which diagonalize them are connected by unitary transformations. This feature is fundamental to both the calculation and the physical interpretation of the theory. The fact that this is no longer true in quantum field theory, as we have amply demonstrated, is a serious difficulty which has yet to be understood and overcome.

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[1]Note that these states are fundamentally different from the usual Glauber coherent states, which are of the form $W(g(\vec{p}))|\phi\rangle$.

[2]For a discussion of this problem see F. Coester and R. Haag, Phys. Rev. 117, 1137 (1960). For another possible approach see K. O. Friedrichs *et al.*, *Integration of Functionals* (New York University Institute of Mathematical Sciences, 1957).

[3]States somewhat like the boson and fermion states constructed here have been used before by Schiff and others. They, however, all work on a lattice space, and to my knowledge the continuum case has never been treated before. For examples of this lattice-space treatment, see L. I. Schiff, Phys. Rev. 92, 766 (1953); D. H. Holland, *ibid.* 98, 788 (1955).

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Quantum Corrections to Stress Tensors and Conformal Invariance*†

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Quantum corrections to the energy-momentum tensors of particles are calculated for these theories: (1) scalar electrodynamics, (2) spinor electrodynamics, and (3) scalar particles with $\lambda\varphi^4$ self-interaction. It is shown that source theory provides a much more satisfactory approach than conventional means. The primary result of the investigation is the clear establishment of the favored role of the so-called "conformal" stress tensor – that is, the stress tensor that in the zero-mass limit transforms covariantly under the action of the conformal group. In terms of this tensor, unsubtracted spectral forms for the modifications are written down. A heuristic "proof" that this should be generally possible is provided. It is argued that broken scale invariance does not affect this subtraction-free property, and this is confirmed by explicit calculation of the order-λ^2 modifications in the $\lambda\varphi^4$ theory.

I. INTRODUCTION

The energy-momentum tensor, or stress tensor, appears in classical and quantum theories as the local measure of the mechanical properties of a particle or field system: It represents the flux of energy and momentum. But this property does not uniquely serve to define the tensor[1]; even the requirement of symmetry in the tensor indices leaves available a wide class of tensors,[2] of which the canonical one of Belinfante[3] is only one. On the other hand, the dynamical significance of the stress tensor emerges when the coupling with gravity is considered. Einstein's classical theory of gravity provides, through the variational principle, a unique coupling of gravity with matter and energy through the conventional stress tensor[4,5] and the same form persists when the simplest quantum-mechanical realization of the gravitational field in terms of gravitons is made.[2,6,7] But certainly it is possible to obtain coupling through a different stress tensor if one is willing to complicate the form of the action.[2,5]

Our program here is to calculate the lowest-order modifications to the stress tensors of spin-0 and spin-$\frac{1}{2}$ particles in three theories: $\lambda\varphi^4$ interaction, scalar electrodynamics, and spinor electrodynamics. We will perform the computations both in terms of the conventional tensor couplings and in terms of the coupling with the tensor suggested by consideration of conformal invariance.[2,5] The transcendent virtues of the latter will become quite evident. In this respect this paper may be considered complementary to the recent one of Callan, Coleman, and Jackiw (CCJ).[5] However, the method of computation adopted here differs from that used in their paper, and is, we feel, more convenient and satisfactory: That is, we will base our calculation upon the foundation of Schwinger's source theory.[2,6-10] To compare the two methods, for the simple example of the $\lambda\varphi^4$ modification we shall indicate how the calculation is performed both in source theory and in conventional operator field theory. In doing so, as well as in the electrodynamic calculations, we will see that the situation is rather more pleasing than was envisaged by CCJ. Moreover, by explicit computation of the λ^2 effects in the case of the $\lambda\varphi^4$ interaction we will study the consequences (or more properly the lack thereof) of broken scale invariance on the improvement brought about by using the "conformal" coupling, and we will indicate why we suspect this breakdown is irrelevant to the contact-term considerations central to this paper. But if these theoretical delights do not satisfy a reader's palate, we advise him to turn elsewhere for sustenance, for we can only echo the masters in saying that there is no hope of ever finding experimental tests for the results herein discussed.[6,11]

II. STRESS TENSORS AND CONFORMAL INVARIANCE

In line with the comments above, we begin by studying the primitive coupling of scalar particles to gravity. By itself, the gravitational action is[12]

$$W_g=\int (dx)[T^{\mu\nu}(x)h_{\mu\nu}(x)+\mathcal{L}(h,\Gamma)(x)]\,, \tag{2.1}$$

valid for weak fields. Here $h_{\mu\nu}$ is the gravitational field, $\Gamma^{\mu}_{\alpha\beta}$ the corresponding Christoffel variable, and $T^{\mu\nu}$ the graviton source. The masslessness of the graviton demands the conservation statement

$$\partial_\mu T^{\mu\nu}=0. \tag{2.2}$$

Now, outside the particle source the (symmetrical) stress tensor of the spin-0 particle obeys the same equation; thus we introduce the primitive interaction in (2.1) by the replacement in that equation of

$$T^{\mu\nu}\to T^{\mu\nu}+t^{\mu\nu}. \tag{2.3}$$

For a noninteracting particle of zero spin the canonical stress tensor is

$$t^{\mu\nu}_{(0)}=\partial^\mu\varphi\partial^\nu\varphi+g^{\mu\nu}\mathcal{L}_{(0)}\,, \tag{2.4}$$

with

$$\mathcal{L}_{(0)}=-\tfrac{1}{2}(\partial_\lambda\varphi\partial^\lambda\varphi+m^2\varphi^2). \tag{2.5}$$

To construct the modifications to this primitive interaction, it is necessary to consider multiparticle exchange mechanisms.[9,10] Since the free propgation of particles presents no mysteries, this problem can be easily solved once the effective sources for the possible intermediate multiparticle states are obtained. An extended gravition source can, in its most simple aspect, produce particle-antiparticle pairs. From (2.1) and (2.4) it is an easy matter to identify the effective two-particle source. The interaction term in the action is

$$\int (dx)[\partial^\mu\varphi\partial^\nu\varphi+g^{\mu\nu}\tfrac{1}{2}(-\partial_\lambda\varphi\partial^\lambda\varphi-m^2\varphi^2)]h_{\mu\nu}\,, \tag{2.6}$$

which implies the effective pair source

$$\tfrac{1}{2}iK(x')K(x'')|_{\rm eff}=\int (dx)h_{\mu\nu}(x)\{\partial^\mu\delta(x-x')\partial^\nu\delta(x-x'')-\tfrac{1}{2}g^{\mu\nu}[m^2\delta(x-x')\delta(x-x'')+\partial_\lambda\delta(x-x')\partial^\lambda\delta(x-x'')]\}. \tag{2.7}$$

This possesses the Fourier transform

$$iK(p)K(p')|_{\rm eff}=\int (dx')(dx'')e^{-ipx'}e^{-ipx''}iK(x')K(x'')|_{\rm eff}=-h_{\mu\nu}(p+p')[(m^2-pp')g^{\mu\nu}+p^\mu p'^\nu+p'^\mu p^\nu]. \tag{2.8}$$

Since we will want to compute electrodynamic corrections, it is necessary to remember that photons also couple to gravity through

$$t^{\mu\nu}_{(1)}=F^{\mu\lambda}F^{\nu}{}_{\lambda}-\tfrac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}. \tag{2.9}$$

The explicit coupling terms in the action are

$$\begin{aligned}W=\int (dx)(dx')(dx'')h_{\mu\nu}(x)\big([\partial^\mu\delta(x-x')A^\lambda(x')-\partial^\lambda\delta(x-x')A^\mu(x')][\partial^\nu\delta(x-x'')A_\lambda(x'')-\partial_\lambda\delta(x-x'')A^\nu(x'')]\\ -\tfrac{1}{4}g^{\mu\nu}\{[\partial^\alpha\delta(x-x')A^\beta(x')-\partial^\beta\delta(x-x')A^\alpha(x')][\partial_\alpha\delta(x-x'')A_\beta(x'')-\partial_\beta\delta(x-x'')A_\alpha(x'')]\}\big),\end{aligned} \tag{2.10}$$

from which the two-photon effective source can be immediately inferred,

$$\begin{aligned}\tfrac{1}{2}iJ_\alpha(x')J_\beta(x'')|_{\rm eff}=\int(dx)h_{\mu\nu}(x)\{\partial^\mu\delta(x-x')\partial^\nu\delta(x-x'')g_{\alpha\beta}-\partial^\mu\delta(x-x')\partial_\alpha\delta(x-x'')\delta^\nu_\beta-\partial_\beta\delta(x-x')\partial^\nu\delta(x-x'')\delta^\mu_\alpha\\ +\partial^\lambda\delta(x-x')\partial_\lambda\delta(x-x'')\delta^\mu_\alpha\delta^\nu_\beta-\tfrac{1}{2}g^{\mu\nu}[\partial^\lambda\delta(x-x')\partial_\lambda\delta(x-x'')g_{\alpha\beta}-\partial_\beta\delta(x-x')\partial_\alpha\delta(x-x'')]\},\end{aligned} \tag{2.11}$$

or in Fourier space

$$\tfrac{1}{2}iJ_\alpha(k)J_\beta(k')|_{\rm eff}=h_{\mu\nu}(k+k')[-k^\mu k'^\nu g_{\alpha\beta}+k^\mu k'_\alpha\delta^\nu_\beta+k_\beta k'^\nu\delta^\mu_\alpha-kk'\delta^\mu_\alpha\delta^\nu_\beta-\tfrac{1}{2}g^{\mu\nu}(-kk'g_{\alpha\beta}+k_\beta k'_\alpha)]. \tag{2.12}$$

It is evident that these effective photon sources are conserved,

$$k^{\alpha} J_{\alpha}(k) J_{\beta}(k')|_{\mathrm{eff}} = 0, \tag{2.13}$$

and further (2.12) is symmetric under the interchange

$$k \leftrightarrow k', \quad \alpha \leftrightarrow \beta. \tag{2.14}$$

The quantity in square brackets in (2.12) should be symmetrized in μ and ν, but the symmetrical field $h_{\mu\nu}$ projects out the appropriate part anyway.

