[5]R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation*, edited by L. Witten (Wiley, New York, 1962), Chap. 7.

[6]A similar formulation of Einstein's equations, though not as complete in its discussion, can be found in P. A. M. Dirac, Proc. Roy. Soc. (London) A246, 333 (1958).

[7]We follow the notation of Schutz, Ref. 2. In particular, Greek indices run from 0 to 3, while Latin indices run from 1 to 3, with these exceptions: a and b label the velocity potentials and run from 1 to 5, while m and n

label the Dirac φ equations and run from 1 to 4.

[8]Actually, Dirac's proof (Ref. 4) is for a system with finitely many degrees of freedom. We have generalized it to fields using the results of the Appendix.

[9]B. F. Schutz, Jr., in Proceedings of the Pittsburgh Conference on Relativity (Springer, Berlin, to be published).

[10]See Appendix B of Schutz, Ref. 2.

[11]J. L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1966).

New Equation of Motion for Classical Charged Particles*

Tse Chin Mo and C. H. Papas

Electrical Engineering Department, California Institute of Technology, Pasadena, California 91109

(Received 15 March 1971)

With the intuitive new ideas that (1) in classical electrodynamics, radiation reaction should be expressible by the external field and the charge's kinematics, (2) a charge experiences, in addition to the Lorentz forces, another "small" external force $e_1 F^{\mu\lambda} \dot{u}_\lambda$ proportional to its acceleration, and (3) inertia plus radiation is balanced by these two external forces, we propose the new equation of motion,

$$m\dot{u}^\mu - (2e^3/3m) F_{\text{ext}}^{\lambda\alpha} \dot{u}_\lambda u_\alpha u^\mu = e F_{\text{ext}}^{\mu\lambda} u_\lambda + e_1 F_{\text{ext}}^{\mu\lambda} \dot{u}_\lambda \, ,$$

where mass conservation requires $e_1 = 2e^3/3m$. (The particle's spin is not considered in this work.) This equation for a classical charge is free from all the well-known difficulties of the Lorentz-Dirac equation. It conserves energy and momentum in a modified form in which the energy-momentum tensor contains a part $t^{\mu\nu}(x)$ made of a new field-charge interaction $\phi^\mu(x)$, in addition to the conventional "local" part made of $F_{\text{ret}}^{\mu\nu}(x)$ and $F_{\text{ext}}^{\mu\nu}(x)$ only, and therefore it no longer satisfies the conventional "local" conservation laws. It predicts correct radiation damping, as demonstrated here by applying it to various cases of basic physical importance. Also, it implies that a massless particle follows a null geodesic and cannot interact with the electromagnetic field whether it be charged or not; this implication may add a new degree of freedom to the charge-conservation law.

I. INTRODUCTION

The equation of motion of a charged particle has been a subject of interest for many years.[1] The equation now generally accepted was obtained by Dirac by decomposing the energy-momentum tensor of the retarded self-field into a sum that renormalizes mass and a difference that gives reaction.[2] An explanation and rederivation based on an absorber mechanism was provided by Wheeler and Feynman.[3] However, as is well recognized, the Lorentz-Dirac equation has certain inherent difficulties. First, it involves the derivative of the acceleration and hence needs one extra condition, in addition to the Newtonian initial conditions, to determine the motion. Second, it gives runaway solutions which can be avoided only by artificially presenting a preacceleration.[4] Third, in certain cases it implies that the external energy supplied to the particle goes only into kinetic energy, and

radiation is created from an "acceleration self-energy" which becomes more and more negative and is unphysical. It is the purpose of this work to obtain a new equation that is free from these difficulties, agrees with existing laboratory results, and predicts new phenomena that can distinguish the new equation from the old one and test its validity.

II. THE NEW EQUATION

By following the old idea of expressing the radiation reaction of a charged particle only by its kinematical quantities, it is not possible to construct an equation that includes reaction in a form simpler and more satisfactory than the Lorentz-Dirac one. However, in classical electrodynamics in an inertial frame[5] the only field that can accelerate a charged particle and make it radiate is the external electromagnetic field $F_{\text{ext}}^{\mu\nu}$. Accordingly, radiation reaction should be expressible by $F_{\text{ext}}^{\mu\nu}$
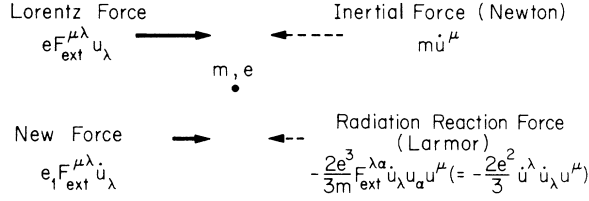
Lorentz Force

$eF^{\mu\lambda}_{ext} u_\lambda$ ——→   ←- - - - Inertial Force (Newton)

