$$(r^2-2mr+e^2+a^2)[p_\varphi{}^2+\sin^2\theta(r^2+a^2\cos^2\theta)\mu^2] = \sin^2\theta(\epsilon er+ap_\varphi)^2. \quad (14)$$

When the transformation is reversible, the energy-extraction process has its maximum possible efficiency. Repetition of an energy-extraction process with maximum possible efficiency results in conversion into energy of 50% of the mass of an extreme charged black hole and 29% of that of an extreme rotating black hole. Thus, black holes appear to be the "largest storehouse of energy in the universe."[14]

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Motion of Particles in Einstein's Relativistic Field Theory. III. Coordinate Conditions

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In two earlier papers the author developed an approximation procedure for finding the Lorentz-covariant equations of structure and motion of interacting particles (represented by singularities) in Einstein's theory of the nonsymmetric field and in Einstein's theory of the gravitational field. In the earlier papers the author worked exclusively in a specific set of coordinate systems, the harmonic coordinate systems. In this paper the author shows that the procedure developed in the earlier papers can be used without imposing any conditions on the coordinates except that the fundamental field be the Minkowski metric in the absence of particles. The author also shows that up to any finite order of approximation the use of harmonic coordinates does not reduce the set of invariantly distinct solutions to Einstein's field equations.

I. INTRODUCTION

In two earlier papers[1] the author developed an approximation procedure for finding the Lorentz-covariant equations of structure[2] and motion of interacting particles (represented by singularities) in Einstein's theory of the nonsymmetric field and, because it is a special case of that theory, in Einstein's theory of the gravitational field. The procedure was developed for the most general particles which could be represented by singularities in a perfectly isolated region of the space-time continuum. Such particles were called ideal particles. The terms "ideal particle" and "perfectly isolated region of the continuum" were defined in the earlier papers.

The approximation procedure developed by the author allows one to find the equations of structure and motion step by step with respect to the powers of a parameter κ which measures the "strength" of the singularities associated with each particle in the region being investigated. The approximation procedure for finding these equations is only valid at points which are relatively far from each particle, and one finds meaningful equations of motion only for particles which are not too near one another.

In the above-mentioned papers[1] the author worked almost exclusively in a specific set of coordinate systems – the harmonic coordinate systems. In this paper the author will show that the procedure developed in the previous papers can be used to find the equations of structure and motion for particles in any coordinate system where the fundamental field $g_{\mu\nu}$ – this is the field which describes the structure of the space-time continuum in Einstein's theory – can be expanded in a power series in κ and where this field approaches the Minkowski metric $\eta_{\mu\nu}$ as κ goes to zero. The author will also show in this paper that the use of harmonic coordinates over a region of the continuum does not reduce the set of invariantly distinct solutions to Einstein's field equations – at least up to any finite order of approximation.

Unless otherwise stated the notation used in this paper will be identical to that used in the two earlier papers, Papers I and II.[1] It will be assumed that the reader is familiar with these papers.

II. COORDINATES AND THE METHOD OF APPROXIMATION

If no coordinate conditions are imposed on the four-dimensional region of the continuum we are investigating except for the condition that $g_{\mu\nu}$ be expandable in a power series in κ of the form[3]

$$g_{\mu\nu}=\sum_k \kappa^k\, {}_{(k)}g_{\mu\nu}\,, \quad {}_{(0)}g_{\mu\nu}=\eta_{\mu\nu}\,, \tag{1}$$

then the field equations of Einstein's theory of the nonsymmetric field can be put into the form[4]

$$\begin{gathered}\Box^2 j_\mu = s_\mu\,, \quad j_\mu{}^{,\mu}=0\,,\\ \gamma^*_{[\mu\nu]}{}^{,\nu}=j_\mu\,, \quad \gamma^*_{[\mu\nu,\rho]}=0\,,\\ \Box^2\gamma_{(\mu\nu)}-\gamma_{(\mu\rho)}{}^{,\rho}_{,\nu}-\gamma_{(\nu\rho)}{}^{,\rho}_{,\mu}+\eta_{\mu\nu}\gamma_{(\rho\sigma)}{}^{,\rho\sigma}=t_{\mu\nu}\,,\end{gathered} \tag{2}$$

where

$$\begin{gathered}\gamma_{\mu\nu}=\eta_{\mu\rho}\eta_{\nu\sigma}\mathfrak{g}^{\rho\sigma}-\eta_{\mu\nu}\,,\\ \gamma_{(\mu\nu)}=\eta_{\mu\rho}\eta_{\nu\sigma}\mathfrak{g}^{(\rho\sigma)}-\eta_{\mu\nu}\,,\\ \gamma^*_{[\mu\nu]}=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathfrak{g}^{[\rho\sigma]}\,,\end{gathered} \tag{3}$$

and

$$\begin{gathered}s_\mu=-\tfrac{1}{3}\eta_{\mu\rho}\epsilon^{\rho\sigma\kappa\lambda}R^{N'}_{[\kappa\lambda,\sigma]}\,,\\ t_{\mu\nu}=-2(R^{N'}_{(\mu\nu)}-\tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}R^{N'}_{(\rho\sigma)}).\end{gathered} \tag{4}$$

