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†Current address: Max Planck Institut für Physik und Astrophysik, 8 München 23, Föhringer Ring 6, West Germany.

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[16]See Ref. 15, Eq. (58.01).

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# Reversible Transformations of a Charged Black Hole*

Demetrios Christodoulou
*Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540*

and

Remo Ruffini
*Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540, and Institute for Advanced Study, Princeton, New Jersey 08540*

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A formula is derived for the mass of a black hole as a function of its "irreducible mass," its angular momentum, and its charge. It is shown that 50% of the mass of an extreme charged black hole can be converted into energy as contrasted with 29% for an extreme rotating black hole.

The mass $m$ of a rotating black hole can be increased and (Penrose[1]) decreased by the addition of a particle and so can its angular momentum $L$; but (Christodoulou[2]) there is no way whatsoever to decrease the irreducible mass $m_{ir}$ in the equation

$$E^2 - p^2 = m^2 = m_{ir}^2 + L^2/4m_{ir}^2 \tag{1}$$

for the mass of a black hole. The concept of reversible ($m_{ir}$ unchanged) and irreversible transformations ($m_{ir}$ increases), which was introduced and exploited by one of us to obtain this result, is extended here to the case where the object also has charge, to yield the following four conclusions:

(1) The rest mass of a black hole is given in

terms of its irreducible mass and its angular momentum $L$ and charge $e$ by the formula[3]

$$m^2 = (m_{\rm ir} + e^2/4m_{\rm ir})^2 + L^2/4m_{\rm ir}^{\;2}\,. \tag{2}$$

(2) Reversibility implies and demands zero separation between the "negative-root states" and "positive-root states" of the particle defined by a quadratic equation of the form

$$\alpha E^2 - 2\beta E + \gamma = 0\,, \tag{3}$$

a requirement which is met and can only be met at the horizon itself.

(3) There exists a one-to-one connection between (a) the irreducible mass (as defined here and previously exclusively through the theory of reversible and irreversible transformations), and (b) the proper surface area $S$ of the horizon (shown by Hawking[4] never to decrease),

$$S = 16\pi m_{\rm ir}^{\;2}\,. \tag{4}$$

(4) The innermost stable circular orbit is the simplest place for a black hole to hold a particle bound and ready for capture. This orbit lies just outside the horizon only when the black hole is an extreme Kerr-Newman[5] black hole in the sense

$$(L^2/m^2) + e^2 \;(= a^2 + e^2 \text{ in the notation of Kerr}) = m^2 \tag{5}$$

or parametrically

$$L = 2m_{\rm ir}^{\;2}\cos\chi\,, \quad e = \pm 2m_{\rm ir}(\sin\chi)^{1/2}\,, \quad m = 2^{1/2}m_{\rm ir}(1+\sin\chi)^{1/2} \tag{6}$$

($\chi$ has any value from 0 to $\frac{1}{2}\pi$). The binding energies of a particle in this most-bound stable orbit are given by the formula

$$\frac{E}{\mu} = \frac{(2a^2 - m^2)\lambda + a(\lambda^2 m^2 + 4a^2 - m^2)^{1/2}}{4a^2 - m^2}\,, \tag{7}$$

where $\lambda = \epsilon e/\mu m$ (cf. Fig. 1). The transformation on the black hole affected by capture of a particle from such an orbit becomes reversible only when the charge-to-mass ratios of the particle ($\epsilon/\mu$) and the black hole ($e/m$) attain the limits $|\epsilon/\mu| \to \infty$ and parameter $\chi \to 0$ ($e/m \to 0$, $L \to m^2$) such that $\epsilon e/m\mu \to -\infty$. The binding of the particle in this orbit is 100% of its rest mass.