                              $m\dot{u}^\mu$

                    m, e
                      •

New Force ——→   ←- -   Radiation Reaction Force (Larmor)

$e_1 F^{\mu\lambda}_{ext} \dot{u}_\lambda$ $-\dfrac{2e^3}{3m} F^{\lambda\alpha}_{ext} \dot{u}_\lambda u_\alpha u^\mu (= -\dfrac{2e^2}{3}\dot{u}^\lambda \dot{u}_\lambda u^\mu)$

FIG. 1. The physical interpretation of the new equation.

and the particle kinematics. On the other hand, since a charge e at rest experiences only an electric force $e\vec{E}$ and in motion experiences an additional magnetic force $e\vec{v}\times\vec{B}$, which together make up $eF^{\mu\lambda}_{ext} u_\lambda$, it is natural to assume that when accelerating, a charge experiences still another aspect of the same external electromagnetic field – a new force $e_1 F^{\mu\lambda}_{ext}\dot{u}_\lambda$ with e_1 a small constant (i.e., $e_1 F^{\mu\lambda}_{ext}\dot{u}_\lambda \ll eF^{\mu\lambda}_{ext} u_\lambda$ in most physical cases) and $\dot{u}^\lambda \equiv du^\lambda/ds$.[6] We use geometrized units, $c=1$, $G=1$, $k=1$, with signature $\eta_{\mu\nu}=(1,-1,-1,-1)$,

$\{x^0, x^1, x^2, x^3\} \equiv \{t, x, y, z\}$, s the proper time and u^μ the four-velocity.

Now, given the motion $u^\mu(s)$ of a charge, the rate of radiated energy-momentum $(-2e^2/3)\dot{u}_\lambda\dot{u}^\lambda u^\mu$ is obtained by integrating Poynting's vector over the far-zone retarded sphere.[7] By using the radiation-neglected equation $m\dot{u}^\mu = eF^{\mu\lambda}_{ext} u_\lambda$, this rate can be expressed as $(-2e^3/3m)F^{\lambda\alpha}_{ext}\dot{u}_\lambda u_\alpha u^\mu$, which is roughly the expression for radiation in terms of $F^{\mu\nu}_{ext}$. Equating the inertia and radiation to the forces the charge sees through $F^{\mu\nu}_{ext}$, we have the new equation of motion (see Fig. 1)

$$m\dot{u}^\mu - (2e^3/3m)F^{\lambda\alpha}_{ext}\dot{u}_\lambda u_\alpha u^\mu = eF^{\mu\lambda}_{ext} u_\lambda + e_1 F^{\mu\lambda}_{ext}\dot{u}_\lambda,$$

(1)

where the requirement that (1) be an identity after scalar multiplication by u_μ implies that $e_1 \equiv 2e^3/3m$ is indeed a small constant. For a system of charges, in (1) for the ith charge $F^{\mu\nu}_{ext(i)}$ becomes $\sum_{j\neq i} F^{\mu\nu}_{ret(j)}$, where $F^{\mu\nu}_{ret(j)}$ is the retarded field of the jth charge.

Also, just for reference, the MKS 3-vector form of (1) reads

$$m\frac{d}{dt}\frac{\vec{v}}{(1-\beta^2)^{1/2}} + \frac{e^2}{6\pi\epsilon_0 c^3}\frac{\vec{v}}{(1-\beta^2)^3}\left[\left(\frac{d}{dt}\vec{\beta}\right)^2 - \left(\vec{\beta}\times\frac{d}{dt}\vec{\beta}\right)^2\right]$$

$$= e(\vec{E}+\vec{v}\times\vec{B}) + \frac{e^3}{6\pi m\epsilon_0 c^3}\left[\vec{E}\frac{d}{dt}\frac{1}{(1-\beta^2)^{1/2}} + \left(\frac{d}{dt}\frac{\vec{v}}{(1-\beta^2)^{1/2}}\right)\times\vec{B}\right],$$

(1')

where $\beta \equiv v/c$.

III. GENERAL PROPERTIES OF THE NEW EQUATION

(a) Mass conservation. Scalar multiplication by u_μ gives an identity and hence m is constant.

(b) Self-evident radiation term. Scalar multiplication by $\dot{u}_\mu$ gives

$$(-2e^3/3m)F^{\lambda\alpha}_{ext}\dot{u}_\lambda u_\alpha u^\mu = (-2e^2/3)\dot{u}_\lambda\dot{u}^\lambda u^\mu,$$

which always represents radiation. This justifies the second term on the left-hand side of (1) as radiation reaction with u^μ determined by (1).

(c) Newtonian motion. No more than the first derivative of velocity is involved and, accordingly, motion is determined by the initial velocity and position and by $F^{\mu\nu}_{ext}$.

(d) No runaway solutions (see below).

(e) No preacceleration.