The field $\mathfrak{g}^{\mu\nu}$ is the conventional contravariant tensor density[5] associated with the fundamental field $g_{\mu\nu}$, and $R^{N'}_{\mu\nu}$ is that part of the contracted curvature tensor $R_{\mu\nu}$ which is nonlinear in $\gamma_{\mu\nu}$. Equations (2) replace Eqs. (4.26) of Paper I where we had assumed harmonic coordinates in the region under investigation.

It should be noted that the expansion (1) can only be valid at points which are sufficiently "distant" from the world lines of particles. The order of magnitude of such "distances" is discussed in the earlier papers.[1] When investigating the physical consequences of Eqs. (2) it is to be understood that we are investigating fields only at points sufficiently far from the world lines of particles so that (1) can be considered valid.

Equations (2) can be solved at each order only if the conditions

$$s_\mu{}^{,\mu}=0\,, \tag{5a}$$

$$t_{(\mu\nu)}{}^{,\nu}=0 \tag{5b}$$

are satisfied at each order. It is immediately obvious from its definition in (4) that s_μ satisfies Eq. (5a). If we use the same arguments involving the Bianchi identities as were used in Paper I, we find that if the field equations (2) have been satisfied in all lower orders, then Eqs. (5b) are satisfied at any given order. Using Eqs. (2) and proceeding step by step, assuming such a procedure converges, we can determine the field $\gamma_{\mu\nu}$ – and through the use of (3) and the definition of $\mathfrak{g}^{\mu\nu}$, the field $g_{\mu\nu}$ – to any order of approximation desired.

We now want to discuss a specific method for solving Eqs. (2). First let us define a field ϵ^*_μ through the equations

$$\Box^2\epsilon^*_\mu=\gamma_{(\mu\rho)}{}^{,\rho}\,. \tag{6}$$

These equations always have a solution over a finite region of the continuum – remember we are only interested in solutions to (6) at "distances" from the world lines associated with particles so that (1) is valid. We can also assume without any loss in generality – at least as long as we are only interested in solutions to (2) up to an arbitrary but finite order of approximation – that ϵ^*_μ can be expanded in a power series in κ such that $\epsilon^*_\mu\to 0$ as $\kappa\to 0$. This follows from the fact that the right-hand side of (6) can be expanded in such a power series in κ.

If we make use of Eqs. (6), we see that the field equations (2) take the form

$$\begin{gathered}\Box^2 j_\mu=s_\mu\,, \quad j_\mu{}^{,\mu}=0\,,\\ \gamma^*_{[\mu\nu]}{}^{,\nu}=j_\mu\,, \quad \gamma^*_{[\mu\nu,\rho]}=0\,,\\ \Box^2\gamma^*_{(\mu\nu)}=t_{\mu\nu}\,, \quad \gamma^*_{(\mu\nu)}{}^{,\nu}=0\,,\end{gathered} \tag{7}$$

where[6]

$$\gamma_{(\mu\nu)} = \gamma^*_{(\mu\nu)} + \epsilon^*_{\mu,\,\nu} + \epsilon^*_{\nu,\,\mu} - \eta_{\mu\nu}\epsilon^{*\,,\,\rho}_{\rho} \,. \tag{8}$$

Equations (7)–(8) are identical in form to the field equations of Paper I, Eqs. (4.26), and can be analyzed in the same way. From (8), however, we see that at each order in κ the field to that order[7] ${}_{[k]}\gamma_{\mu\nu}$ is not completely restricted by the field equations in that an additional term $\delta\,{}_{[k]}\gamma_{\mu\nu}$ of the form

$$\delta\,{}_{[k]}\gamma_{\mu\nu} = \kappa^k \left({}_{(k)}\epsilon^*_{\mu,\,\nu} + {}_{(k)}\epsilon^*_{\nu,\,\mu} - \eta_{\mu\nu}\,{}_{(k)}\epsilon^{*\,,\,\rho}_{\rho}\right), \tag{9}$$

where ${}_{(k)}\epsilon^*_{\mu}$ is arbitrary, can always be added to the field. If, as in Paper I, one imposes the condition

$$\gamma_{(\mu\nu)}{}^{,\,\nu} = 0 \tag{10}$$

on the field equations – that is, one insists on using harmonic coordinates in the region under investigation – then ${}_{(k)}\epsilon^*_{\mu}$ is not arbitrary but must satisfy the equations

$$\Box^2\,{}_{(k)}\epsilon^*_{\mu} = 0 \,. \tag{11}$$

We shall now discuss the source of the arbitrariness we have just found in the solution to Eqs. (2). Because of conditions (1) which were imposed on the field $g_{\mu\nu}$ before arriving at the field equations (2), the field equations (2) are not covariant under the group of general coordinate transformations. They are however still covariant under the group of Lorentz transformations and under gauge transformations. By a gauge transformation we mean a transformation of the form

