An Electromagnetic Thirring Problem*

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A neutral rotating mass shell surrounds a concentric stationary electrically charged insulating shell. The dipole-like magnetic field induced by (and proportional to) the rotation of the neutral shell is calculated on the basis of the coupled linearized Einstein-Maxwell field equations of general relativity. This field is apparently at variance with a conjecture made on Machian grounds, for which a possible explanation is suggested. The corresponding induced quadrupolar electric field is calculated for the region within the charged shell, and the potential is given for this field everywhere. Though understandable on mutually inconsistent elementary grounds, we regard this field as a useful example of a solution of linearized general relativity.

I. INTRODUCTION

Thirring[1] was the first to investigate on the basis of general relativity an effect which has since become known as the "dragging of inertial frames" by rotating massive bodies. There is, of course, no such effect in Newton's gravitational theory. Though minute for all conceivable laboratory situations, the dragging of inertial frames is by no means negligible, for example, in the neighborhood of fast-rotating neutron stars. Indeed, according to the theory, some of these stars could drag the inertial frame near their surface around several times per second.[2]

Thirring's particular investigation concerned the gravitational field inside a massive rotating shell placed in an otherwise empty and asymptotically flat space-time. Part of the motivation stemmed from what Einstein had termed "Mach's principle," according to which a gravitational theory should be so constructed that the same consequences result from assuming, for example, that a certain mechanical system is rotating within a fixed universe, or that the universe is rotating and the system is at rest. In other words, all that should matter are the masses and *relative* motions of *all* the bodies in the universe, and no extraneous entities like "absolute space" should enter any problem. (Such statements, of course, assume the *a priori* existence of a Galilean space-time metric and ignore the independent existence of fields and the difficulties of defining relative motion at a distance in general relativity[3]; yet although Mach's principle was originally a proposal to modify classical mechanics, its basic idea has nevertheless proved a source of fruitful and often true conjectures in general relativity.) According to this principle, the rotation of the universe relative to the earth would gravitationally cause the Coriolis and centrifugal fields experienced on earth. By fractionization, one might expect the field inside a rotating mass shell to have weak Coriolis and centrifugal features. And this, indeed, was Thirring's finding.[4]

Analogously, one of us conjectured[5] that an electrically charged stationary shell inside the large rotating shell would be surrounded by a dipole-like magnetic field. Quite apart from Mach's principle, to which one may subscribe or not, this conjecture poses a well-definable problem in general relativity which we call an "electromagnetic Thirring problem": to find the coupled gravitational and electromagnetic fields in an asymptotically flat space-time whose only content is a charged spherical shell surrounded by a concentric neutral massive shell, the first at rest and the second in uniform rotation relative to infinity. In the present paper we solve this problem to first order in the gravitational constant k, and to second order in the angular velocity ω. The magnetic field has already been reported by us without proof.[6]

We have since then learned of the work of Cohen,[7] who treated a related problem by a different method. Using the full nonlinear Einstein gravitational-field equations, but restricting himself to lowest order in the rotation rates, Cohen computes the electromagnetic *test* field due to a *rotating* charged shell of negligible mass and a concentric rotating neutral shell of arbitrary mass. His main concern is the confirmation of the Machian expectation that, when the radius of the outer shell approximates its Schwarzschild radius, the electromagnetic field inside that shell depends only on the relative angular velocity of the two shells.