(f) Additional "effective" external field. Taking the radiation term in (1) to the right-hand side and combining it with $e_1 F^{\mu\lambda}\dot{u}_\lambda$, we can interpret the total acceleration-and-external-field–dependent

force as obtained from an effective field,[8]

$$f^{\mu\nu} \equiv (2e^2/3m)\dot{u}_\lambda(F^{\mu\lambda}_{ext} u^\nu - F^{\nu\lambda}_{ext} u^\mu),$$

(2)

in addition to $F^{\mu\nu}_{ext}$, which the charge feels through the Lorentz force.

(g) Energy-momentum conservation. Since energy-momentum conservation and Maxwell's equations imply the equation of motion, (1) causes changes in at least one of them. For radiation to be of the Larmor form as in (b), the conventional Maxwell's equations must be retained. Thus the electromagnetic energy-momentum tensor T_{em} should be modified as

$$T^{\mu\nu}_{em} = T^{\mu\nu}_{tot} - \sum_i T^{\mu\nu}_{ret(i)} + t^{\mu\nu},$$

(3)

where $t^{\mu\nu}$ is defined to satisfy (see Appendix)

$$t^{\mu\nu}{}_{,\nu} = \frac{2e^2}{3m}\sum_{(i)}\dot{u}_{\nu(i)}(F^{\mu\nu}_{ext}u^\lambda_{(i)} - F^{\lambda\nu}_{ext}u^\mu_{(i)})J_{\lambda(i)} \equiv V^\mu$$

(4)

and the symmetry requirement. Here $T^{\mu\nu}_{tot}$ and $T^{\mu\nu}_{ret(i)}$ have just the familiar form of $F^{\mu\lambda}F^\nu{}_\lambda - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$, but with $F^{\mu\nu}_{tot} \equiv F^{\mu\nu}_{incident\ wave} + \sum_i F^{\mu\nu}_{ret(i)}$ and $F^{\mu\nu}_{ret(i)}$ inserted in them, respectively. The sig-

nificance of (3) and (4) is that by integrating $T^{\mu\nu}_{em,\nu}$ + $T^{\mu\nu}_{mass,\nu} = 0$ over the 3-space of the ith charge, then[9]

$$\int_{(i)} (T^{\mu\nu}_{em} + T^{\mu\nu}_{mass})_{,\nu} d^3x = 0 \Rightarrow \text{Eq. (1)}, \qquad (5)$$

where Maxwell's equations and $T^{\mu\nu}_{mass} \equiv \mu u^\mu u^\nu$ with proper mass density μ have been used, and no mass renormalization was involved.[10]

IV. SPECIAL CASES AND COMPARISON WITH THE LORENTZ-DIRAC EQUATION

Now we shall examine the implication of the new equation (1) for certain basic physical situations and compare the results with those[11] of the Lorentz-Dirac equation

$$m\dot{u}^\mu = eF^{\mu\nu}_{ext} u_\lambda + \tfrac{2}{3}e^2(\ddot{u}^\mu + \dot{u}_\lambda \dot{u}^\lambda u^\mu). \qquad (6)$$

(a) *No external field, $F^{\mu\nu}_{ext} = 0$.* Equation (1) directly gives $u^\mu = \text{const}$, but for (6) this solution has to be "physically" singled out from the infinity of runaway solutions.

(b) *Constant uniform electric field, $\vec{E} = \vec{e}_x E$.* The new equation (1) gives $u^\mu = (\cosh\eta, \sinh\eta, 0, 0)$, where $\eta \equiv C_1 + eEs/m$, for initial velocity $\vec{v} = \vec{e}_x \tanh C_1$; the Lorentz-Dirac equation (6) gives $u^\mu = (\cosh\xi, \sinh\xi, 0, 0)$, where $\xi \equiv K_1 + K_2 e^{s/\tau} + eEs/m$ and $\tau \equiv 2e^2/3m$, which with the "physical" requirement $\dot{u}^\mu = 0$ when $E = 0$ implies $K_2 \equiv 0$. Thus (1) yields the same solution as (6), but (1) works all by itself. Also, in this case, from (1) the radiation $(-2e^2/3)\dot{u}_\lambda \dot{u}^\lambda u^\mu = (2e^4E^2/3m^2)u^\mu$ is equal to the new external force $e_1 F^{\mu\lambda}_{ext}\dot{u}_\lambda$; but from the Lorentz-Dirac equation, (6), it is supplied by the negative "self-acceleration energy" term $(2e^2/3)\ddot{u}^\mu$ and the "puzzle" of radiating without radiation reaction,
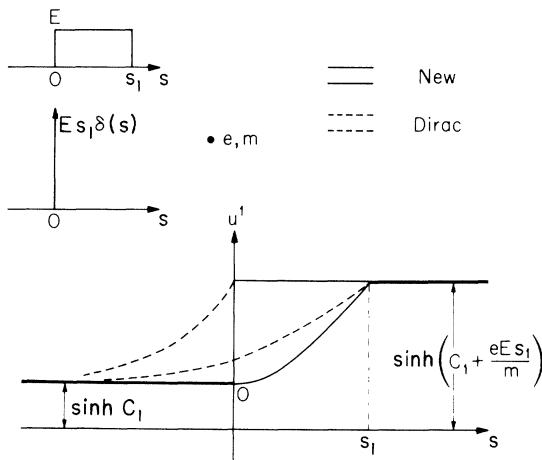
which really demonstrates another defect of Eq. (6), arises.[12]

