p, q) = t(-p^2)$. Now we can integrate over q at once, according to (4.7). Equation (B15) then becomes

$$-(4\pi)^{-4}\int_{4m^2}^{\infty} d\mu^2 \rho(\mu^2, m^2)P(\mu^2)\Delta_+(x-y;\mu^2), \tag{B16}$$

where

$$P(\mu^2) = \frac{1}{2m^2}\int_0^1 \frac{dx}{z(x)} W(z). \tag{B17}$$

$W(z)$ is given in (6.9), and $z(x) = 1 - \mu^2 x(1-x)/m^2$. Equation (B17) is derived from (B14) by using the Feynman parametrization

$$(abc)^{-1} = \int_0^1 dx \int_0^1 2y dy \{[ax + b(1-x)]y + c(1-y)\}^{-3}.$$

The integrand of (B17) has a singularity only at those values of x that make $z(x) = 0$. But that singularity is logarithmic, and therefore the contribution from the neighborhood of it can be made vanishingly small. This can be seen by changing the variable of integration from x to z, and noting that the integrand is symmetric about $x = \frac{1}{2}$. Thus $P(\mu^2)$ is finite for all μ^2, and it can be shown that $P(\mu^2) \to 0$ as $\mu^2 \to \infty$. A final change of variable, from z to $\nu^2 \equiv \mu^2/(1-z)$, gives us (6.8).

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