Whereas our method is inferior to that of Cohen in that we cannot treat arbitrarily strong gravitational fields, it nevertheless has the following advantages. Our flat-space *form* of the Maxwell equations for the perturbed fields[8] exhibits fictitious currents and charges which allow one to recognize at once the structure of the gravitationally induced electromagnetic fields in lowest order and to compute these fields by the standard methods of electrodynamics. Thus for our particular problem one verifies immediately the essentially dipolar structure of the magnetic field and gets the magnetic moment expected on Machian grounds via Thirring's dragging-velocity formula. (However, as we shall point out and discuss below, the dipolar field occurs in a "wrong" frame.) In addition, our form of the fictitious source terms shows that the apparent *currents* arise from the space-time components $g_{4\lambda}$ of the metric and hence their leading terms are proportional to ω, whereas the apparent ("vacuum-polarization") *charges* are due to the remaining metric components. The current potentials $g_{4\lambda}$ are independent of the reaction of the metric to the Maxwell field. The deviations of the charge potentials $(g_{\lambda\mu}, g_{44})$ from their flat-space values consist of terms independent of ω but due to reaction with the Maxwell field, and of other terms proportional to ω^2; all of which are therefore absent in Cohen's work. We find that the ω^2-dependent gravitationally induced electric field appears only in conjunction with a mass quadrupole moment of the outer shell, and accordingly has a quadrupole angular dependence; in particular, this field extends into the interior of the charged shell. Finally, in our method we need not restrict the relative magnitudes of the two shell masses and of the charge.

The following is an outline of our procedure. If the electromagnetic units are Gaussian and the speed of light is taken to be unity, as it will be throughout this paper, the coupled Einstein-Maxwell field equations are

$$G_{ij} = \kappa(T_{ij} + S_{ij}) \quad (\kappa = 8\pi k) \tag{1.1}$$

and

$$F^{ik}{}_{;k} = 4\pi J^i , \tag{1.2a}$$

$$F_{[ij,k]} = 0 , \tag{1.2b}$$

where G_{ij} = Einstein tensor; k = Newton's constant of gravitation; T_{ij} = material-energy tensor; S_{ij} = electromagnetic-energy tensor defined by

$$S_{ij} = (1/4\pi)[F_{hi}F^h{}_j - \tfrac{1}{4}g_{ij}(F_{hk}F^{hk})] ; \tag{1.3}$$

F_{ij} = electromagnetic-field tensor; and J^i = current-density vector. We use ", k" and "; k" to denote partial and covariant differentiation, respectively, with respect to the coordinate x^k, and [ ] to denote skew-symmetrization; Latin indices run from one to four and the Greek indices in the sequel run from one to three.

To find the coupled fields of a given distribution of matter and charge, according to general relativity, it is therefore necessary to find functions g_{ij} and F_{ij} satisfying (1.1) and (1.2) and possibly certain boundary conditions, so that the corresponding T_{ij} and J^i describe that distribution of matter and charge. One well-known solution of these equations is the Reissner-Nordström metric and field, characteristic of the vacuum outside any spherically symmetric distribution of matter and charge.[9]

If one considers the reaction of the metric to the electromagnetic field as negligible, one omits S_{ij} from Eq. (1.1); the electromagnetic field is then a "test" field. This is the curved-space equivalent of the way in which electromagnetic problems are solved as a matter of course in special relativity.

In order to linearize (1.1) and (1.2) in k, one assumes[10] the existence of a one-parameter family of solutions of these equations, depending smoothly on the gravitational constant k as parameter. When $k=0$ (no gravity), the solution is assumed to be $g_{ij} = \eta_{ij} = \mathrm{diag}(1, 1, 1, -1)$ (flat space), and $F_{ij} = E_{ij}$, where E_{ij} is an appropriate flat-space solution of the Maxwell equations (1.2). One then decomposes g_{ij} and F_{ij} as follows:

$$g_{ij} = \eta_{ij} + kh_{ij} + O(k^2) , \tag{1.4}$$

$$F_{ij} = E_{ij} + kH_{ij} + O(k^2) , \tag{1.5}$$

h_{ij} and H_{ij} being defined by these equations. Similarly, and in an obvious notation, one has

$$T_{ij} = T_{ij}^{(0)} + kT_{ij}^{(1)} + O(k^2) , \quad S_{ij} = S_{ij}^{(0)} + kS_{ij}^{(1)} + O(k^2) . \tag{1.6}$$