(c) *Incident rectangular pulse $\vec{E} = \vec{e}_x E$ for $0 < s < s_1$.* Equation (1) gives

$$u^\mu_{(N)} = \begin{cases} (\cosh C_1, \sinh C_1, 0, 0), & s < 0 \\ (\cosh\psi, \sinh\psi, 0, 0), & 0 < s < s_1 \\ (\cosh\zeta, \sinh\zeta, 0, 0), & s_1 < s, \end{cases} \qquad (7)$$

where $\psi \equiv eEs/m + C_1$ and $\zeta \equiv eEs_1/m + C_1$. But (6), with the requirement $\dot{u}^i(\infty) = 0$, gives

$$u^\mu_{(LD)} = \begin{cases} (\cosh\alpha, \sinh\alpha, 0, 0), & s < 0 \\ (\cosh\sigma, \sinh\sigma, 0, 0), & 0 < s < s_1 \\ (\cosh\zeta, \sinh\zeta, 0, 0), & s_1 < s, \end{cases} \qquad (8)$$
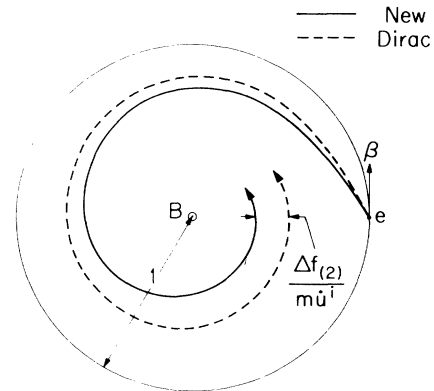
where

$$\alpha \equiv C_1 + (eE\tau/m)(1 - e^{-s_1/\tau}) e^{s/\tau}$$

and

$$\sigma \equiv C_1 + (eE\tau/m)(1 - e^{(s-s_1)/\tau}) + eEs/m.$$

Thus (8) represents preacceleration, whereas (7) shows that the electron does not respond until the pulse hits it. The limiting case of a δ pulse[2] is easily obtained by letting $s_1 \to 0$ and keeping Es_1 constant. For this limit (7) gives simply a step jump in velocity which is due only to idealizing the incident wave as a δ function, whereas (8) gives a purely preaccelerational motion (see Fig. 2).

FIG. 2. Responses of the new equation and the Lorentz-Dirac equation to rectangular and δ incident electric field.

$$\frac{\Delta f_{(2)}}{m\dot{u}^i} = \max\left[\frac{\beta^2}{(1-\beta^2)^2}\left(\frac{r_c}{r_1}\right)^2, \frac{\beta^4}{(1-\beta^2)^3}\left(\frac{r_c}{r_1}\right)^2\right]$$

$$\sim 10^{-8} \text{ for } e^- \text{ at } \sim 5 \text{ BeV, } r_1 \sim 1 \text{ meter}$$

$$\gtrsim 1 \text{ if } \frac{1}{1-\beta^2}\left(\frac{e^2}{m\epsilon_0 c^2}\right)\left(\frac{mc}{eB}\right)^{-1} \gtrsim 1,$$

e.g.: e^- at $\sim 10^n$ BeV, $B \sim 10^b$ gauss, $b + 2n \gtrsim 10$

FIG. 3. Motions in perpendicular magnetic field and their deviations.

(d) Motion perpendicular to uniform magnetic field $\vec{B} = \vec{e}_x B$. In this case exact analytic solutions cannot be found for (1) and (6), but a perturbation method can be used to obtain and compare their total correctional forces which spiral the circular orbit inward (see Fig. 3) as a result of synchrotron radiation.[13] Now the first-order corrections,

$$f^{\mu}_{(N1)} \equiv \frac{2e^3}{3m}\, \dot{u}_{\lambda(1)}(F^{\mu\lambda}_{\text{ext}} + F^{\lambda\alpha}_{\text{ext}}\, u_{\alpha(1)}u^{\mu}_{(1)}) = -\frac{2e^4B^2}{3m^2}\,\frac{\beta}{(1-\beta^2)^{1/2}}\left[\frac{\beta}{(1-\beta^2)^{1/2}}\, u^{\mu}_{(1)} + \left(0,0,\sin\frac{eBs}{m},\cos\frac{eBs}{m}\right)\right]$$