Now W_g possesses invariance under the gravitational gauge transformation

$$h_{\mu\nu} \to h_{\mu\nu} + (\partial_{\mu} \xi_{\nu} + \partial_{\nu} \xi_{\mu}), \tag{2.15}$$

and this is maintained by the particle and photon couplings since in source-free regions

$$\partial_{\mu} t^{\mu\nu} = 0. \tag{2.16}$$

Explicitly it is trivial to verify that (2.8) and (2.12) are left unchanged by (2.15). [Note that, effectively, terms in $J_{\alpha}(k) J_{\beta}(k')|_{\mathrm{eff}}$ proportional to k_{α} or k'_{β} are zero, since $ek = e'k' = 0$.] All these properties could be made manifest by use of the projection factors employed by Radkowski,[7] who has presented these scalar and vector couplings in great detail, but all that machinery seems hardly necessary for our limited considerations. Projection factors would seem useful only when one has more than one of them.

For spin $\frac{1}{2}$ we have the usual symmetric stress tensor

$$t^{\mu\nu}_{(1/2)} = \tfrac{1}{2} \psi \gamma^0 \frac{1}{2} \left(\gamma^{\mu} \frac{1}{i} \partial^{\nu} + \gamma^{\nu} \frac{1}{i} \partial^{\mu} \right) \psi + g^{\mu\nu} \mathfrak{L}_{(1/2)}, \tag{2.17}$$

with the free electronic Lagrangian

$$\mathfrak{L}_{(1/2)} = -\tfrac{1}{2} \psi \gamma^0 \left(\gamma^{\mu} \frac{1}{i} \partial_{\mu} + m \right) \psi. \tag{2.18}$$

We recognize that outside the source for the electrons

$$\partial_{\mu} t^{\mu\nu}_{(1/2)} = 0. \tag{2.19}$$

And so $t^{\mu\nu}_{(1/2)}$ is an appropriate effective source for gravitons, and generates the primitive electronic-gravitational interaction through its introduction in W_g [Eq. (2.1)]. Since for real electrons, $\mathfrak{L}_{(1/2)}$ numerically vanishes, we can read off the effective pair source from the interaction term in the action

$$\begin{aligned} W_{eg} &= \int (dx) \tfrac{1}{2} \psi \gamma^0 \gamma^{\mu} \frac{1}{i} \partial^{\nu} \psi h_{\mu\nu}(x) \\ &= \int (dx)(dx') \tfrac{1}{2} \psi(x) \gamma^0 \gamma^{\mu} h_{\mu\nu}(x) \frac{1}{i} \partial^{\nu} \delta(x - x') \psi(x'). \end{aligned} \tag{2.20}$$

That is,

$$i\eta(x)\eta(x')\gamma^0|_{\mathrm{eff}} = \tfrac{1}{2}[h_{\mu\nu}(x) + h_{\mu\nu}(x')]\gamma^{\mu} \frac{1}{i} \partial^{\nu} \delta(x - x') \tag{2.21}$$

(since $\psi\gamma^0\gamma^{\mu}\psi$ is antisymmetric) or

$$i\eta(p)\eta(p')\gamma^0|_{\mathrm{eff}} = \tfrac{1}{4} h_{\mu\nu}(p + p')[\gamma^{\mu}(p - p')^{\nu} + \gamma^{\nu}(p - p')^{\mu}]. \tag{2.22}$$

In the following sections, we will discover, as CCJ did, that there are certain advantages in using a different stress tensor for scalar particles, one that is suggested by considerations of the conformal group. So we turn to the consideration of this group. It is a 15 parameter SO(4, 2) group which contains the Poincaré group as a subgroup. The remaining transformations are the isotropic dilations, or conformal transformations in the narrow sense: Their infinitesimal action on the coordinates is

$$\delta x^{\mu} = \delta a\, x^{\mu} + \delta b_{\nu}(2x^{\mu} x^{\nu} - g^{\mu\nu} x^2). \tag{2.23}$$

Scale transformations [the $\delta a x^{\mu}$ terms in (2.23)] are included as a special case. For a reasonable class of theories, it can be shown that any Poincaré-invariant theory that also possesses scale invariance is invariant under the full conformal group (see Ref. 2, p. 226 and Ref. 5). But it is hardly our intention here to enter into a discussion of these abstruse matters. We merely point out that when scale invariance implies conformal invariance, it is possible to find a traceless stress tensor, in source-free regions, to which we as always restrict our attention. Then the response of the action to a conformal transformation is

$$-\int (dx) t^{\mu\nu} \tfrac{1}{2}(\partial_{\mu} \delta x_{\nu} + \partial_{\nu} \delta x_{\mu}) = -\tfrac{1}{2} \int (dx) t \delta\varphi(x), \tag{2.24}$$

where $\delta\varphi(x) = 2\delta a + 4\delta b_{\nu} x^{\nu}$, which vanishes as claimed. But now if we allow δa and δb_{ν} to be functions of position, the action responds as

$$\delta W = -\int (dx)[t^{\mu\nu} x_{\nu} \partial_{\mu} \delta a(x) + t^{\mu}{}_{\lambda}(2x^{\lambda} x^{\nu} - g^{\lambda\nu} x^2) \partial_{\mu} \delta b_{\nu}] \tag{2.25}$$

in which we see the appearance and conservation of the scale and conformal currents

$$c^{\mu} = t^{\mu\nu} x_{\nu}, \quad c^{\mu\nu} = t^{\mu}{}_{\lambda}(2x^{\lambda} x^{\nu} - g^{\lambda\nu} x^2), \tag{2.26}$$

in which conservation is again only made possible because

$$t = t^{\mu}{}_{\mu} = 0. \tag{2.27}$$

Such a traceless $t^{\mu\nu}$, although very nice for such considerations, is by no means the only stress tensor associated with the particle of interest. It does, however, have the additional attractive feature of transforming covariantly under conformal transformations,[5]

$$\delta_c t^{\mu\nu}(0)=0. \tag{2.28}$$

Let us now specialize to the spin-0 example. The theory discussed in the beginning of this section becomes scale-invariant in the limit as $m^2\to 0$. The stress tensor we used there [Eq. (2.4)], which we will now denote with a tilde, is neither traceless as $m^2\to 0$,

$$\tilde{t}_{(0)\mu}^{\ \mu}=-\partial^\mu\varphi\partial_\mu\varphi-2m^2\varphi^2, \tag{2.29}$$

nor does it transform in a covariant manner when the field is varied according to [Ref. 2, Eq.(3-7.71)]

$$\delta_c\varphi=\delta b_\mu(2x^\mu x^\nu-g^{\mu\nu}x^2)\partial_\nu\varphi+2\delta b_\mu x^\mu\varphi. \tag{2.30}$$

For when $m^2=0$,

$$\delta_c\tilde{t}_{(0)}^{\mu\nu}=2[\delta b^\nu(\partial^\mu\varphi)\varphi+\delta b^\mu(\partial^\nu\varphi)\varphi-g^{\mu\nu}\varphi\delta b_\lambda\partial^\lambda\varphi]. \tag{2.31}$$

But if we define a new stress tensor by adding to $\tilde{t}_{(0)}^{\mu\nu}$ a suitable identically conserved term,

$$t_{(0)}^{\mu\nu}=\tilde{t}_{(0)}^{\mu\nu}-\tfrac{1}{6}(\partial^\mu\partial^\nu-g^{\mu\nu})\varphi^2, \tag{2.32}$$

we find using the equation of motion,

$$t_{(0)\mu}^{\ \mu}=-m^2\varphi^2 \tag{2.33}$$

and

$$\delta_c t_{(0)}^{\mu\nu}=0 \quad\text{at}\quad m^2=0. \tag{2.34}$$

To use this tensor in practice we need, as before, the effective particle-pair source: In place of (2.8),

$$iK(p)K(p')|_{\rm eff}=-h_{\mu\nu}(p+p')[\tfrac{1}{3}(m^2-pp')g^{\mu\nu}+\tfrac{2}{3}(p^\mu p'^\nu+p^\nu p'^\mu)-\tfrac{1}{3}(p^\mu p^\nu+p'^\mu p'^\nu)]. \tag{2.35}$$

III. THE CONFORMAL SCALAR THEORY-ELECTRODYNAMICS

We will now use the effective sources presented in Sec. III to calculate, to lowest nontrivial order, the graviton-particle vertex functions. A general schematic formulation of this problem has been presented by the author elsewhere,[13] but our work here will be completely self-contained. Since the primitive interaction is through the stress tensor, this computation provides a particularly simple and illuminating way to calculate the modification of this tensor.

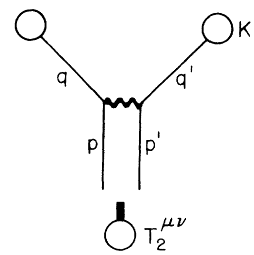

FIG. 1. Causal diagram giving particle contribution to stress-tensor modification in electrodynamics.