In the linear approximation one works to first order in k only. It is convenient to introduce the new coefficients

$$\gamma_{ij} = h_{ij} - \tfrac{1}{2}\eta_{ij}h , \tag{1.7}$$

where

$$h = \eta^{ij}h_{ij} , \quad \eta^{ij} = \eta_{ij} , \tag{1.8}$$

and, if necessary, to make a transformation $x^i \to x^i + k\xi^i$ to new coordinates which satisfy the coordinate conditions

$$\gamma^{ij}{}_{,j} = 0 \quad (\gamma^{ij} = \eta^{ip}\eta^{jq}\gamma_{pq}) . \tag{1.9}$$

If, in particular, the solution is stationary, then, in virtue of (1.1), the γ_{ij} satisfy the Poisson-type equations

$$\Delta\gamma_{ij} = -16\pi(T^{(0)}_{ij} + S^{(0)}_{ij}), \tag{1.10}$$

which one can solve in the familiar way, using boundary and jump conditions.

In applying this method to the electromagnetic Thirring problem at hand we first consider the energy tensors of the outer and inner shells (the latter to include the electromagnetic field), each in the absence of the other shell. Since a modification of either tensor, due to the distribution described by the other, can occur only if gravity is present, such a modification can only affect the k-dependent terms in (1.6). Thus in (1.10) we simply insert the sum of the *undisturbed* energy tensors of the two shells. Consequently, the γ's we seek are the sums of the γ's corresponding to the pure Thirring problem and the Reissner-Nordström problem, respectively. The availability of the exact Reissner-Nordström solution is a piece of good fortune in the present problem, but in other problems of a similar nature the γ's can be worked out *ab initio*. After the γ's are found, the perturbed electromagnetic field is obtained by solving Maxwell's equations (1.2) in their linearized form.

II. SUMMARY OF RESULTS OF THE THIRRING PROBLEM

Consider a thin rigidly rotating neutral mass shell in an otherwise empty and asymptotically flat space-time. Let the shell be spherical with radius R in the zeroth approximation, and let its angular speed as judged by an observer at infinity be ω = const. Idealizing the shell as being infinitely thin, assuming the proper surface density of proper mass ρ to be constant on the shell, and neglecting terms of higher than second order in ω, one obtains a unique solution of the linearized Einstein field equations, as shown in varying degrees of completeness by Thirring,[1] Bass and Pirani,[11] Hönl and Maue,[12] and Ehlers.[13]

The relevant metric can be written, in linearized harmonic coordinates $(\vec{r}, t) = (x^\lambda, t)$, as

$$\Theta = (1-2U)d\vec{r}^2 + 2dt\,\vec{V}\cdot d\vec{r} - (1+2U)dt^2, \tag{2.1}$$

where the gravitational potentials U, $\vec{V}$ are defined as follows: If the shell has total mass

$$M = 4\pi R^2\rho[1+\tfrac{2}{3}(\omega R)^2], \tag{2.2}$$

angular momentum vector

$$\vec{L} = \tfrac{2}{3}MR^2\vec{\omega}, \tag{2.3}$$

and quadrupole-moment tensor[14]

$$\vec{D} = \tfrac{3}{5}MR^4(\tfrac{1}{3}\omega^2 - \vec{\omega}\vec{\omega}), \tag{2.4}$$

and if u is related to the radial coordinate by

$$u := \begin{cases} 1/R & \text{if } r \le R \\ 1/r & \text{if } r > R \end{cases} \tag{2.5}$$

(where convenient, we use := and =: for "is defined by" and "defines," respectively), then

$$U = -kMu - \tfrac{1}{2}k(\vec{r}\cdot\vec{D}\cdot\vec{r})u^5, \tag{2.6}$$

$$\vec{V} = 2k(\vec{r}\times\vec{L})u^3. \tag{2.7}$$

Further details of the solution, which we do not need in the sequel, are described in Ref. 13.