$$= \frac{2e^2}{3}\,(\ddot{u}^{\mu}_{(1)} + \dot{u}_{\lambda(1)}\dot{u}^{\lambda}_{(1)}u^{\mu}_{(1)}) \equiv f^{\mu}_{(\text{LD1})}\,, \tag{9}$$

are equal. Here $m\dot{u}^{\mu}_{(1)} \equiv eF^{\mu\lambda}_{\text{ext}}u_{\lambda(1)}$ and $u^{\mu}_{(1)}$ represent circular motion without radiation perturbation. The second-order corrections are

$$f^{\mu}_{(N2)} \equiv \frac{2e^3}{3m}\,\left[\dot{u}_{\lambda(2)}(F^{\mu\lambda}_{\text{ext}} + F^{\lambda\alpha}_{\text{ext}}\,u_{\alpha(1)}u^{\mu}_{(1)}) + \dot{u}_{\lambda(1)}F^{\lambda\alpha}_{\text{ext}}(u_{\alpha(1)}u^{\mu}_{(2)} + u_{\alpha(2)}u^{\mu}_{(1)})\right]$$

$$= \frac{2e^3}{3m}\,F^{\lambda\alpha}_{\text{ext}}\dot{u}_{\lambda(1)}u_{\alpha(1)}u^{\mu}_{(2)} + \frac{4}{9}\,\frac{e^7B^3}{m^4}\,\frac{\beta}{(1-\beta^2)^{3/2}}\,\frac{eBs}{m}\left[\frac{2\beta}{(1-\beta^2)^{1/2}}\,u^{\mu}_{(1)} + \left(0,0,\sin\frac{eBs}{m},\cos\frac{eBs}{m} + \frac{m}{eBs}\,\sin\frac{eBs}{m}\right)\right], \tag{10a}$$

$$f^{\mu}_{(\text{LD2})} \equiv \tfrac{2}{3}e^2(\ddot{u}^{\mu}_{(2)} + 2\dot{u}_{\lambda(1)}\dot{u}^{\lambda}_{(2)}u^{\mu}_{(1)} + \dot{u}_{\lambda(1)}\dot{u}^{\lambda}_{(1)}u^{\mu}_{(2)})$$

$$= \tfrac{2}{3}e^2\dot{u}_{\lambda(1)}\dot{u}^{\lambda}_{(1)}u^{\mu}_{(2)} + \frac{4}{9}\,\frac{e^7B^3}{m^4}\,\frac{\beta}{(1-\beta^2)^{3/2}}\,\frac{eBs}{m}\left[\frac{2\beta}{(1-\beta^2)^{1/2}}\left(1 + \frac{m}{2eBs}\,\sin\frac{2eBs}{m}\right)u^{\mu}_{(1)}\right.$$

$$\left. + \left(0,0,\sin\frac{eBs}{m} - \frac{2m}{eBs}\cos\frac{eBs}{m},\cos\frac{eBs}{m} + \frac{m}{eBs}\sin\frac{eBs}{m}\right)\right]. \tag{10b}$$

Here the second-order total solution $u^{\mu}_{(2t)} \equiv u^{\mu}_{(1)} + u^{\mu}_{(2)}$ which satisfies $m\dot{u}^{\mu}_{(2t)} = eF^{\mu\lambda}_{\text{ext}}u_{\lambda(2t)} + f^{\mu}_{(N1)}$ is the same for (1) and (6). Comparing the difference of second-order forces $\Delta f^{i}_{(2)} \equiv f^{i}_{(\text{LD2})} - f^{i}_{(N2)}$ and the first-order correction force $f^{i}_{(1)}$ to the main force $m\dot{u}^{i}$, we get

$\Delta f^{i}_{(2)}$	$f^{i}_{(1)}$	$m\dot{u}^{i}_{(1)}$		
$\dfrac{\beta^2}{(1-\beta^2)^2}\left(\dfrac{r_c}{r_1}\right)^2$	$\dfrac{\beta}{(1-\beta^2)^{1/2}}\left(\dfrac{r_c}{r_1}\right)$	1	for	$\dfrac{\beta^2}{1-\beta^2} \lesssim 1$
$\dfrac{\beta^4}{(1-\beta^2)^3}\left(\dfrac{r_c}{r_1}\right)^2$	$\dfrac{\beta^3}{(1-\beta^2)^{3/2}}\left(\dfrac{r_c}{r_1}\right)$	1	for	$\dfrac{\beta^2}{1-\beta^2} \gtrsim 1.$