In this section, we will perform the calculation with the "conformal" stress tensor (2.32) providing the coupling to gravity. We begin by considering the amplitude for the causal process illustrated in Fig. 1. The picture corresponds to a process in which an extended graviton source $T_2^{\mu\nu}$ produces two real charged scalar particles, which subsequently scatter electromagnetically before they are detected by their sources K_1. (Annihilation is impossible by angular momentum conservation, of course.) The scattering is described by the vacuum-amplitude term

$$i\tfrac{1}{2}\int(dx)(dx')j_{12}^\mu(x)D_+(x-x')j_{12\mu}(x'), \tag{3.1}$$

where

$$j_{12}^\mu(x)=i\partial^\mu\varphi_1(x)e\underline{q}\varphi_2(x)-\varphi_1(x)e\underline{q}i\partial^\mu\varphi_2(x), \tag{3.2}$$

$\underline{q}$ being the charge matrix

$$\underline{q}=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \tag{3.3}$$

and the fields being now corresponding two-component objects. We relate φ_2 to its source:

$$\varphi_2(x)=i\int(dx')\Delta^{(+)}(x-x')K_2(x'). \tag{3.4}$$

Putting the pieces together and using the labeling of Fig. 1, the vacuum amplitude (3.1) is

$$-i\tfrac{1}{2}\int\frac{(dq)}{(2\pi)^4}\frac{(dq')}{(2\pi)^4}\varphi_1(-q')\int d\omega_p d\omega_{p'}e\underline{q}K_2(p')K_2(p)(-e\underline{q})(2\pi)^4\delta(p+p'-q-q')\frac{(p+q)(p'+q')}{(p-q)^2}\varphi_1(-q). \tag{3.5}$$

To obtain the desired process, we merely express the source product by its effective realization in (2.35). Thus the vacuum amplitude of interest is

$$-i\int\frac{(dq)}{(2\pi)^4}\frac{(dq')}{(2\pi)^4}\frac{1}{2}\varphi(-q)\varphi(-q')\Pi^{\mu\nu}h_{\mu\nu}(q+q'), \tag{3.6}$$

where

$$\Pi^{\mu\nu}=-ie^2\int d\omega_p d\omega_{p'}(2\pi)^4\delta(p+p'-q-q')\frac{(p+q)(p'+q')}{(p-q)^2}\left[\tfrac{1}{3}(m^2-pp')g^{\mu\nu}+\tfrac{2}{3}(p^\mu p'^\nu+p^\nu p'^\mu)-\tfrac{1}{3}(p^\mu p^\nu+p'^\mu p'^\nu)\right]. \tag{3.7}$$

As we (implicitly) observed in Sec. II,

$$(q+q')^\nu\Pi_{\mu\nu}=0, \tag{3.8}$$

so the only possible tensor form for $\Pi^{\mu\nu}$, symmetric in q and q', is

$$\Pi^{\mu\nu}=\Pi_1(q-q')^\mu(q-q')^\nu+\Pi_2(Q^\mu Q^\nu-g^{\mu\nu}Q^2), \tag{3.9}$$

where $Q=q+q'$ is the injected momentum. To determine the two invariants Π_1 and Π_2 it is merely necessary to evaluate (3.7) in the rest frame of Q, where if

$$Q^2=-M^2, \tag{3.10}$$

$$\Pi^{\mu\nu}=ie^2\int\frac{d\Omega}{32\pi^2}\left[1-\left(\frac{2m}{M}\right)^2\right]^{1/2}\frac{2/\zeta^2+1+\cos\theta}{1-\cos\theta+\gamma}\left[\tfrac{1}{6}M^2g^{\mu\nu}+\tfrac{2}{3}(p^\mu p'^\nu+p^\nu p'^\mu)-\tfrac{1}{3}(p^\mu p^\nu+p'^\mu p'^\nu)\right], \tag{3.11}$$

where

$$\zeta^2=\frac{M^2-(2m)^2}{M^2},\quad \gamma=\frac{\mu^2}{2(\frac{1}{4}M^2-m^2)}. \tag{3.12}$$

A small photon mass μ has been introduced here to control the infrared problem in the usual way. A simple evaluation of (3.11) provides us with the results (as $\mu\to 0$),

$$\Pi_1=\frac{i\alpha}{4}\frac{1}{M^2}\frac{2(M^2-2m^2)}{(1-4m^2/M^2)^{1/2}}\left(3+\ln\frac{\mu^2}{M^2-4m^2}\right), \tag{3.13}$$

$$\Pi_2=-\frac{i\alpha}{4M^2}\left(1-\frac{4m^2}{M^2}\right)^{1/2}\times\left(2M^2-\frac{8}{3}m^2+\frac{2M^2}{3}\frac{M^2-2m^2}{M^2-4m^2}\ln\frac{\mu^2}{M^2-4m^2}\right), \tag{3.14}$$

which posses singularities, but not nonintegrable ones, at $M^2=4m^2$.

It remains only to effect the space-time generalization. The essential observation is that in (3.6),

$$\int\frac{(dq)}{(2\pi)^4}\frac{(dq')}{(2\pi)^4}e^{iqx}e^{iq'x'}\Pi(M^2)h_{\mu\nu}(Q)=\delta(x-x')\int(dX)\int\frac{(dQ)}{(2\pi)^4}e^{iQ(x-X)}\Pi(M^2)h_{\mu\nu}(X), \tag{3.15}$$

where the causal restriction $x^0>X^0$ is removed as usual,

$$\int\frac{(dQ)}{(2\pi)^4}e^{iQ(x-X)}\Pi(M^2)=\int\frac{d\omega_Q dM^2}{2\pi}e^{iQ(x-X)}\Pi(M^2)=\int\frac{dM^2}{2\pi i}\Delta_+(x-X,M^2)\Pi(M^2), \tag{3.16}$$

where the last form, by space-time uniformity, can be taken to apply generally, apart from contact terms. In addition, in generalizing, we want to realize gauge invariance of the modified action (3.6), so that the original action correctly expresses the gauge variance of the various fields; to which end, we can try making a gauge transformation thusly,

$$h_{\mu\nu}(Q)\to\frac{Q^\lambda}{M^2}\left[-Q_\lambda h_{\mu\nu}(Q)+Q_\nu h_{\mu\lambda}(Q)+Q_\mu h_{\nu\lambda}(Q)\right]=\frac{1}{M^2}Q^\lambda i\Gamma_{\mu\nu\lambda}(Q), \tag{3.17}$$

but even this is gauge-invariant only up to second derivatives; when

$$h_{\mu\nu}\to h_{\mu\nu}+\partial_\mu\xi_\nu+\partial_\nu\xi_\mu,\qquad \Gamma_{\mu\nu\lambda}(x)\to\Gamma_{\mu\nu\lambda}(x)+2\partial_\mu\partial_\nu\xi_\lambda(x). \tag{3.18}$$

But we can add a further gauge term,

$$h_{\mu\nu}(Q)\to\frac{i}{M^2}\left[Q^\lambda\Gamma_{\mu\nu\lambda}(Q)-Q_{\{\mu,}\Gamma_{\nu\}}(Q)\right]=\frac{R_{\mu\nu}(Q)}{M^2}, \tag{3.19}$$

where $\Gamma_\mu=\Gamma_{\mu\lambda}{}^\lambda=\partial_\mu h$ transforms according to

$$\Gamma_\mu(x)\to\Gamma_\mu(x)+2\partial_\mu\partial_\lambda\xi^\lambda(x), \tag{3.20}$$

so the resulting structure is completely invariant under gravitational gauge transformations, which is signaled by the appearance of the Riemann-Christoffel tensor.

So we write in place of (3.9)

$$\Pi^{\mu\nu}=i\Pi(M^2)\tfrac{1}{2}\left[-\tfrac{1}{3}Q^2g^{\mu\nu}+\tfrac{1}{3}(q+q')^\mu(q+q')^\nu-(q-q')^\mu(q-q')^\nu\right]+i\Lambda(M^2)(-Q^2g^{\mu\nu}+Q^\mu Q^\nu), \tag{3.21}$$

with

$$i\Pi=-2\Pi_1,\qquad i\Lambda=\Pi_2+\tfrac{1}{3}\Pi_1. \tag{3.22}$$

We can express our generalized result as a modification of the action,

$$\int (dx)(d\xi)h_{\mu\nu}(x)F_1^{\mu\nu}{}_{\lambda\sigma}(x-\xi)t_{(0)}^{\lambda\sigma}(\xi)$$
$$+\int (dx)(d\xi)h_{\mu\nu}(x)G_1(x-\xi)z_{(0)}^{\mu\nu}(\xi). \tag{3.23}$$

Here

$$z_{(0)}^{\mu\nu}(\xi)=(\partial^\mu\partial^\nu-g^{\mu\nu}\partial^2)\tfrac{1}{2}\varphi^2(\xi) \tag{3.24}$$

which is identically conserved. The two form factors have the representations

$$F_1^{\mu\nu}{}_{\lambda\sigma}(Q)=(-Q^2g^\mu_\lambda g^\nu_\sigma+Q_\sigma Q^\nu g^\mu_\lambda+Q_\lambda Q^\mu g^\nu_\sigma-Q_\sigma Q_\lambda g^{\mu\nu})$$
$$\times\frac{1}{2\pi}\int_{(2m)^2}^\infty \frac{dM^2}{M^2}\frac{\Pi(M^2)}{Q^2+M^2-i\epsilon}, \tag{3.25}$$

$$G_1(Q)=\frac{1}{2\pi}\int_{(2m)^2}^\infty dM^2\frac{\Lambda(M^2)}{Q^2+M^2-i\epsilon},$$

where explicitly now from (3.13) and (3.14)

$$\Pi(M^2)=-\frac{\alpha}{M^2}\left[1-\left(\frac{2m}{M}\right)^2\right]^{-1/2}$$
$$\times(M^2-2m^2)\left(3+\ln\frac{\mu^2}{M^2-4m^2}\right) \tag{3.26}$$

and

$$\Lambda(M^2)=-\frac{\alpha}{4}\left[1-\left(\frac{2m}{M}\right)^2\right]^{1/2}\left(-\frac{8}{3}\frac{m^2}{M^2}-\frac{4m^2}{M^2-4m^2}\right). \tag{3.27}$$

The threshold factors assure us that the lower limit of the integrals is indeed $4m^2$. Note the highly amusing feature that G_1, the form factor for the identically conserved $z_{(0)}^{\mu\nu}$, does not possess an infrared divergence as $\mu\to 0$; like the electric form factor of spin-$\frac{1}{2}$ quantum electrodynamics (QED), only $F_1^{\mu\nu}{}_{\lambda\sigma}$ depends on μ. Of course, only the term proportional to Q^2 survives in $F_1^{\mu\nu}{}_{\lambda\sigma}$ when we employ the gravitational Lorentz gauge, where

$$\partial_\mu(h^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}h)=\partial^2\xi^\nu=0.$$