Since $\omega R < 1$, it is seen from the above formulas that the linear approximation is reasonable provided

$$\frac{kM}{R} \ll 1, \tag{2.8}$$

this being the condition for U and $\vec{V}$ to be small compared to unity. Hence, in the domain of validity of this solution, the Thirring angular velocity with which the inertial frame in the interior of the shell rotates relative to that at infinity,

$$-\vec{\Omega} = \tfrac{4}{3}\frac{kM}{R}\vec{\omega}, \tag{2.9}$$

is small compared to the rotation rate of the shell.

III. SUPPLEMENTS TO THE REISSNER-NORDSTRÖM PROBLEM

The Reissner-Nordström metric has the form[9]

$$\Phi = \left(1 - \frac{N}{\rho} + \frac{Q}{\rho^2}\right)^{-1} d\rho^2 + \rho^2 d\omega^2 - \left(1 - \frac{N}{\rho} + \frac{Q}{\rho^2}\right)dt^2, \tag{3.1}$$

where

$$d\omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2, \tag{3.2}$$

$$N = 2km, \quad Q = kq^2, \tag{3.3}$$

and m and q are the mass and charge, respectively, of the central object. If that is a spherical *shell*, the argument leading to the exterior metric (3.1) can be applied without modification to the empty interior, except for the final identification (3.3) of the constants N and Q. These must be taken to be zero if we seek an everywhere-regular *interior* solution. We conclude that the interior is flat. However, the form of the metric (3.1) has to be modified by a coordinate transformation before it can be joined continuously to a suitable flat interior metric, so as to provide a satisfactory solution to the complete shell problem.

Since we are interested in the linearized metric of the charged shell, we first reexpress Φ in harmonic coordinates[15] so that its linearization will automatically satisfy the coordinate conditions

(1.9). Consequently we seek a new radial coordinate

$$r = f(\rho)\rho\,, \qquad (3.4)$$

where f is to be determined such that our ultimate three coordinates

$$x = r\sin\theta\cos\phi\,,\quad y = r\sin\theta\sin\phi\,,\quad z = r\cos\theta \qquad (3.5)$$

are harmonic with respect to Φ. The condition for a function V to be harmonic with respect to a metric g_{ij} is

$$\Box V \equiv |g|^{-1/2}\partial_i(|g|^{1/2}g^{ij}\partial_j V) = 0\,, \qquad (3.6)$$

where $\partial_i = \partial/\partial x^i$. One can easily check that the time coordinate t of Φ is already harmonic. The condition for the harmonicity of x can be reduced to

$$[(f'\rho^3 + f\rho^2)X]' - 2f\rho = 0\,, \qquad (3.7)$$

where X is the coefficient of $-dt^2$ in (3.1) and the prime denotes $d/d\rho$. It is clear that this condition will at the same time assure the harmonicity of y and z, because Eq. (3.6) is preserved under "rotations" $(x, y, z) \to (y, z, x) \to (z, x, y)$. Explicitly, Eq. (3.7) reads

$$f''(\rho^3 - N\rho^2 + Q\rho) + f'(4\rho^2 - 3N\rho + 2Q) - fN = 0\,. \qquad (3.8)$$

Considering f as a power series in ρ^{-1},

$$f = \sum C_n \rho^{-n}\,, \qquad (3.9)$$

and substituting this into Eq. (3.8), we find

$$C_{n+1} = \frac{(n-1)^2}{(n+1)(n-2)} N C_n - \frac{n-1}{n+1} Q C_{n-1}\,, \qquad (3.10)$$

provided $n > 2$. By equating to zero the three highest-order terms, we find

$$C_0, C_3 = \text{arbitrary}\,,\quad C_2 = 0\,,\quad C_1 = -\tfrac{1}{2}NC_0\,. \qquad (3.11)$$