In black-hole physics one has reversibility without reversibility. As compared to such frictional processes as a brick sliding on a pavement, or an accelerated charged particle radiating, or a freely falling deformed droplet of molasses reverting to sphericity, it is difficult to name an act more impressively lacking in reversibility than capture of a particle by a black hole. Lost beyond recall is not a part of the mass-energy of the moving system but all of it; and not only mass-energy but also identity. The resulting black hole, like the original black hole, is characterized by three "independent determinants," mass, charge, and angular momentum, and by nothing else; all particularities (anomalies in higher multipole moments; also baryon number, lepton number, and strangeness) are erased, according to all available indications.[6-11] To reverse the change in a black hole brought about by the addition of a particle with a given rest mass, charge, and angular momentum ($\mu$, $\epsilon$, $p_\varphi$) one does not and cannot cause the black hole to reexpel the particle. Nor is there any such thing as a particle with a negative rest mass that one can add to cancel the first addition. Add instead ($B$ in Fig. 1) a particle of the original rest mass $\mu$ but of charge $-\epsilon$ and angular momentum $-p_\varphi$. This addition restores the determinants $m$,

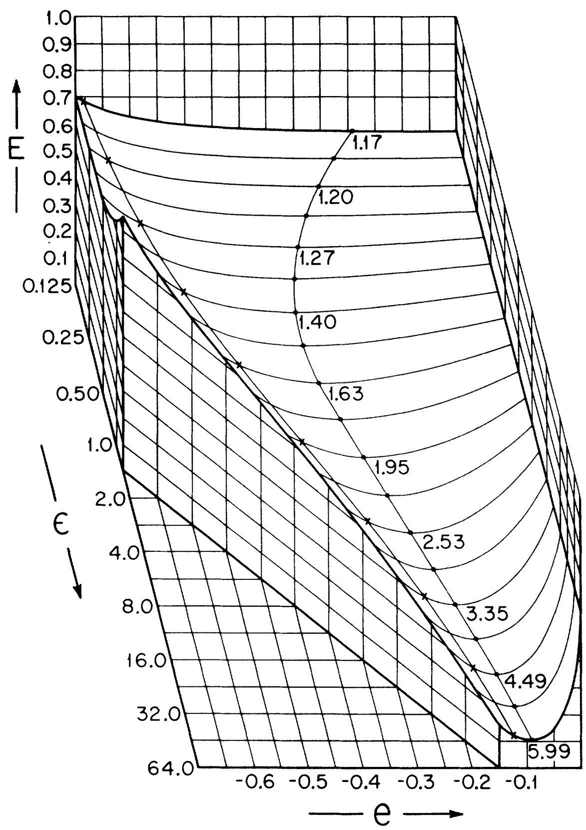


FIG. 1. Energy of a test particle of specified charge $\epsilon$, corresponding to the circular orbit touching the one-way membrane of a black hole of specified charge $e$ of the limiting configurations $a^2 + e^2 = m^2$, versus $\epsilon$ and $e$. Such orbits will be stable and will be the orbits of lowest energy if $E \le e(\mu/m)$ for $e\epsilon < 0$. The crossed curve is the boundary between stable and unstable orbits. The numbers on the energy minima correspond to the values of the angular momentum for the most-bound orbit.

$e$, $L$ of the black hole to their original value when and only when positive- and negative-root states have zero separation, a condition that is fulfilled only at the horizon itself, $r = r_{\text{horizon}} = r_+ = m + (m^2 - a^2 - e^2)^{1/2}$. At the horizon the positive- and negative-root surfaces $E = E_\pm(p_\varphi, \epsilon)$ meet at a "knife edge" (cf. Fig. 18 in Ruffini and Wheeler[12]) and the two "hyperbolas" of Fig. 2 degenerate to the "straight line" (acquires one more dimension and becomes a plane when the charge as well as the angular momentum of the test particle is taken into account),

$$E = \frac{ap_\varphi + e\epsilon r_+}{a^2 + r_+^2}\ . \tag{8}$$

Details follow.