For notational simplicity we restrict ourselves to such a gauge in all the following. Then our results may be expressed as a modification of the particle stress tensor,

$$t_{(0)}^{\mu\nu}\to t_{(0)}^{\mu\nu}+\delta_1 t_{(0)}^{\mu\nu}, \tag{3.28}$$

where

$$\delta_1 t_{(0)}^{\mu\nu}(x)=\int(d\xi)F_1(x-\xi)t_{(0)}^{\mu\nu}(\xi)+\int(d\xi)G_1(x-\xi)z_{(0)}^{\mu\nu}(\xi), \tag{3.29}$$

where now

$$F_1(Q^2)=\frac{-Q^2}{2\pi}\int_{(2m)^2}^\infty\frac{dM^2}{M^2}\frac{\Pi(M^2)}{Q^2+M^2-i\epsilon}. \tag{3.30}$$

It does not hurt to reemphasize that, properly considered, our results involve no contact terms, for the leading Q^2 in F_1 is demanded by the condition that the normalization of the primitive interaction through $t_{(0)}^{\mu\nu}$ not be disturbed when more elaborate processes are considered. However, as a new kind of term, $G_1z_{(0)}^{\mu\nu}$ is not so restricted *a priori*, and indeed is manifestly independent of gauge. Observe finally that the existence of the two spectral forms (3.25) is not in doubt.

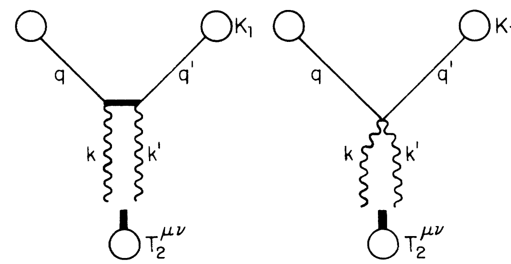

FIG. 2. Causal diagrams for photon contribution in scalar electrodynamics.

Now we turn to those processes where the photons couple to the graviton. The two specific processes are illustrated in Fig. 2, and correspond to the causal exchange of photons. The particle pair creation by the photons is described by the action [Ref. 2, Eq. (3-12.134)]

$$\tfrac{1}{2}\int(dx)(dx')\varphi_1(x)e\underline{q}2pA_2(x)\Delta_+(x-x')e\underline{q}2pA_2(x')\varphi_1(x')$$
$$-\tfrac{1}{2}\int(dx)\varphi_1(x)e^2[A_2(x)]^2\varphi_1(x). \tag{3.31}$$

The photon fields are related back to their sources, which in this context are given by (2.12). Again the resulting amplitude has the form (3.6) but this time (with tensors and invariants distinguished from the previous ones by primes)

$$\Pi'^{\mu\nu}=ie^2\int d\omega_k d\omega_{k'}(2\pi)^4\delta(k+k'-q-q')\frac{4}{(q-k)^2+m^2}$$
$$\times\{-k^\mu k'^\nu qq'+k^\mu k'q\,q'^\nu+k'^\nu kq'\,q^\mu-kk'\,q^\mu q'^\nu-\tfrac{1}{2}g^{\mu\nu}[-(kk')(qq')+(k'q)(kq')]\}$$
$$+ie^2\int d\omega_k d\omega_{k'}(2\pi)^4\delta(k+k'-q-q')(-2k^\mu k'^\nu+\tfrac{1}{2}kk'g^{\mu\nu})+(\mu\leftrightarrow\nu). \tag{3.32}$$

Since again $(q+q')^\mu \Pi'_{\mu\nu}=0$, which can also be easily directly verified, the form (3.9) holds here also,

$$\Pi'^{\mu\nu}=\Pi'_1(q-q)^\mu(q-q')^\nu+\Pi'_2(Q^\mu Q^\nu-g^{\mu\nu}Q^2). \tag{3.33}$$

It is convenient to express the results in terms of the variable defined in (3.12),

$$\zeta=(1-4m^2/M^2)^{1/2}. \tag{3.34}$$

Then a simple calculation in the rest frame of Q yields

$$\Pi'_1=\frac{i\alpha}{4\zeta^4}\left(3-5\zeta^2-\frac{3(1-\zeta^2)^2}{\zeta}\tanh^{-1}\zeta\right) \tag{3.35}$$

and

$$\Pi'_2=-\frac{i\alpha}{12}+\frac{i\alpha}{4\zeta^2}\left(-1+2\zeta^2+\frac{(1-\zeta^2)^2}{\zeta}\tanh^{-1}\zeta\right). \tag{3.36}$$

The various asymptotic limits are

$$\Pi'_1\sim i\alpha\frac{2}{\zeta}\ \text{ as }\ \zeta\to 0\ \ (M^2\to 4m^2), \tag{3.37}$$

$$\Pi'_1\sim -\frac{i\alpha}{2}\ \text{ as }\ \zeta\to 1\ \ (M^2\to\infty), \tag{3.38}$$

$$\Pi'_1\sim -\frac{3i\alpha}{8}\pi\frac{1}{i\zeta}\sim\frac{3\pi}{8}i\alpha\frac{M}{2m}\ \text{ as }\ \zeta\to i\infty\ \ (M^2\to 0), \tag{3.39}$$

on the one hand, and

$$\Pi'_2\sim 0\ \text{ as }\ \zeta\to 0, \tag{3.40}$$

$$\Pi'_2\sim\frac{i\alpha}{6}\ \text{ as }\ \zeta\to 1, \tag{3.41}$$

$$\Pi'_2\sim -\frac{\pi}{8}\alpha\zeta\sim -\frac{i\pi}{8}\alpha\frac{2m}{M}\ \text{ as }\ \zeta\to i\infty\ \ (M^2\to 0), \tag{3.42}$$

on the other.

These considerations are essential for properly performing the space-time generalizations, following the scheme of (3.16). Equation (3.38) tells us that a factor of $-Q^2/M^2$ will be required when the generalization is performed on that term–this is just the factor required by gauge invariance as we saw above. Then, as far as Π'_1 is concerned, (3.39) tells us that there is nothing to stop us from letting the M^2 integration range all the way down to zero, where there is an integrable singularity. This is the expected threshold behavior. The same cannot be done with Π'_2, but rather we must construct the combination that appears in Eq. (3.22), namely, $\Pi'_2+\frac{1}{3}\Pi'_1$, and that is very convenient: It requires no contact term for convergence, and hence from (3.42) the limit of the M^2 integration may also be extended down to zero. Then the corrections found here for the particle stress tensor are

$$\delta_2 t^{\mu\nu}_{(0)}(x)=\int(d\xi)G_2(x-\xi)z^{\mu\nu}_{(0)}(\xi)+\int(d\xi)F_2(x-\xi)t^{\mu\nu}_{(0)}(\xi), \tag{3.43}$$

where again $z^{\mu\nu}_{(0)}$ is given by (3.24), and $t^{\mu\nu}_{(0)}$ by (2.32). In (3.43),

$$F_2(Q^2)=-\frac{Q^2}{2\pi}\int_0^\infty\frac{dM^2}{M^2}\frac{\Pi'(M^2)}{Q^2+M^2-i\epsilon},\quad \Pi'=2i\Pi'_1 \tag{3.44}$$

while

$$G_2(Q^2)=\frac{1}{2\pi}\int_0^\infty dM^2\,\frac{\Lambda'(M^2)}{Q^2+M^2-i\epsilon},\quad \Lambda'=-i(\Pi'_2+\tfrac{1}{3}\Pi'_1). \tag{3.45}$$

In all, the complete modified stress tensor to first order in α is given by (3.29) and (3.43),

$$\bar{t}^{\mu\nu}_{(0)}=t^{\mu\nu}_{(0)}+\delta_1 t^{\mu\nu}_{(0)}+\delta_2 t^{\mu\nu}_{(0)}. \tag{3.46}$$

All of this is an eminently satisfactory result, particularly when it is noted that the conformal-invariance requirement remains satisfied: The trace of $t^{\mu\nu}_{(0)}$ is zero, $\delta_c t^{\mu\nu}_{(0)}=0$, and $\Lambda(M^2)$ and $\Lambda'(M^2)$ vanish when $m^2=0$, according to (3.27), (3.38), and (3.41). So indeed we see that the claims of Callan, Coleman, and Jackiw[5] have been verified. As Sec. IV will emphasize, and the reader will anticipate, the conformal tensor does play a privileged role, and it is really not hard to see why. For suppose we have a scalar conformally invariant theory, as the mass tends to zero. Then in terms of the conformal $t^{\mu\nu}_{(0)}$, the corrections to this tensor can always be written as (before space-time generalization)

$$\Pi t^{\mu\nu}_{(0)}+\Lambda z^{\mu\nu}_{(0)}. \tag{3.47}$$

This must become traceless as m^2 goes to zero; if Π then grows no worse than logarithmically, which is satisfied if the minimal use of contact terms is sufficient – that is, if F_1 requires no more than the one subtraction required by gauge invariance [compare (3.30)], then Λ must tend to zero. But the spectral weight function Λ is a function only of m^2/M^2 so it must vanish as $M^2\to\infty$. Therefore, the unsubtracted single spectral form will exist. This, in our view, is the CCJ theorem. (Of course, one must proceed with caution when the photon mass also appears, but as long as it occurs only inside logarithms it will not affect the argument.) [Naturally, this argument may be based equally well upon the covariance requirement, Eq. (2.28).]

These results turned out even better than one might have hoped. It had been suggested that perhaps the CCJ theorem was satisfied only because of a cancellation between the electronic and photonic parts, but instead both parts individually require nothing other than the contact terms necessary for gauge invariance. Each contribution is individually conformally invariant, and so the "proof" of the CCJ theorem sketched above applies to each alone.