Since for large values of r we wish r to have its usual metric significance, we choose $C_0 = 1$, which implies $C_1 = -\frac{1}{2}N$. The choice $C_3 = 0$ implies, through Eq. (3.10), that $C_n = 0$ for $n \geq 3$. Thus $f = 1 - \frac{1}{2}N/\rho$ is an exact solution of Eq. (3.8), and so the simplest harmonic form of Φ is

$$\Phi = \left(1 - \frac{N}{r + \frac{1}{2}N} + \frac{Q}{(r + \frac{1}{2}N)^2}\right)^{-1} dr^2 + (r + \tfrac{1}{2}N)^2 d\omega^2 - \left(1 - \frac{N}{r + \frac{1}{2}N} + \frac{Q}{(r + \frac{1}{2}N)^2}\right) dt^2\,. \qquad (3.12)$$

If here we set $Q = 0$, we recover a well-known harmonic form of the Schwarzschild metric.[16]

For our purpose of continuing the metric into the interior of the shell, however, we need the freedom afforded by choosing $C_3 = C$, not necessarily zero.[17] Since N and Q contain k as a factor, and since the same will turn out to be true of C, it is clear from Eq. (3.10) that all C_n with $n > 3$ will contain a factor k^2. Consequently they drop out in the linearization. With this cutoff, Eq. (3.4) becomes

$$r = (1 - \tfrac{1}{2}N\rho^{-1} + C\rho^{-3})\rho\,, \qquad (3.13)$$

and its inverse,

$$\rho = (1 + \tfrac{1}{2}Nr^{-1} - Cr^{-3})r\,. \qquad (3.14)$$

The substitution of Eq. (3.14) into the original metric (3.1) leads to a form of that metric which is, of course, exact, but harmonic only to first order in k. A slightly tedious calculation then shows that this exact metric can be joined continuously (i.e., with all g_{ij} continuous) to an interior flat-space metric

$$\Psi = \mu^2(dr^2 + r^2 d\omega^2) - \lambda^2 dt^2 \qquad (3.15)$$

across the sphere $r = a$ (>0) by taking

$$C = a^3\left[\left(1 + \frac{Q}{3a^2} - \frac{N^2}{12a^2}\right)^{1/2} - 1\right], \qquad (3.16a)$$

$$\mu = 2 + \frac{N}{2a} - \left(1 + \frac{Q}{3a^2} - \frac{N^2}{12a^2}\right)^{1/2}, \qquad (3.16b)$$

$$\lambda^2 = 1 - \frac{N}{a\mu} + \frac{Q}{a^2\mu^2}\,. \qquad (3.16c)$$

We observe that C is indeed a multiple of k when expanded in powers of k, as anticipated.

The form (3.1) has no horizons[18] if $N^2 - 4Q < 0$. In that case C and μ are automatically real and $\lambda^2 > 0$; moreover, $\mu \neq 0$ if

$$3a > \sqrt{Q} - N\,. \qquad (3.17)$$

If the form (3.1) *has* horizons, a must evidently be chosen outside of them. The condition for C and μ to be real is then $12a^2 \geq N^2 - 4Q$, which also ensures $\mu \neq 0$; but to ensure $\lambda^2 > 0$ it must be strengthened to

$$9a^2 > N^2 - 4Q\,. \qquad (3.18)$$

Another condition on a, namely,

$$a > Q/N\,, \qquad (3.19)$$

will be seen to follow from energy considerations.

We can now calculate the linear harmonic form of the metric of the charged shell. Discarding the nonlinear powers of k from the metric (3.1) under the harmonizing condition (3.14) yields

$$\Phi' = \left(1 - \frac{N}{r} - \frac{Q}{r^2} + \frac{4C}{r^3}\right) dr^2 + \left(1 + \frac{N}{r} - \frac{2C}{r^3}\right) r^2 d\omega^2 - \left(1 - \frac{N}{r} + \frac{Q}{r^2}\right) dt^2\,. \qquad (3.20)$$