$$\tag{11}$$

Here r_1 is the radius of the circular orbit for $u^{\mu}_{(1)}$ and r_c is the classical radius of the charged particle. Thus the new equation (1) predicts a faster inward spiraling than does Eq. (6) by the deviation $\Delta f^{i}_{(2)}/m\dot{u}^{i}_{(1)}$ compared to the main unperturbed orbit (Fig. 3). For a typical electron synchrotron of 5 BeV, $r_1 \sim 5$ m, $r_c = 2.8 \times 10^{-15}$ m, this deviation is 10^{-8} – far below the quantum fluctuation of synchrotron photon emission.[14] However, for highly energetic charged particles in a very strong electromagnetic field, as in astrophysical applications,[15] where $(1 - \beta^2)^{-1}(e^2/m\epsilon_0 c^2)(mc/eB)^{-1} \gtrsim 1$ (e.g., $b + 2n \gtrsim 10$ for electrons of energy 10^n BeV in a $B = 10^b$ G field) the deviation is large. In such strong fields the new equation (1) predicts orbits quite different than Eq. (6).

(e) Motion in Coulomb field $\vec{E} = (q/r^2)\vec{e}_r$. By a perturbation method as above, the first-order corrections are

$$f^{\mu}_{(N1)} = \frac{2m^2}{3q^2}\,\frac{\beta^8}{(1-\beta^2)^{5/2}}\left((1,0,0,0) - \frac{1}{(1-\beta^2)^{1/2}}\,u^{\mu}_{(1)}\right),$$

$$\tag{12}$$

$$f^{\mu}_{(\text{LD1})} = \frac{2m^2}{3q^2}\,\frac{\beta^7}{(1-\beta^2)^{5/2}}$$

$$\times \left((0,-\sin\Omega s,\cos\Omega s,0) - \frac{\beta}{(1-\beta^2)^{1/2}}\,u^{\mu}_{(1)}\right),$$

where $\Omega \equiv -m\beta^3/eq(1-\beta^2)$. It follows that $f^{i}_{(\text{LD1})}$ has more backward tangential correction than $f^{i}_{(N1)}$ and thus the Lorentz-Dirac orbit collapses faster than

the new orbit by (Fig. 4)

$$\frac{\Delta f^i_{(1)}}{m\dot{u}^i_{(1)}} \equiv \frac{(2e^3/3m)\dot{F}^{i\lambda}_{ext}u_{\lambda(1)}}{m\dot{u}^i_{(1)}} \sim \frac{e}{q}\frac{\beta^3}{(1-\beta^2)} \quad . \tag{13}$$
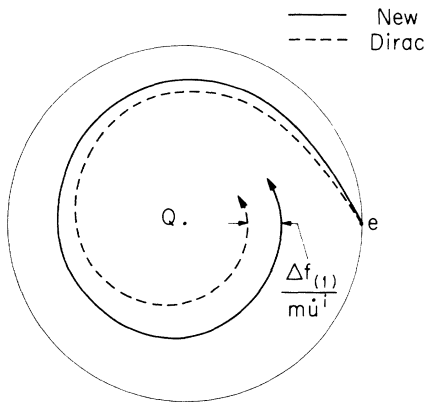
There are no experimental data on this deviation.

(f) Oscillating electric field, $\vec{E} = \vec{e}_x \cos\omega t$. (See Ref. 16.) For initial velocity zero, the new equation (1) gives exactly

$$u^\mu_{(N)} = (\{1 + [(eE/m\omega)\sin\omega t]^2\}^{1/2}, (eE/m\omega)\sin\omega t, 0, 0), \tag{14}$$

which shows no damping because of the continuous supply of energy from the oscillating fields. Also the motion (14) is the same as that obtained from the radiation-neglected Newton's equation $m\dot{u}^\mu = eF^{\mu\lambda}_{ext}u_\lambda$ because in this special case the radiation $(2e^4E^2/3m^2)\cos^2\omega t\, u^\mu$ is again equal to the additional external power-force $e_1F^{\mu\lambda}_{ext}\dot{u}_\lambda$. This result agrees with the usual Thomson scattering[17] and says the latter is exact up to the order of neglecting the magnetic force from the incident wave. For Eq. (6), a perturbation force

$$f^\mu_{(LD1)} = -\frac{2e^3E}{3m}\sin\omega t\, u^0_{(N)}(u^1_{(N)}, u^0_{(N)}, 0, 0) \tag{15}$$

shifts the oscillation phase forward and decreases the amplitude[18] which deviates the motion from $u^\mu_{(N)}$ when $\omega\tau \gtrsim 1$ ($\tau \sim 10^{-24}$ sec for e^-). But this cannot be checked by classical experimental methods because such highly energetic Compton scattering must be treated quantum electrodynamically.

$$\frac{\Delta f_{(1)}}{m\dot{u}^i} \equiv \frac{\frac{2e^3}{3m}\dot{F}^{i\lambda}_{ext}u_{\lambda(1)}}{m\dot{u}^i} \sim \frac{e}{Q}\frac{\beta^3}{(1-\beta^2)}$$

FIG. 4. Motions in a Coulomb field and their deviations.