IV. CONVENTIONAL COUPLING: SCALAR ELECTRODYNAMICS

To see clearly the virtue of the conformal theory presented in Sec. III, it is necessary to contrast it with the conventional theory. Indeed, when the same calculation is performed there, it is found that things are much less satisfactory. The change is in the effective particle pair source: In place of (2.35) we are to use (2.8), which makes an obvious change in (3.7). With reference to Eqs. (3.6) and (3.9), the results of the integration are now

$$\Pi_1 = \frac{i\alpha}{4M^2}\left[1-\left(\frac{2m}{M}\right)^2\right]^{-1/2} 2(M^2-2m^2)\left(3+\ln\frac{\mu^2}{M^2-4m^2}\right), \tag{4.1}$$

$$\Pi_2 = -\frac{i\alpha}{4M^2}\left[1-\left(\frac{2m}{M}\right)^2\right]^{1/2} \times\left(\tfrac{8}{3}(M^2-m^2)+\frac{2M^2(M^2-2m^2)}{M^2-4m^2}\ln\frac{\mu^2}{M^2-4m^2}\right). \tag{4.2}$$

Note that Π_1 is precisely the same as (3.13). Again, we want to group the terms so as to form the form factors for the stress tensor and the identically conserved tensor; this time of course the stress tensor is (2.4). With this in mind the result is (3.29), with

$$\Pi(M^2) = -\frac{\alpha}{M^2}\left[1-\left(\frac{2m}{M}\right)^2\right]^{-1/2} \times(M^2-2m^2)\left(3+\ln\frac{\mu^2}{M^2-4m^2}\right) \tag{4.3}$$

and

$$\Lambda(M^2) = -\tfrac{1}{2}\alpha\left[1-\left(\frac{2m}{M}\right)^2\right]^{1/2} \times\left(\frac{4}{3}\frac{M^2-m^2}{M^2}-\frac{3(M^2-2m^2)}{M^2-(2m)^2}\right), \tag{4.4}$$

in terms of which

$$F_1(Q^2) = -\frac{Q^2}{2\pi}\int_{(2m)^2}^{\infty}\frac{dM^2}{M^2}\frac{\Pi(M^2)}{Q^2+M^2-i\epsilon}, \tag{4.5}$$

$$G_1(Q^2) = -\frac{Q^2}{2\pi}\int_{(2m)^2}^{\infty}\frac{dM^2}{M^2}\frac{\Lambda(M^2)}{Q^2+M^2-i\epsilon}. \tag{4.6}$$

Now it has been necessary to insert a contact term into G_1 in order to make it converge, but this is hardly a very pleasing state of affairs. As before, G_1 does not possess an infrared divergence as $\mu \to 0$; but this now seems less significant, since both form factors require contact terms, so that any number of equivalent groupings of $\delta_1 t^{\mu\nu}$ is possible.

Perhaps even worse is the situation with regard to the photonic contributions now. For that calculation does not depend upon how the particles couple to gravity; and yet we have seen that the expression of the results seems to demand the introduction of the conformal tensor [cf. Eqs. (3.45)]. According to (3.38) and (3.41), any other grouping of the invariants Π_1' and Π_2' would require the introduction of a factor of $-Q^2/M^2$ to ensure convergence when space-time generalization was performed; but then it would be impossible to extrapolate the spectral mass down to zero [cf. (3.42)]. There is really no question but that (3.45) is correct. But this means, surely, that the conventional theory is completely inadequate.

V. SPIN-$\frac{1}{2}$ ELECTRODYNAMICS

The same sort of modifications occur for the electronic stress tensor, and the calculation proceeds in exactly similar fashion, but is, of course, somewhat more complicated. The aim is first to calculate the process shown in Fig. 1, for this case. The vacuum amplitude is still given by (3.1), but now the current is the electronic one,

$$j_\mu = \tfrac{1}{2}\psi\gamma^0 e\underline{q}\gamma_\mu\psi. \tag{5.1}$$

The causal relation of the incident electrons to their sources is

$$\psi_2(x) = i\int d\omega_p\, e^{ipx}(m-\gamma p)\eta_2(p) \tag{5.2}$$

and

$$\psi_2(x)\gamma^0 = i\int d\omega_{p'}\, e^{ip'x}\eta_2(p')\gamma^0(m+\gamma p'). \tag{5.3}$$

The separation into initial and final sources can be done in two ways for each current, so (3.1) corresponds to the term in the vacuum amplitude (V.A.),

$$-\tfrac{1}{2}\int \frac{(dq)}{(2\pi)^4}\frac{(dq')}{(2\pi)^4}\, d\omega_p d\omega_{p'}(2\pi)^4\delta(p+p'-q-q')\psi_1(-q)\gamma^0 e\underline{q}\gamma_\mu(m-\gamma p)\frac{\eta_2(p)\eta_2(p')\gamma^0}{(q-p)^2}(m+\gamma p')e\underline{q}\gamma^\mu\psi_1(-q'), \tag{5.4}$$

in which we are now to insert the effective sources (2.22). Thus

$$\text{V.A.}=\cdots+\tfrac{1}{8}i\int\frac{(dq)}{(2\pi)^4}\frac{(dq')}{(2\pi)^4}\psi_1(-q)\gamma^0\Pi^{\mu\nu}\psi_1(-q')h_{2\mu\nu}(q+q'), \tag{5.5}$$

where

$$\Pi^{\mu\nu}=ie^2\int d\omega_p d\omega_{p'}(2\pi)^4\delta(p+p'-q-q')\gamma_\rho(m-\gamma p)\frac{\gamma^\mu(p-p')^\nu+\gamma^\nu(p-p')^\mu}{(q-p)^2}(m+\gamma p')\gamma^\rho\ . \tag{5.6}$$

Note that $(q+q')_\mu\Pi^{\mu\nu}=0$, as was necessary by construction, and is required by gauge invariance, because of the projection factors $m-\gamma p$ and $m+\gamma p'$. Thus since the final electronic states are also real (so that γq on the left can be replaced by $-m$, $\gamma q'$ on the right by m), $\Pi^{\mu\nu}$ can be composed of only three tensors, compatible with the fact that $\gamma^0\Pi^{\mu\nu}$ is completely antisymmetric, in q and q', and in the spinor indices,

$$\begin{aligned}\Pi^{\mu\nu}=\;&\Pi_1[\gamma^\mu(q-q')^\nu+\gamma^\nu(q-q')^\mu]\\&+\Pi_2(q-q')^\mu(q-q')^\nu\\&+\Pi_3(Q^\mu Q^\nu-g^{\mu\nu}Q^2).\end{aligned} \tag{5.7}$$

In the rest frame of $Q=q+q'$ one has only to perform an angular integration as in the spin-0 case [cf. (3.11)], and after some straightforward if slightly tedious algebra one finds the invariants

$$\Pi_1(M^2)=\frac{-i\alpha}{4(1-4m^2/M^2)^{1/2}}\left[\frac{34}{3}-\frac{88}{3}\frac{m^2}{M^2}+4\left(1-\frac{2m^2}{M^2}\right)\ln\frac{\mu^2}{M^2-4m^2}\right], \tag{5.8}$$

$$\Pi_2(M^2)=-2i\alpha\frac{m}{M^2}\frac{1}{(1-4m^2/M^2)^{1/2}}\,\frac{2}{3}\ , \tag{5.9}$$

$$\Pi_3(M^2)=-2i\alpha\frac{m}{M^2}(1-4m^2/M^2)^{1/2}\,\tfrac{5}{3}\ . \tag{5.10}$$

Observe this time that only Π_1 requires a contact term, so as in Sec. III our naive suppositions are satisfied. We will say more about this below. For now we merely perform a space-time generalization without hesitation. With $z^{\mu\nu}_{(1/2)}$ and $y^{\mu\nu}_{(1/2)}$ defined by

$$y^{\mu\nu}_{(1/2)}=\tfrac{1}{4}(\psi\gamma^0\partial^\mu\partial^\nu\psi-\partial^\mu\psi\gamma^0\partial^\nu\psi) \tag{5.11}$$

and

$$z^{\mu\nu}_{(1/2)}=(\partial^\mu\partial^\nu-g^{\mu\nu}\partial^2)\tfrac{1}{8}\psi\gamma^0\psi, \tag{5.12}$$

and $t^{\mu\nu}_{(1/2)}$ being (2.17), of course, the resulting stress-tensor modification is

$$\begin{aligned}\delta_1 t^{\mu\nu}_{(1/2)}=&\int(d\xi)F_1(x-\xi)t^{\mu\nu}_{(1/2)}(\xi)+\int(d\xi)F_2(x-\xi)y^{\mu\nu}_{(1/2)}(\xi)\\&+\int(d\xi)F_3(x-\xi)z^{\mu\nu}_{(1/2)}(\xi),\end{aligned} \tag{5.13}$$

where

$$F_1(Q^2)=-\frac{Q^2}{2\pi i}\int_{(2m)^2}^\infty\frac{dM^2}{M^2}\frac{\Pi_1(M^2)}{Q^2+M^2-i\epsilon}\ , \tag{5.14}$$

$$F_2(Q^2)=-\frac{1}{2\pi i}\int_{(2m)^2}^\infty dM^2\,\frac{\Pi_2(M^2)}{Q^2+M^2-i\epsilon}\ , \tag{5.15}$$

$$F_3(Q^2)=-\frac{1}{2\pi i}\int_{(2m)^2}^\infty dM^2\,\frac{\Pi_3(M^2)}{Q^2+M^2-i\epsilon}\ . \tag{5.16}$$

It is quite remarkable that the photon mass appears in but one of the terms, that of the stress-tensor form factor. In fact, note that the dependence upon μ is the same in (5.14) and (3.30) and is the same as that of the electrodynamic form factor[9]: This is hardly surprising, since the same soft-photon processes are involved.