The linearized junction conditions (3.16) read

$$C = \tfrac{1}{6}aQ\,, \tag{3.21a}$$

$$\mu^2 = 1 + \frac{N}{a} - \frac{Q}{3a^2}\,, \tag{3.21b}$$

$$\lambda^2 = 1 - \frac{N}{a} + \frac{Q}{a^2}\,. \tag{3.21c}$$

The relevant coordinates for the linear theory, however, are those defined in Eqs. (3.5). If $\sum$ denotes summation of three analogous terms in x, y, z, we have

$$r^2 = \sum x^2, \quad r\,dr = \sum x\,dx, \quad dr^2 + r^2 d\omega^2 = \sum dx^2\,, \tag{3.22}$$

whence (3.20) becomes, by reference also to (3.21a),

$$\Phi' = \left(1 + \frac{N}{r} - \frac{aQ}{3r^3}\right)\sum dx^2 + \left(\frac{aQ}{r^3} - \frac{Q}{r^2}\right)\frac{(\sum x\,dx)^2}{r^2} - \left(1 - \frac{N}{r} + \frac{Q}{r^2}\right)dt^2\,. \tag{3.23}$$

From the continuity of the transformation equations it is clear that, under the conditions (3.21b) and (3.21c), Φ' as given by (3.23) is now continuously joined to the interior metric,

$$\Psi = \mu^2 \sum dx^2 - \lambda^2 dt^2\,. \tag{3.24}$$

The appearance of the non-unit factors λ^2, μ^2 in the interior metric tells us something about the meaning of the coordinates that cover the space-time continuously, and thus also something about the physics. For example, there will be a red shift $\lambda:1$ for photons traveling from the interior to infinity.

It remains to find the linearized electromagnetic field F_{ij} of the shell in the linearized metric consisting of (3.23) for $r \geq a$ and (3.24) for $r < a$. It is clear that there can be no such field in the flat interior, since every nonzero field produces a positive electromagnetic-energy density, but that must vanish in flat space-time. In the exterior metric (3.1) the electromagnetic field is known[9] to be of exact Coulomb form in the coordinate ρ,

$$F^{i'j'} = 0 \quad (i', j' \neq 1,\ 4)\,, \quad F^{4'1'} = q/\rho^2\,. \tag{3.25}$$

Applying to $F^{i'j'}$ the tensor transformation corresponding to the coordinate transformation $(\rho, \theta, \phi, t) \to (x, y, z, t)$, and then lowering indices, we find, to first order in k,

$$F_{ij} = 0 \quad (i, j \neq 4)\,, \quad F_{\lambda 4} = \frac{qx^\lambda}{r^3}\left(1 - \frac{N}{r} + \frac{2aQ}{3r^2}\right). \tag{3.26}$$

The 4-current associated with this field is, to first order in k,

$$\begin{aligned} J^i &= (4\pi\sqrt{-g}\,)^{-1}(\sqrt{-g}\,F^{ij})_{,j} \\ &= (4\pi)^{-1}(1 - N/r)[(qx^\lambda/r^3)H(r-a)]_{,\lambda}\delta_4^i \\ &= (q/4\pi a^2)(1 - N/a)\delta_4^i\,\delta(r-a)\,. \end{aligned} \tag{3.27}$$

The stress-energy-momentum tensor of the shell, which is determined by reading Einstein's field equations "backwards," is, in zeroth order in k, given by

$$T^{ij} = \tau^{ij}\delta(r-a)\,, \tag{3.28}$$

with energy density

$$\tau^{44} = \frac{m}{4\pi a^2} - \frac{q^2}{8\pi a^3} \tag{3.29}$$

and tangential stress

$$-\frac{\tau_{\theta\theta}}{a^2} = -\frac{\tau_{\phi\phi}}{a^2\sin^2\theta} = \frac{q^2}{16\pi a^3}\,, \tag{3.30}$$

all other components being zero.[19] The physically required positive character of the shell density τ^{44} imposes the condition (3.19) on the shell radius. In the limit $\tau^{44} = 0$, we have $\lambda = 1$ and thus no red shift between the interior and infinity.