V. CONCLUSION

The fact that (1) overcomes all former difficulties and predicts results not experimentally distinguishable from the Lorentz-Dirac equation in all laboratory cases of basic importance and the intuitive simplicity and soundness of the new ideas on which it is based lead us to suggest that the new equation (1) correctly accounts for radiation reaction in the motion of classical charged particles and should replace the celebrated Lorentz-Dirac equation of motion (6).

The new equation can manifest itself by predicting different motions and thus different radiation rates and patterns for highly energetic charges in very strong electromagnetic fields, e.g., as in astrophysical cases for electrons with 10^n BeV in 10^b -G fields such that $b + 2n \gtrsim 10$. At present it does not seem trivial to find an action integral for (1). However, there is no valid action integral[19] that leads to (6).

Also it can be shown that for $m = 0$, Eq. (1) gives $\dot{u}^\mu = 0$ and $u_\lambda u^\lambda \equiv 0$ independent of $F^{\mu\nu}_{ext}$. Thus a massless particle follows a null geodesic and cannot interact with the electromagnetic field whether it be charged or not. This might add a new degree of freedom to the charge-conservation law. The additional force appearing in (1) alters the conventional interaction $-J_\mu A^\mu_{ext}$. Thus this work is a first step of a new way toward including radiation reaction in curved space-time[20] and may possibly lead to changes in quantum theory.

APPENDIX

We investigate here the possibility of defining a $t^{\mu\nu}$ by (4), the construction of such a $t^{\mu\nu}$, and the indeterminacy of it. Now, if

$$t^{\mu\nu}_{,\nu} = V^\mu(x), \tag{A1}$$

where $V^\mu(x)$ is an arbitrary vector field, then a nontrivial symmetric tensor field solution $t^{\mu\nu}$ always exists (of course not unique) without any integrability condition, and its particular integral is

$$t^{\mu\nu}_{(p)} = 2\phi^{(\mu,\nu)} - \eta^{\mu\nu}\phi^\lambda_{,\lambda} \quad , \tag{A2}$$

where the parenthesis means symmetrization and ϕ^μ satisfies the familiar equation

$$\Box\phi^\mu \equiv \phi^{\mu,\lambda}{}_{,\lambda} = V^\mu. \tag{A3}$$

The radiation and retardation conditions are included in the ϕ^μ. Equation (A1) in three dimensions is exactly the well-known stress equilibrium equation in continuum mechanics.

Thus, the general solution of (A1) is

$$t^{\mu\nu} = t^{\mu\nu}_{(p)} + \Delta^{\mu\nu}, \tag{A4}$$

where $\Delta^{\mu\nu}$ is any symmetric tensor satisfying $\Delta^{\mu\nu}{}_{,\nu} = 0$. One way of choosing such $\Delta^{\mu\nu}$ harmonics is by using (A2) with a ϕ^λ that satisfies (A3) with $V^\mu = 0$. Also the six degrees of indeterminacy of $t^{\mu\nu}$ are accounted for by these indeterminacies in $\Delta^{\mu\nu}$.

For the purpose of energy-momentum conservation (5), it is sufficient to let $\Delta^{\mu\nu} = 0$ and choose $t^{\mu\nu}$ as the particular integral of (A4), i.e.,

$$t^{\mu\nu} = 2\phi^{(\mu,\nu)} - \eta^{\mu\nu}\phi^\lambda{}_{,\lambda}, \tag{A5}$$

where

$$\phi^\mu(x) = \frac{1}{4\pi}\int d^4x' \frac{\delta(t' - t + |\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|} V^\mu(x') \tag{A6}$$

with $V^\mu(x)$ defined in (4). Thus the additional energy-momentum change $t^{\mu\nu}$ in (3), which is defined by (4) and enters energy-momentum conservation as (5), indeed exists and is explicitly expressed by the field and currents as (A5) and (A6). Of course, the modified conservation law (5) includes this new $t^{\mu\nu}(x)$, thus differs from the conventional conservation law which involves only $F^{\mu\nu}_{\text{ext}}(x)$ and $F^{\mu\nu}_{\text{ret}}(x)$, and therefore it no longer satisfies the conventional "local" conservation laws. But the modified new conservation law is still a local one in the sense that it is a pointwise differential relation.

---

*Based partly on Caltech Antenna Laboratory Report No. 53, 1970 (unpublished), and on the presentation by T. C. Mo at the 1971 American Physical Society Meeting in New York [Bull. Am. Phys. Soc. 16, 118 (1971)] and the presentation by C. H. Papas before Academy of Sciences, Yerevan, U.S.S.R., 20 September 1970 [Izvestiia Akad. Nauk Arm. SSR, Fiz. 5, 402 (1970)].