Now again we must turn to those processes where photons are exchanged. The one relevant to our contemplation is shown in Fig. 3 and has the same significance as the corresponding processes in the scalar "electron" case. The appropriate action term is [Ref. 2, Eq. (3-13.66)]

$$\tfrac{1}{2}\int(dx)(dx')\psi_1(x)\gamma^0 e\underline{q}\gamma A_2(x)\,G_+(x-x')e\underline{q}\gamma A_2(x')\,\psi_1(x'). \tag{5.17}$$

The effective photon source is still given by (2.12). The resulting vacuum amplitude can still find its expression in (5.5) but now

$$\begin{aligned}\Pi'^{\mu\nu}=4ie^2\int d\omega_k d\omega_{k'}(2\pi)^4\delta(k+k'-q-q')\frac{\gamma^\alpha[m-\gamma(q-k)]\gamma^\beta}{(q-k)^2+m^2}\\ \times[-k^\mu k'^\nu g_{\alpha\beta}+k^\mu k'_\alpha\delta^\nu_\beta+k_\beta k'^\nu\delta^\mu_\alpha-kk'\,\delta^\mu_\alpha\delta^\nu_\beta-\tfrac{1}{2}g^{\mu\nu}(-kk'g_{\alpha\beta}+k_\beta k'_\alpha)]+(\mu\longleftrightarrow\nu),\end{aligned} \tag{5.18}$$

which, since it is conserved, necessarily has the form of (5.7),

$$\Pi'^{\mu\nu}=\Pi_1'[\gamma^\mu(q-q')^\nu+\gamma^\nu(q-q')^\mu]+\Pi_2'(q-q')^\mu(q-q')^\nu+\Pi_3'(Q^\mu Q^\nu-g^{\mu\nu}Q^2). \tag{5.19}$$

The computation is even simpler than that for the invariants of Eq. (5.7). One finds, using the ζ variable of (3.34),

$$\Pi_1'=\frac{i\alpha}{\zeta^4}\left[1+\tfrac{1}{3}\zeta^2-\frac{(1-\zeta^2)^2}{\zeta}\tanh^{-1}\zeta\right], \tag{5.20}$$

which takes on the limiting forms

$$\Pi_1'\sim i\alpha\tfrac{4}{5}\quad\text{as}\quad\zeta\to 0\quad(M^2\to 4m^2), \tag{5.21}$$

$$\Pi_1'\sim i\alpha\tfrac{4}{3}\quad\text{as}\quad\zeta\to 1\quad(M^2\to\infty), \tag{5.22}$$

$$\Pi_1'\sim-\frac{\alpha}{\zeta}\frac{\pi}{2}\sim i\,\frac{\alpha\pi}{4}\frac{M}{m}\quad\text{as}\quad\zeta\to i\infty\quad(M^2\to 0); \tag{5.23}$$

while

$$\Pi_2'=\frac{i\alpha}{\zeta^6}\frac{1-\zeta^2}{4m}[10-\tfrac{26}{3}\zeta^2-(2/\zeta)(1-\zeta^2)(5-\zeta^2)\tanh^{-1}\zeta] \tag{5.24}$$

has the limits

$$\Pi_2'\sim\frac{i\alpha}{4m}\frac{32}{105}\quad\text{as}\quad\zeta\to 0, \tag{5.25}$$

$$\Pi_2'\sim\frac{i\alpha}{3m}(1-\zeta^2)\quad\text{as}\quad\zeta\to 1, \tag{5.26}$$

$$\Pi_2'\sim\frac{i\alpha}{4m}i\pi\frac{1}{\zeta}\sim\frac{i\alpha\pi}{8}\frac{M}{m}\quad\text{as}\quad M^2\to 0; \tag{5.27}$$

and finally

$$\Pi_3'=\frac{i\alpha}{4m}\frac{1-\zeta^2}{\zeta^4}[-2+\tfrac{10}{3}\zeta^2+(2/\zeta)(1-\zeta^2)^2\tanh^{-1}\zeta] \tag{5.28}$$

corresponding to which

$$\Pi_3'\sim\frac{i\alpha}{m}\frac{4}{15}\quad\text{as}\quad\zeta\to 0, \tag{5.29}$$

$$\Pi_3'\sim\frac{i\alpha}{3m}(1-\zeta^2)\quad\text{as}\quad\zeta\to 1, \tag{5.30}$$

$$\Pi_3'\sim-\frac{i\alpha}{4m}i\pi\zeta\sim\frac{i\alpha}{2}\pi\frac{1}{M}\quad\text{as}\quad M^2\to 0. \tag{5.31}$$

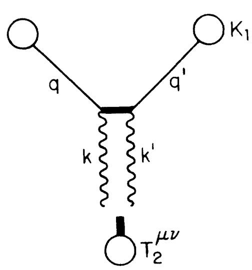

FIG. 3. Causal diagram for photonic contribution in spinor electrodynamics.

Equations (5.21), (5.25), and (5.29) tell us that none of the weight functions have singularities or zeros at $4m^2$, so we would expect the spectral integrals to begin at $M^2=0$. Indeed, only Π_1' will require a contact term for gauge invariance and convergence, and it is sufficiently well behaved at $M^2=0$ [Eq. (5.23)] that a nonintegrable singularity will not result in this way. We are fortunate that Π_3' does not require a contact term, for (5.31) shows then that disaster would befall at $M^2=0$. Everything works out here as well as it did in the conformal scalar case.

Thus our efforts are crowned with success: The complete order-α electronic–stress-tensor modification is

$$\bar{t}^{\mu\nu}_{(1/2)}=t^{\mu\nu}_{(1/2)}+\delta_1 t^{\mu\nu}_{(1/2)}+\delta_2 t^{\mu\nu}_{(1/2)}, \tag{5.32}$$

where $\delta_1 t^{\mu\nu}_{(1/2)}$ was given by (5.13) and $\delta_2 t^{\mu\nu}_{(1/2)}$ is

$$\delta_2 t^{\mu\nu}_{(1/2)}=\int(d\xi)G_1(x-\xi)t^{\mu\nu}_{(1/2)}(\xi)+\int(d\xi)G_2(x-\xi)y^{\mu\nu}_{(1/2)}(\xi)+\int(d\xi)G_3(x-\xi)z^{\mu\nu}_{(1/2)}(\xi), \tag{5.33}$$

in which

$$G_1(Q^2)=-\frac{Q^2}{2\pi i}\int_0^\infty\frac{dM^2}{M^2}\frac{\Pi_1'(M^2)}{Q^2+M^2-i\epsilon}, \tag{5.34}$$

$$G_2(Q^2)=-\frac{1}{2\pi i}\int_0^\infty dM^2\,\frac{\Pi_2'(M^2)}{Q^2+M^2-i\epsilon}, \tag{5.35}$$

and

$$G_3(Q^2)=-\frac{1}{2\pi i}\int_0^\infty dM^2\,\frac{\Pi_3'(M^2)}{Q^2+M^2-i\epsilon}. \tag{5.36}$$

Finally we consider the scaling limit of the spin-$\frac{1}{2}$ theory. The canonical tensor is already the one appropriate for studying conformal invariance, for with $m^2=0$, (2.27) and (2.28) hold. [The demonstration of the latter point merely involves the contrast of the symmetry of $\gamma^0\gamma^\mu$ with the antisymmetry of $\psi\psi$. A general formula for the conformal transformation of fields of arbitrary spin appears in Ref. 5.] Since $t_{(1/2)}{}^\mu{}_\mu$ is zero, only Π_2 and Π_3 can contribute in the zero-m limit, but (5.9) shows that these weight functions vanish there. Incidentally, note that if a factor of $-Q^2/M^2$ had been inserted in either F_2 or F_3 [Eqs. (5.15), (5.16)], these functions would not have gone to zero with m as $m\ln m$, but would have increased without bound as $1/m$. This provides potent evidence concerning the correctness of our subtraction procedure. The same remarks apply to the photonic contributions [see (5.26) and (5.30)].

VI. STRESS-TENSOR MODIFICATION IN $\lambda\varphi^4$ THEORY

This example is discussed in Ref. 5: Our purpose here, however, is to demonstrate how much more simple and transparent things are in source theory than in operator field theory where it is necessary to employ regulator masses and fields which themselves must couple to gravity. The system we are considering has the primitive Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}(\partial^\nu\varphi\partial_\nu\varphi + m^2\varphi^2) - \lambda\varphi^4\,, \tag{6.1}$$

so scattering is described by the purely local action term

$$-6\lambda\int(dx)[\varphi_1(x)]^2[\varphi_2(x)]^2\,. \tag{6.2}$$

We relate the incoming fields to their sources; the net effect is to replace in (3.5)

$$(e\underline{q})^2\frac{(p+q)(p'+q')}{(p-q)^2} \to 12\lambda\,. \tag{6.3}$$

If we use the conformal stress tensor for the interaction with gravity, (3.11) applies with the above replacement, or

$$\Pi^{\mu\nu} = -\frac{i3\lambda}{4\pi}\int_{-1}^{1} d(\cos\theta)(1-4m^2/M^2)^{1/2}[(M^2/6)g^{\mu\nu} + \tfrac{2}{3}(p^\mu p'^\nu + p^\nu p'^\mu) - \tfrac{1}{3}(p^\mu p^\nu + p'^\mu p'^\nu)]\,, \tag{6.4}$$

which leads to the conclusions for the invariants (3.9),

$$\Pi_1 = 0, \quad \Pi_2 = -\frac{i\lambda}{\pi}\left[1-\left(\frac{2m}{M}\right)^2\right]^{1/2}\frac{m^2}{M^2}\,. \tag{6.5}$$

According to (3.22) there is only the form-factor term corresponding to the identically conserved $z^{\mu\nu}_{(0)}$; and since conformal invariance is maintained, no contact term is required for convergence.