IV. THE PERTURBED ELECTROMAGNETIC FIELD

In this section we first derive the general equations which the linear perturbations H_{ij}, $J^i_{(1)}$ of an electromagnetic field and its current have to satisfy. The unperturbed quantities E_{ij}, $J^i_{(0)}$, and the linear perturbation of the metric, given by γ_{ij}, enter these equations as known functions. We then specialize these equations to stationary states in which the unperturbed field E_{ij} is electrostatic, obtaining decoupled equations for the electric and magnetic components of H_{ij} which are such that the qualitative nature of the solutions can readily be recognized. After these preliminaries, we solve these equations for the electromagnetic Thirring problem and describe briefly the resulting fields.

It is obvious from Eqs. (1.2) and (1.5) that

$$H_{[ij,k]} = 0. \tag{4.1}$$

The perturbed version of Eq. (1.2a) is obtained conveniently be rewriting that equation as

$$(1/\sqrt{-g}\,)(\sqrt{-g}\,g^{ij}g^{kl}F_{jl})_{,k} = 4\pi J^i\,,$$

which simplifies in harmonic coordinates [see Eq. (3.6)] to

$$g^{kl}(g^{ij}F_{jl})_{,k} = 4\pi J^i\,.$$

Differentiating with respect to the perturbation parameter k at $k = 0$ and using the definitions (1.4), (1.5), and (1.7), we obtain

$$H^{ik}{}_{,k} = 4\pi(J^i_{(1)} - \gamma J^i_{(0)} + \gamma^{ik}J_{(0)k}) - \tfrac{1}{2}E^{ik}\gamma_{,k} + \gamma^{ij}{}_{,k}E_j{}^k + \gamma^{jk}E^i{}_{j,k} \quad (\gamma := \eta^{ij}\gamma_{ij}) . \tag{4.2}$$

Similarly one derives from the law of conservation of charge [or from (4.2)] the relation

$$J^k_{(1),k} = \tfrac{1}{2}\gamma_{,k}J^k_{(0)} . \tag{4.3}$$

In these and similar equations, indices are shifted by means of the flat-space metric η_{ij}. (Note that, since H_{ij} is defined as the perturbation of F_{ij} and $H^{ij} := \eta^{ik}\eta^{jl}H_{kl}$, H^{ij} is *not* the perturbation of F^{ij}.)

The perturbed curved-space Maxwell equations (4.1) and (4.2) have the *form* of the ordinary Maxwell equations in flat space, expressed in inertial coordinates; their curved-space origin shows up in the fictitious γ_{ij}-dependent current on the right-hand side of Eq. (4.2), which should only be regarded as a device for computing H_{ij}.

Let us now consider stationary fields and assume that $J^\lambda_{(0)} = 0$, $E_{\lambda\mu} = 0$. If we then put

$$\vec{E}_{(0)} := (E_{\lambda 4}) , \qquad \sigma_{(0)} := J^4_{(0)} , \tag{4.4}$$

$$\vec{E}_{(1)} := k(H_{\lambda 4}) , \qquad \vec{B} := k(H_{23}, H_{31}, H_{12}) , \tag{4.5}$$

$$\vec{J}_{(1)} := k(J^\lambda_{(1)}) , \qquad \sigma_{(1)} := kJ^4_{(1)} , \tag{4.6}$$

$$\Gamma := k(\gamma_{\lambda\mu}) , \tag{4.7}$$

and use U and $\vec{V}$ as in Eq. (2.1), we can write Eqs. (4.1), (4.2), and (4.3) in 3-dimensional vector notation as follows:

$$\nabla\cdot\vec{B} = 0 , \quad \nabla\times\vec{B} = 4\pi(\vec{J}_{(1)} + \sigma_{(0)}\vec{V}) + [\vec{E}_{(0)}, \vec{V}] , \tag{4.8}$$

$$\nabla\times\vec{E}_{(1)} = 0 ,$$
$$\nabla\cdot\vec{E}_{(1)} = 4\pi(\sigma_{(1)} - \sigma_{(0)}\mathrm{tr}\Gamma) + 2\vec{E}_{(0)}\cdot\nabla U + \Gamma : \nabla\vec{E}_{(0)} , \tag{4.9}$$