[1] H. Lorentz, *Theory of Electrons* (Dover, New York, 1915), 2nd ed.; G. Schott, *Electromagnetic Radiation* (Cambridge U. P., Cambridge, England, 1912); L. Landau and E. Lifshitz, *Classical Theory of Fields* (Addison-Wesley, Reading, Mass., 1962), 2nd ed., Secs. 75 and 76; F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, Mass., 1965); V. L. Ginzberg, Usp. Fiz. Nauk 98, 569 (1969) [Soviet Phys. Usp. 12, 565 (1970)]; S. A. Lebedeff, Phys. Rev. D 1, 1583 (1970).

[2] P. A. M. Dirac, Proc. Roy. Soc. (London) A167, 148 (1938).

[3] J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. 17, 157 (1945).

[4] See any text treating classical radiation for the pre-acceleration-included integral-equation form of the Lorentz-Dirac equation.

[5] For noninertial frames, see T. C. Mo, J. Math. Phys. 11, 2589 (1970).

[6] Here, of course, the existence of this new force $e_I F^{\mu\lambda}_{\text{ext}} \dot{u}_\lambda$ is an *ad hoc* postulation.

[7] A. Schild, J. Math. Anal. Appl. 1, 127 (1960).

[8] If preferred, one can express $f^{\mu\nu}$ by field quantities alone as $f^{\mu\nu} \equiv (e/m)(F^\mu_{\text{ext}\lambda} I^{\lambda\nu} - F^\nu_{\text{ext}\lambda} I^{\lambda\mu})$, where $I^{\mu\lambda} \equiv \frac{2}{3}e(\dot{u}^\mu u^\lambda - \dot{u}^\lambda u^\mu)$ or $I^{\mu\lambda} \equiv \frac{1}{2}F^{\mu\lambda}_{(-)}$ at the charge; and interpret $f^{\mu\nu}$ as the interaction of the external field with the self-difference radiation field.

[9] The mathematical detail is just similar to the familiar one; see, e.g., V. Fock, *The Theory of Space, Time and Gravitation* (Pergamon, New York, 1964), 2nd ed., Sec.

63; and J. L. Synge, *General Relativity* (Interscience, New York, 1960), p. 359.

[10] This conservation can be equivalently stated as Eq. (1) and Maxwell's equations $\Rightarrow (T^{\mu\nu}_{\text{em}} + T^{\mu\nu}_{\text{mass}})_{,\nu} = 0$.

[11] G. N. Plass, Rev. Mod. Phys. 33, 37 (1961); C. S. Shen, Phys. Rev. Letters 24, 410 (1970).

[12] For an attempted explanation of this "puzzle," see, e.g., T. Fulton and F. Rohrlich, Ann. Phys. (N.Y.) 9, 499 (1960); D. Leiter, Intern. J. Theor. Phys. 3, 387 (1970).

[13] J. Schwinger, Phys. Rev. 75, 1912 (1949); D. H. Tomboulian and P. L. Hartman, *ibid.* 102, 1423 (1956).

[14] M. Sands, Phys. Rev. 97, 470 (1955); T. Erber, Rev. Mod. Phys. 38, 626 (1966).

[15] H. Y. Chiu and V. Canuto, Astrophys. J. 153, 157(L) (1968); Phys. Rev. Letters 22, 415 (1969); J. E. Gunn and J. P. Ostriker, *ibid.* 22, 728 (1969); F. Occhionero and M. Demianski, *ibid.* 23, 1128 (1969).

[16] Suggested by R. P. Feynman and also by P. A. M. Dirac in a private communication for a basic test in the case of a time-dependent field.

[17] L. Landau and E. Lifshitz, Ref. 1, Sec. 78.

[18] G. N. Plass, Ref. 11, Eqs. (92)-(95).

[19] The procedure of adding $A^\mu_{(-)} J_\mu$, where $A^\mu_{(-)} \equiv \frac{1}{2}(A^\mu_{\text{ret}} - A^\mu_{\text{adv}})$ and $J^\mu \equiv \rho_{(0)} u^\mu$, in the Lagrangian integrand and varying u^μ with $A^\mu_{(-)}$ fixed, then finally evaluating $F_{(-)}$ at the charge to obtain the Lorentz-Dirac equations (6), is not a correct variation principle – see, for example, F. Rohrlich [Phys. Rev. Letters 12, 375 (1964)], since $A^\mu_{(-)}$ is a function of u^μ.

[20] Simply postulating d/ds of the new equation to be replaced by the invariant derivative D/DS in a curved space-time may exclude possible couplings of the Riemann tensor $R^{\mu\nu\alpha\beta}$ into the motion. Also for a work following Dirac's idea, see B. S. DeWitt and R. W. Brehme, Ann. Phys. (N.Y.) 9, 220 (1960).