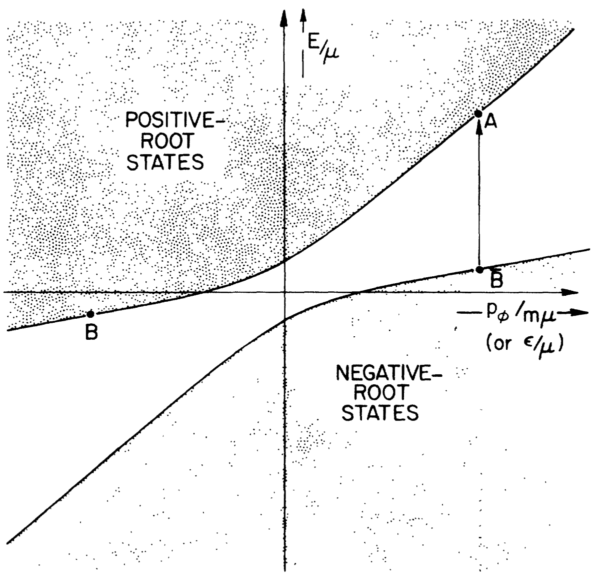


FIG. 2. Reversing the effect of having added to the black hole one particle ($A$) by the Penrose process of adding another particle ($B$) of the same rest mass but of opposite angular momentum and charge in a "positive-root negative-energy state." The diagram shows schematically the energy $E$ of the particle (measured in the Lorentz frame tangent at $r \to \infty$) as a function of the charge $\epsilon$ of the particle and its angular momentum $p_\varphi$ about the axis of the black hole (for simplicity one of these two dimensions has been suppressed in the frame). The particle under examination is in the equatorial plane of the black hole ($\theta = \frac{1}{2}\pi$) at a specified value of $r$. The energy lies on the indicated "positive-root" curve when $p_r$ and $p_\theta$ are zero, otherwise in the dotted region above the curve. Addition of $B$ is equivalent to subtraction of $\bar{B}$. Thus the combined effect of the capture of particles $A$ and $B$ is an increase in the mass of the black hole given by the vector $\bar{B}A$. This vector vanishes and reversibility is achieved when and only when the separation between positive-root states and negative-root states is zero [in this case the hyperbolas coalesce to the straight line given by Eq. (8)].

The Hamilton-Jacobi equation

$$g^{\alpha\beta}\left(\frac{\partial S}{\partial x^\alpha} + \epsilon A_\alpha\right)\left(\frac{\partial S}{\partial x^\beta} + \epsilon A_\beta\right) + \mu^2 = 0 \tag{9}$$

for the motion of the particle, separated and solved by Carter,[13] leads to the quadratic equation (3) for the energy with

$$\begin{aligned}
\alpha &= r^4 + a^2(r^2 + 2mr - e^2)\,,\\
\beta &= (2mr - e^2)ap_\varphi + \epsilon er(r^2 + a^2)\,,\\
\gamma &= -(r^2 - 2mr + e^2)p_\varphi^2 + 2\epsilon earp_\varphi + \epsilon^2 e^2 r^2\\
&\quad - (r^2 - 2mr + a^2 + e^2)(\mu^2 r^2 + Q)\\
&\quad - [(r^2 - 2mr + a^2 + e^2)p_r]^2\,,
\end{aligned} \tag{10}$$

see Fig. 1. Here $Q$ is a constant of the motion (generalization of the usual expression for the square of the angular momentum) related to the polar momentum $p_\theta$ at any angle $\theta$ by the equation

$$Q = \cos^2\theta\,[a^2(\mu^2 - E^2) + p_\varphi^2(\sin\theta)^{-2}] + p_\theta^2\ . \tag{11}$$

The positive- and negative-root solutions of (a) coincide only when the discriminant of this equation vanishes. This condition is satisfied only at the horizon where the energy is given by Eq. (8). The derivation of formula (1) assumes and demands that the in-falling particles make an effectively infinitesimal change in the properties of the black hole. Applying the laws of conservation of the three determinants and writing $E = dm$, $p_\varphi = dL$, $\epsilon = de$, we obtain the partial differential equation

$$dm(L, e) = \frac{(L/m)dL + r_+ e\,de}{r_+^2 + L^2/m^2}\,. \tag{12}$$

Integration gives Eq. (1), provided that the following condition is satisfied:

$$\frac{L^2}{4m_{\text{ir}}^2} + \frac{e^4}{16m_{\text{ir}}^4} \leq 1\ . \tag{13}$$

For it to be possible to reverse the transformation, both the original particle (positive energy) and the added particle (negative energy) must arrive at the horizon with zero radial momentum. Otherwise there is a nonzero kinetic energy that is irretrievably lost.

When one turns from reversibility as a criterion for an interesting transformation to merely the ability to extract energy, it becomes important to specify under what conditions a positive-root state has negative energy. From Eq. (10) it follows that the region (outside the one-way membrane) where positive-root states of negative energy are available to a particle of specified rest mass $\mu$, charge $\epsilon$, and angular momentum $p_\varphi$ extends to the surface

$$(r^2-2mr+e^2+a^2)[p_\varphi{}^2+\sin^2\theta(r^2+a^2\cos^2\theta)\mu^2]$$
$$=\sin^2\theta(\epsilon er+ap_\varphi)^2. \tag{14}$$

When the transformation is reversible, the energy-extraction process has its maximum possible efficiency. Repetition of an energy-extraction process with maximum possible efficiency results in conversion into energy of 50% of the mass of an extreme charged black hole and 29% of that of an extreme rotating black hole. Thus, black holes appear to be the "largest storehouse of energy in the universe."[14]

#### ACKNOWLEDGMENT

We are indebted to John A. Wheeler for stimulating discussions and also for advice on the wording of this paper.