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu} \,, \tag{12}$$

where ϵ^{μ} is arbitrary except that it has a power-series expansion in κ such that $\epsilon^{\mu} \to 0$ as $\kappa \to 0$.[8]

Lorentz transformations are well known and will not be discussed further here. Let us, however, investigate some of the consequences of Eqs. (2) being covariant under gauge transformations. Let us assume we have a solution up to kth order[7] ${}_{[k]}\gamma_{\mu\nu}$ to the field equations (2). Let us then make a coordinate transformation to a new set of coordinates x'^{μ},

$$x'^{\mu} = x^{\mu} + \kappa^k\,{}_{(k)}\epsilon^{\mu} \,, \tag{13}$$

where ${}_{(k)}\epsilon^{\mu}$ is arbitrary. From (13) we find

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \eta^{\mu}_{\nu} + \kappa^k\,{}_{(k)}\epsilon^{\mu}{}_{,\,\nu} \tag{14}$$

and thus

$${}_{[k]}\gamma'_{\mu\nu} = {}_{[k]}\gamma_{\mu\nu} + \kappa^k\left({}_{(k)}\epsilon_{\mu,\,\nu} + {}_{(k)}\epsilon_{\nu,\,\mu} - \eta_{\mu\nu}\,{}_{(k)}\epsilon_{\rho}{}^{,\rho}\right) + O(\kappa^{k+1}) \,. \tag{15}$$

From the above and the fact that

$${}_{[k]}\gamma'_{\mu\nu}(x') = {}_{[k]}\gamma'_{\mu\nu}(x) + O(\kappa^{k+1}) \,, \tag{16}$$

we see that if we have solved the field equations (2) up to kth order, choosing a new solution up to kth order by adding a term $\delta\,{}_{[k]}\gamma_{\mu\nu}$ of the form

$$\delta\,{}_{[k]}\gamma_{\mu\nu} = \kappa^k\left({}_{(k)}\epsilon_{\mu,\,\nu} + {}_{(k)}\epsilon_{\nu,\,\mu} - \eta_{\mu\nu}\,{}_{(k)}\epsilon_{\rho}{}^{,\rho}\right) \tag{17}$$

to the original solution can be regarded as equivalent to keeping the original solution but investigating it in a new coordinate system related to the old system through a transformation of the form (13). We thus see – comparing (17) with (9) – that the arbitrariness in the solutions at each order found earlier associated with Eqs. (2) can be attributed to the freedom beyond a Lorentz transformation one has at each order, except the zeroth order, in the choice of a coordinate system.

Next we show that if one solves Eqs. (2) step by step with respect to powers of κ in any coordinate system in which (1) is valid, one can always introduce a coordinate system (primed) at each order so that

$${}_{[k]}\gamma'_{(\mu\nu)}{}^{,\,\nu} = O(\kappa^{k+1}) \,. \tag{18}$$

All one need do is, at each order after solving Eqs. (2), make a coordinate transformation of the form (13) choosing ${}_{(k)}\epsilon_{\mu}$ to satisfy the equations

$$\Box^2\,{}_{(k)}\epsilon_{\mu} = -{}_{(k)}\gamma_{(\mu\nu)}{}^{,\,\nu} \,. \tag{19}$$

This can always be done as Eqs. (19) always have a solution over a finite region of space-time.[9] In this manner one can step by step introduce a coordinate system so that the field $\gamma'_{(\mu\nu)}$ satisfies Eqs. (18) – at least to any finite order of approximation. The restriction of finite orders of approximation is made necessary because we have not proved the convergence of the infinite series of transformations involved if the procedure is carried to infinite order. To sum up, we have found that if one restricts oneself to solutions of Eqs. (2) satisfying (1), then the use of harmonic coordinates up to any finite order of approximation does not restrict the set of invariantly distinct solutions to the field equations (2). This of course justifies the use of harmonic coordinates in all practical applications of Eqs. (2).