If, on the other hand, the conventional tensor is used, in place of (6.4) we have

$$\Pi^{\mu\nu} = -i12\lambda\int\frac{d\Omega}{32\pi^2}\left[1-\left(\frac{2m}{M}\right)^2\right]^{1/2}(\tfrac{1}{2}M^2g^{\mu\nu} + p'^\mu p^\nu + p'^\nu p^\mu)\,, \tag{6.6}$$

which implies

$$\Pi_1 = 0, \quad \Pi_2 = -i\frac{\lambda}{\pi}\left[1-\left(\frac{2m}{M}\right)^2\right]^{1/2}\left(\frac{1}{2}+\frac{m^2}{M^2}\right)\,. \tag{6.7}$$

Here again we are confronted with the unhappy necessity of inserting a contact term for the sole reason of making the resulting spectral integral convergent.

All of this is quite simple and unsurprising in view of the above. But we want now to use the great simplicity of this interaction to see what the situation is in operator field theory. We recover that case if we perform the space-time generalization process at an earlier stage: That is, if directly in (3.7) we make the formal substitution

$$id\omega_p\, id\omega_{p'} \to \frac{(dp)}{(2\pi)^4}\frac{(dp')}{(2\pi)^4}\frac{1}{p^2+m^2-i\epsilon}\,\frac{1}{p'^2+m^2-i\epsilon}\,. \tag{6.8}$$

But of course this is invalid, for the amplitude then ceases to exist. However, operator field theory has well-known methods of dealing with these divergence difficulties, which make use of counterterms and regulator masses. If we introduce corresponding regulator fields into the theory then, we will obtain for the conformal result here

$$\Pi^{\mu\nu} = -12i\lambda\int\frac{(dp)}{(2\pi)^4}\{g^{\mu\nu}[p(Q-p)-m^2] + p^\mu(Q-p)^\nu + p^\nu(Q-p)^\mu + \tfrac{1}{3}(Q^\mu Q^\nu - g^{\mu\nu}Q^2)\}$$
$$\times\{(p^2+m^2)[(p-Q)^2+m^2]\}^{-1} + \text{R.T.}, \tag{6.9}$$

which is precisely Eq. (4.13) of Ref. 5. (The R.T. are regulator terms.)

Superficially it would not appear that this result is conserved, for the internal particles are not real any longer, but in fact one can verify that it is proportional to $Q^\mu Q^\nu - g^{\mu\nu}Q^2$ by virtue of the regulator terms. In fact, one finds by using the parametric method of Schwinger and Feynman,

$$\Pi^{\mu\nu} = -24i\lambda(Q^\mu Q^\nu - g^{\mu\nu}Q^2)\int_0^1 dx[\tfrac{1}{6} - x(1-x)]\int\frac{(dp)}{(2\pi)^4}[p^2+m^2+Q^2x(1-x)]^{-2} + \text{R.T.}, \tag{6.10}$$

a formula which appears in part in the CCJ paper. It is an elementary matter to perform the momentum integration,

$$\frac{2\pi^2}{(2\pi)^4}\int_0^\infty \tfrac{1}{2}q^2dq^2\left(\frac{1}{[q^2+m^2+Q^2x(1-x)]^2} - \frac{c_1^2}{[q^2+M_1^2+Q^2x(1-x)]^2} - \frac{c_2^2}{[q^2+M_2^2+Q^2x(1-x)]^2}\right)$$
$$= \frac{1}{16\pi^2}\{\ln[Q^2x(1-x)+m^2] - c_1^2\ln[Q^2x(1-x)+M_1^2] - c_2^2\ln[Q^2x(1-x)+M_2^2]\}\,. \tag{6.11}$$

As $M_1^2, M_2^2 \to \infty$, the last two terms do not contribute due to the x integration, and we have

$$\Pi_2 = -\frac{24\lambda}{16\pi^2}\int_0^1 dx[\tfrac{1}{6} - x(1-x)]\ln\left(1+\frac{Q^2}{m^2}x(1-x)\right)$$
$$= \frac{\lambda}{4\pi^2}Q^2\int_0^1 dx\,\frac{(1-2x)^2}{Q^2+m^2/x(1-x)}, \tag{6.12}$$

or if we let

$$M^2 = m^2x^{-1}(1-x)^{-1},$$
$$\Pi_2 = \frac{\lambda}{2\pi^2}Q^2\int_{(2m)^2}^\infty \frac{m^2dM^2}{M^4}\left(1-\frac{4m^2}{M^2}\right)^{1/2}\frac{1}{Q^2+M^2}, \tag{6.13}$$

which differs from the result we found above only in that an additional factor of $-Q^2/M^2$ is inserted in the spectral form; that is, the source-theoretical answer would agree with the operator-field-theory one if a simple contact term were inserted,

$$\frac{1}{Q^2+M^2} \to \frac{1}{Q^2+M^2} - \frac{1}{M^2}\,. \tag{6.14}$$

But surely the unsubtracted form is the proper one to use, since a contact term is required by neither gauge nor convergence considerations. But we become sure that (6.13) is wrong when we observe that conformal invariance is lost. Here $\Pi_2 \sim m^2/m^2 \sim$const as $m^2 \to 0$ instead of going to zero as $m^2\ln m$ for the unsubtracted form, and so the trace of the stress tensor no longer vanishes in the conformal limit.

Why has operator field theory confused a situation that seemed so simple in source theory? Doubtless, it is better to proceed in a way that is always meaningful, rather than trying to attach meaning to divergent integrals. But, of course, the day can be saved for the operator field theorist, for it is only necessary to insert a correct counterterm at the vertex, namely,

$$\Pi_2^{\text{count}} = -\frac{\lambda}{\pi}\int_{(2m)^2}^\infty \frac{dM^2}{2\pi}\,\frac{m^2}{M^4}\left(1-\frac{4m^2}{M^2}\right)^{1/2}, \tag{6.15}$$

which just undoes the extra subtraction. But this is not very satisfying.

CCJ's form (6.13) unnecessarily weakens the impact of the improvement offered by the conformal tensor in another respect. If one proceeds in just the above manner, but with the conventional stress-tensor coupling, it is indeed necessary to employ a third regulator mass, but the result is just what one would expect; in (6.12) $\frac{1}{6} - x(1-x)$ is replaced by $-x(1-x)$ only. But then it is evident that in place of (6.13) we have

$$\Pi_2 = \frac{\lambda}{2\pi^2}Q^2\int_{(2m)^2}^\infty \frac{dM^2}{M^2}\left(1-\frac{4m^2}{M^2}\right)^{1/2}\left(\frac{m^2}{M^2}+\frac{1}{2}\right)\frac{1}{Q^2+M^2}, \tag{6.16}$$

which involves just the weight factor of (6.7) and the required contact term. Why (6.13) should be preferred to this result is not entirely clear.

Because of these apparent inadequacies in operator field theory, we shall not attempt to apply it to the other examples treated in this paper. It would appear that source theory provides a result more quickly and in a more perspicuous form, and hence should enable us to more quickly approach the physics of the problem. Perhaps, however, this discussion sheds some light on the sense in which operator field theory is related to source theory, which infelicitous relation has been considered in passing elsewhere.[9,14]

VII. IMPLICATIONS OF BROKEN SCALE INVARIANCE

Recently, various authors[15-17] have noted that in conventional operator perturbation theory, scale invariance is broken in higher orders, in particular, in second order for φ^4 theory. This would mean that as $m^2 \to 0$, $t \not\to 0$ [see (2.24)] and consequently there would be no reason to suspect the "conformal" tensor coupling would still lead to subtraction-free spectral forms. Here we will in fact confirm that for the φ^4 theory, to order λ^2, the improved situation does in fact still hold, and we will indicate why the general arguments are misleading.

For this simple theory, even the second-order modifications are quite easy to calculate. The causal processes still involve two-particle exchange, but the vertex and scattering amplitudes must be used to greater accuracy. And to avoid the problem of overlap, all that is necessary is to use the complex conjugate of the latter amplitude.[13] We will consider only the coupling with gravity through the "conformal" stress tensor, (2.32).

Of the three terms that arise in this manner, the one (a) involving the vertex correction at the graviton vertex is trivial: It arises merely from the coupling [see (3.23)]

$$i\int (dx)t^{(1)}_{\mu\nu}(x)h^{\mu\nu}(x) = i\int (dx)(d\xi)h_{\mu\nu}(x)G^{(1)}(x-\xi)z^{\mu\nu}(\xi), \tag{7.1}$$

in a slightly simplified notation (the superscript represents the order in λ), where $G^{(1)}$ has the form (3.25) with weight function [cf. (6.5)]

$$\Lambda^{(1)}(M^2) = -\frac{\lambda}{\pi}\left(1-\frac{4m^2}{M^2}\right)^{1/2}\frac{m^2}{M^2}\ . \tag{7.2}$$

Thus, replacing (2.35) we have the effective sources

$$iK(p)K(p')|_{\text{eff}} = -h_{\mu\nu}(p+p')G^{(1)}(Q)(Q^\mu Q^\nu - g^{\mu\nu}Q^2)\,. \tag{7.3}$$

Therefore, from the form of (6.4), we deduce

$$\Pi^{(2,a)} = 0\,,$$
$$\Lambda^{(2,a)} = \frac{3\lambda^2}{4\pi^3}m^2\left(1-\frac{4m^2}{M^2}\right)^{1/2}\int_{4m^2}^{\infty}\frac{dM'^2}{M'^2}\left(1-\frac{4m^2}{M'^2}\right)^{1/2}\frac{1}{M'^2-M^2-i\epsilon}\ . \tag{7.4}$$

For the other processes we need the order-λ^2 scattering amplitude. Under causal circumstances this involves two-particle exchange between the effective sources [see (6.2)],

$$iK(x)K(x')|_{\text{eff}} = -12\lambda\varphi^2(x)\delta(x-x')\,. \tag{7.5}$$

This leads to the V.A. expression

$$-9\frac{\lambda^2}{\pi}\left(1-\frac{4m^2}{M^2}\right)^{1/2}\int\frac{(dP)}{(2\pi)^4}\varphi_1{}^2(-P)\varphi_2{}^2(P) \to -\frac{9}{2}\frac{\lambda^2}{\pi}\int\frac{(dP)}{(2\pi)^4}\varphi^2(-P)\varphi^2(P)(-P^2)\int_{4m^2}^{\infty}\frac{dM^2}{2\pi i}\left(1-\frac{4m^2}{M^2}\right)^{1/2}\frac{1}{M^2}\frac{1}{P^2+M^2-i\epsilon}\,, \tag{7.6}$$

where space-time generalization was performed with the aid of a factor $(-P^2/M^2)$ to ensure normalization of the charge λ. Now the graviton effective particle source (2.35) is employed in (7.6) with the sign of ϵ reversed (this is the complex conjugation referred to above)[13] in order to deduce the remaining two contributions to $\Pi^{(2)\mu\nu}$.