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# Motion of Particles in Einstein's Relativistic Field Theory. III. Coordinate Conditions

Coates R. Johnson
*Department of Physics and Astronomy, University of Georgia, Athens, Georgia 30601*

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In two earlier papers the author developed an approximation procedure for finding the Lorentz-covariant equations of structure and motion of interacting particles (represented by singularities) in Einstein's theory of the nonsymmetric field and in Einstein's theory of the gravitational field. In the earlier papers the author worked exclusively in a specific set of coordinate systems, the harmonic coordinate systems. In this paper the author shows that the procedure developed in the earlier papers can be used without imposing any conditions on the coordinates except that the fundamental field be the Minkowski metric in the absence of particles. The author also shows that up to any finite order of approximation the use of harmonic coordinates does not reduce the set of invariantly distinct solutions to Einstein's field equations.

## I. INTRODUCTION

In two earlier papers[1] the author developed an approximation procedure for finding the Lorentz-covariant equations of structure[2] and motion of interacting particles (represented by singularities) in Einstein's theory of the nonsymmetric field and, because it is a special case of that theory, in Einstein's theory of the gravitational field. The procedure was developed for the most general particles which could be represented by singularities in a perfectly isolated region of the space-time continuum. Such particles were called ideal particles. The terms "ideal particle" and "perfectly isolated region of the continuum" were defined in the earlier papers.

The approximation procedure developed by the author allows one to find the equations of structure and motion step by step with respect to the powers of a parameter $\kappa$ which measures the "strength"

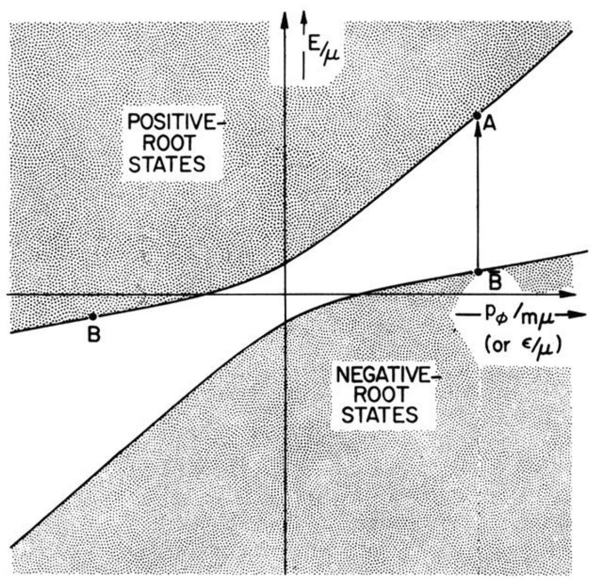


FIG. 2. Reversing the effect of having added to the black hole one particle ($A$) by the Penrose process of adding another particle ($B$) of the same rest mass but of opposite angular momentum and charge in a "positive-root negative-energy state." The diagram shows schematically the energy $E$ of the particle (measured in the Lorentz frame tangent at $r \to \infty$) as a function of the charge $\epsilon$ of the particle and its angular momentum $p_\varphi$ about the axis of the black hole (for simplicity one of these two dimensions has been suppressed in the frame). The particle under examination is in the equatorial plane of the black hole ($\theta = \frac{1}{2}\pi$) at a specified value of $r$. The energy lies on the indicated "positive-root" curve when $p_r$ and $p_\theta$ are zero, otherwise in the dotted region above the curve. Addition of $B$ is equivalent to subtraction of $\bar{B}$. Thus the combined effect of the capture of particles $A$ and $B$ is an increase in the mass of the black hole given by the vector $\bar{B}A$. This vector vanishes and reversibility is achieved when and only when the separation between positive-root states and negative-root states is zero [in this case the hyperbolas coalesce to the straight line given by Eq. (8)].