III. INERTIAL SYSTEMS

It should be noted that the general solution to the homogeneous equations associated with Eqs. (2) – that is, to Eqs. (2) where we set s_{μ}, j_{μ}, and $t_{\mu\nu}$ equal to zero – will only differ from the general solution to the corresponding homogeneous equations in Paper I by a term $\delta\gamma^H_{\mu\nu}$ of the form

$$\delta\gamma^H_{\mu\nu} = \epsilon^{*H}_{\mu,\nu} + \epsilon^{*H}_{\nu,\mu} - \eta_{\mu\nu}\epsilon^{*H,\rho}_{\rho}\,, \qquad (20)$$

where ϵ^{*H}_{μ} is arbitrary except that it can be expanded in a power series in κ such that $\epsilon^{*H}_{\mu} \to 0$ as $\kappa \to 0$.[10] The form of the solutions to the homogeneous equations associated with (2) is important since one finds using the approximation procedure described in Paper I that the character of the entire solution to the field equations depends on the choice of functions one takes as solutions to the associated homogeneous equations.[11]

One result of the analysis of this paper is that one can always choose a coordinate system at each order so that the coordinate-dependent term (20) is absent from any chosen solution to the homogeneous equations associated with (2). This suggests that we define an inertial system within our approximation procedure as a coordinate system in which (1) is valid and in which the solutions to the homogeneous equations associated with (2) take the form given in Paper I through (5.16)–(5.26). Note that this is not the definition of an inertial system presented in Sec. V C of Paper I, but a generalization of that definition to include coordinate systems which are not necessarily harmonic.

Using this new definition of an inertial system one finds that the equations of structure and motion for particles in an inertial system will still take the general form given in (6.4)–(6.9) of Paper I. Also, the explicit low-order equations of structure and motion found in Paper II to hold in an inertial system, that is, Eqs. (1.51)–(1.58) and Eqs. (1.68)–(1.73), retain the same form under the new definition of an inertial system.

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[2]Various quantities which characterize the structure of a particle, such as mass, charge, spin, etc., will vary in time. The equations satisfied by these quantities will be known as equations of structure.

[3]A subscript (k) to the left of a field indicates order.

[4]Unless otherwise indicated all raising and lowering of indices will be performed with the Minkowski metric $\eta_{\mu\nu}$.

[5]The field $\mathfrak{g}^{\mu\nu}$ is defined through the equation $\mathfrak{g}^{\mu\nu} = (-g)^{1/2}g^{\mu\nu}$, where $g^{\mu\nu}$ is defined through $g_{\mu\rho}g^{\nu\rho} = g_{\rho\mu}\,g^{\rho\nu} = \delta^{\nu}_{\mu}$, and g denotes the determinant of $g_{\mu\nu}$.

[6]We assume that the quantities ϵ^*_μ can always be chosen so that the magnitudes of the components of $\epsilon^*_{\mu,\nu}$ are all much less than 1 over those portions of the continuum–sufficiently far from the world lines of particles–where we seek solutions to Einstein's field equations. That this assumption is reasonable follows from the smallness with respect to 1 of the magnitudes of the components of $\gamma_{(\mu\nu)}$ over such portions of the continuum. Note that from (6), $\Box^2\epsilon^*_{\mu,\nu} = \gamma_{(\mu\rho),\nu}{}^{,\rho}$.

[7]A subscript $[k]$ to the left of a field will mean that the field with this subscript is identical up to terms of order k to the field without the subscript.

[8]This implies ϵ^μ is "small." We shall restrict ϵ^μ to functions such that the magnitudes of the components of $\epsilon_{\mu,\nu}$ are all much less than 1. With this restriction transformations (12) are one-to-one and have a nonvanishing Jacobian.

[9]We assume that transformations (13) for which (19) is valid can always be chosen so that they are one-to-one and have a nonvanishing Jacobian over those portions of the continuum–sufficiently far from the world lines of particles–where we seek solutions to Einstein's field equations. That this is a reasonable assumption follows from the smallness with respect to 1 of the magnitudes of the components of $\kappa^k{}_{(k)}\gamma_{(\mu\nu)}$ over such portions of the continuum.

[10]We assume that the quantities ϵ^{*H}_μ can always be chosen so that the magnitudes of the components of $\epsilon^{*H}_{\mu,\nu}$ are all much less than 1 over those portions of the continuum–sufficiently far from the world lines of particles–where we seek solutions to Einstein's field equations. Arguments for the reasonableness of this assumption are entirely analogous to those given in Ref. 6 and involving ϵ^*_μ.

[11]Of course the equations of structure and motion associated with the solutions to the homogeneous equations are not identical to the equations of structure and motion which are associated with the solutions to the full field equations. In using the approximation procedure developed by the author, the functional forms of the solutions to the homogeneous equations are retained in the approximation procedure but the equations of structure and motion associated with these solutions are not retained and are not imposed upon the particles. See Sec. V B of Paper I.