The first employs (7.6) with $-P^2$ being the injected mass, so simply

$$\Pi^{(2,b)\mu\nu} = i\frac{9}{16}\frac{\lambda^2}{\pi^3}\int_{-1}^{1}d(\cos\theta)\left(1-\frac{4m^2}{M^2}\right)^{1/2}M^2$$
$$\times\int\frac{dM'^2}{M'^2}\left(1-\frac{4m^2}{M'^2}\right)^{1/2}\frac{1}{M'^2-M^2+i\epsilon}\left[\tfrac{1}{6}M^2g^{\mu\nu}+\tfrac{2}{3}(p^\mu p'^\nu+p^\nu p'^\mu)-\tfrac{1}{3}(p^\mu p^\nu+p'^\mu p'^\nu)\right], \tag{7.7}$$

which gives

$$\Pi^{(2,b)}=0\,,$$

$$\Lambda^{(2,b)}=\frac{3\lambda^2}{4\pi^3}\left(1-\frac{4m^2}{M^2}\right)^{1/2} m^2\int\frac{dM'^2}{M'^2}\left(1-\frac{4m^2}{M'^2}\right)^{1/2}\frac{1}{M'^2-M^2+i\epsilon}\,, \tag{7.8}$$

which is precisely the complex conjugate of (7.4). Thus we see [using (3.34)]

$$\Lambda^{(2,a)}+\Lambda^{(2,b)}=\frac{3\lambda^2}{2\pi^3}m^2\left(1-\frac{4m^2}{M^2}\right)^{1/2}P\int_{4m^2}^{\infty}\frac{dM'^2}{M'^2}\left(1-\frac{4m^2}{M'^2}\right)^{1/2}\frac{1}{M'^2-M^2}$$

$$=\frac{3\lambda^2}{\pi^3}\frac{m^2}{M^2}\zeta\left(1-\frac{1}{2}\zeta\ln\frac{1+\zeta}{1-\zeta}\right). \tag{7.9}$$

Evidently this vanishes with m^2/M^2; so there will be no question but that the unsubtracted spectral form of (3.25) will exist, with (7.9) as the weight function.

The final term (c) involves (7.6) again, but in the crossed channel. Thus [cf. (3.11)] we have

$$\Pi^{(2,c)}_{\mu\nu}=-i\frac{9}{8}\frac{\lambda^2}{\pi^3}\int_{-1}^{1}d(\cos\theta)\left(1-\frac{4m^2}{M^2}\right)^{1/2}$$

$$\times\int\frac{dM'^2}{M'^2}\left(1-\frac{4m^2}{M'^2}\right)^{1/2}\frac{1-\cos\theta}{1-\cos\theta+\frac{1}{2}M'^2(\frac{1}{4}M^2-m^2)^{-1}}\left[\tfrac{1}{6}M^2g_{\mu\nu}+\tfrac{2}{3}(p_\mu p'_\nu+p_\nu p'_\mu)-\tfrac{1}{3}(p_\mu p_\nu+p'_\mu p'_\nu)\right]. \tag{7.10}$$

It is most convenient to carry out the M'^2 integration first, using

$$v^2=1-4m^2/M'^2 \quad\text{and}\quad \zeta^2=1-4m^2/M^2\,. \tag{7.11}$$

Then we find

$$\int_{4m^2}^{\infty}\frac{dM'^2}{M'^2}\left(1-\frac{4m^2}{M'^2}\right)^{1/2}\frac{1-x}{1-x+\frac{1}{2}M'^2(\frac{1}{4}M^2-m^2)^{-1}}=2\left(-1+\tfrac{1}{2}\omega\ln\frac{\omega+1}{\omega-1}\right)\equiv f(x,\zeta^2)\,, \tag{7.12}$$

where

$$\omega^2=1+2(\zeta^{-2}-1)(1-x)^{-1}\,. \tag{7.13}$$

The asymptotic form of this function is

$$f(x,\zeta^2)\sim-\ln\frac{1-\zeta^2}{2(1-x)}\,,\quad \zeta\to1,\quad\text{or}\sim\ln[(1-x)M^2/2m^2],\quad M^2\to\infty\,. \tag{7.14}$$

The combinations that appear in (3.22) are the following:

$$\Pi^{(2,c)}=\frac{9}{4}\frac{\lambda^2}{\pi^3}\zeta^{-1}M^{-2}\int_{-1}^{1}dx\,f(x,\zeta^2)(\tfrac{1}{4}M^2-m^2)(1-3x^2)\,, \tag{7.15}$$

so

$$\Pi^{(2,c)}\sim\tfrac{3}{8}\lambda^2\pi^{-3}\quad\text{as}\quad M^2\to\infty\,, \tag{7.16}$$

and

$$\Lambda^{(2,c)}=-\tfrac{9}{8}\lambda^2\pi^{-3}\zeta M^{-2}\int_{-1}^{1}dx\,f(x,\zeta^2)\tfrac{2}{3}m^2\,, \tag{7.17}$$

so

$$\Lambda^{(2,c)}\sim0\quad\text{as}\quad M^2\to\infty\,. \tag{7.18}$$

Our complete results in order λ^2 are then

$$\Pi^{(2)}=\tfrac{9}{16}\lambda^2\pi^{-3}\zeta\int_{-1}^{1}dx\,f(x,\zeta^2)(1-3x^2) \tag{7.19}$$

and

$$\Lambda^{(2)}=3\lambda^2\pi^{-3}\frac{m^2}{M^2}\zeta\left(1-\tfrac{1}{2}\zeta\ln\frac{1+\zeta}{1-\zeta}-\tfrac{1}{4}\int_{-1}^{1}dx\,f(x,\zeta^2)\right). \tag{7.20}$$

Evidently subtractions will be necessary only with $\Pi^{(2)}$ and there only insofar as required by gauge invariance. The $z^{\mu\nu}$ form factor requires no contact term. And, consequently, $t\to0$ with m^2. Everything is just as satisfactory as in the lowest-order cases considered previously.

How can this be reconciled with the arguments of Wilson[15] and Callan[16]? The point to recognize is that we are not really considering a conformally invariant theory; only after forming the trace do we take the zero-mass limit. This was all that was required for the general arguments in Sec. III. The actual zero-mass case is, of course, infrared singular, and all our spectral functions involving Π [cf. (3.25) and (3.26)] diverge in the $m^2 \to 0$ limit. Presumably, it is because these divergences are only logarithmic that our demand that the trace have a smooth conformal limit is satisfied: $t \sim m^2$.

What these other authors set out to do, in effect, is to construct a consistent theory of massless particles interacting with themselves, which at least requires great care if one is to avoid replacing ultraviolet divergences with infrared ones. They accomplish this by introducing a parameter κ having dimensions of a mass, which in effect means that subtractions are to be performed at κ rather than at zero (although, in source theory, at least, this would not seem to be possible for the propagation function). Naturally this destroys scale invariance; but it will not affect our ultraviolet-convergence considerations, which are the point of this paper. But it is worth emphasizing that those subtractions required by gauge invariance can only be made at zero. And, as far as we have seen, no other contract terms are necessary or can be inserted.

Certainly, from our viewpoint there is considerable artificiality associated with the introduction of κ. Possibly, source theory provides an alternative, if trivial, way of making a massless theory consistent. For the $m=0$ case, $\Pi =$const. Then [cf. (3.30)]

$$F \sim Q^2 \int \frac{dM^2}{M^2} \frac{1}{M^2+Q^2-i\epsilon} = \int dM^2\left(\frac{1}{M^2}-\frac{1}{M^2+Q^2}\right). \tag{7.21}$$

Since there is no threshold singularity, it is suggested that the generalization results in letting M^2 range from $-\infty$ to $+\infty$, whence, if the singularities have the same structure, $F=0$. This "dynamical" result seems most sensible, for in the absence of dimensional parameters, the form factor $\bar{F}$ can be a function only of λ (and not of Q^2) and so by charge normalization is unity. (An identical argument applies to the propagator, only that mass normalization is the guiding principle there.) Such an interpretation of the zero-mass case is consistent with scale invariance, naturally, but is hardly useful or illuminating. It seems that however a massless theory is constructed, it is discontinuously related to the massive one, and hence irrelevant to the considerations of this paper.

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[1]See, for example, L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley, Reading, Mass., 1962).

[2]Julian Schwinger, *Particles, Sources and Fields* (Addison-Wesley, Reading, Mass., 1970).

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[11]R. P. Feynman, Acta Phys. Polon. 24, 697 (1963).

[12]Since we are doing source theory, fields and the like are not operators, but numerical quantities: commuting ones for boson fields and anticommuting ones in the case of fermions. The metric tensor used is

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and we will frequently employ the measure on the mass-μ hyperboloid,

$$d\omega_p^\mu = \frac{(d\vec{p})}{(2\pi)^3}\frac{1}{2p^0}, \quad p^0 = (\vec{p}^{\,2}+\mu^2)^{1/2}.$$

For other notation see, for example, Ref. 2.

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