$$\nabla\cdot\vec{J}_{(1)} = 0 . \tag{4.10}$$

[The last term in Eq. (4.8) is the Lie commutator $(\vec{E}_{(0)}\cdot\nabla)\vec{V} - (\vec{V}\cdot\nabla)\vec{E}_{(0)}$ of $\vec{E}_{(0)}$ and $\vec{V}$, and the last term in the second Eq. (4.9) is the trace of the matrix product of Γ and $\nabla\vec{E}_{(0)}$.] These decoupled equations for $\vec{B}$ and $\vec{E}_{(1)}$ exhibit the influence of the gravitational potentials U, $\vec{V}$, and Γ upon the magnetic and electric fields, respectively, as described in the Introduction.

We now apply these equations to the electromagnetic Thirring problem, in which $\vec{J}_{(1)} = 0$.

The Magnetic Field

Since $\vec{V}$ vanishes in the linearized Reissner-Nordström metric, $\vec{V}$ in Eq. (4.8) has to be taken from the Thirring metric alone. With $\sigma_{(0)} = (q/4\pi a^2) \times\delta(r-a)$ and $\vec{V}$ from Eq. (2.7), we find immediately that the first effective current in Eq. (4.8) is nothing but the convection current which would arise if the charged sphere were rotating, relative to the stationary frame of reference used so far, with the angular velocity $\vec{\Omega}$ defined in Eq. (2.9). The corresponding magnetic field is well known; it is given by

$$\vec{B}^{\mathrm{I}} = \begin{cases} \frac{2}{3}qa^{-1}\vec{\Omega} & (r<a) \\ qa^2[(\vec{\Omega}\cdot\vec{r})r^{-5}\vec{r} - \frac{1}{3}r^{-3}\vec{\Omega}] & (r>a) . \end{cases} \tag{4.11}$$

Outside the shell it is a dipole field of moment $\frac{1}{3}qa^2\vec{\Omega}$ and inside it is homogeneous.

The second effective-current term in Eq. (4.8) vanishes inside the rotating shell, and outside it is

$$[\vec{E}_{(0)}, \vec{V}] = -qR^3[r^{-3}\vec{r}, (\vec{r}\times\vec{\Omega})u^3]$$
$$= 3qR^3r^{-6}(\vec{r}\times\vec{\Omega}) .$$

This represents circular currents flowing in the sense of the shell's rotation, and rapidly decreasing with increasing distance from the shell. The corresponding magnetic field,

$$\vec{B}^{\mathrm{II}} = \begin{cases} -\frac{1}{2}qR^{-1}\vec{\Omega} & (r\leq R) \\ -3(R/a)^2\vec{B}^{\mathrm{I}} + 3qR^3r^{-4}[(\vec{\Omega}\cdot\vec{r})r^{-2}\vec{r} - \frac{1}{2}\vec{\Omega}] & (r>R) , \end{cases} \tag{4.12}$$

is homogeneous inside the rotating shell, and outside of it the leading (r^{-3}) terms constitute a dipole field of moment $-qR^2\vec{\Omega}$ (see Fig. 1).

Inside and near the charged shell $\vec{B}^{\mathrm{I}}$ dominates $\vec{B}^{\mathrm{II}}$ by a factor $\sim R/a$, and well outside the rotating shell $\vec{B}^{\mathrm{II}}$ dominates $\vec{B}^{\mathrm{I}}$ by a factor $\simeq -3(R/a)^2$. The total dipole moment of $\vec{B}^{\mathrm{I}} + \vec{B}^{\mathrm{II}}$ at infinity is $-q\vec{\Omega} \times(R^2 - \frac{1}{3}a^2)$.

In the limiting case of a point charge $(a\to 0)$, $\vec{B}^{\mathrm{I}}$ vanishes, so that $\vec{B}^{\mathrm{II}}$ describes the whole field. This field was already obtained by Hofmann.[20]

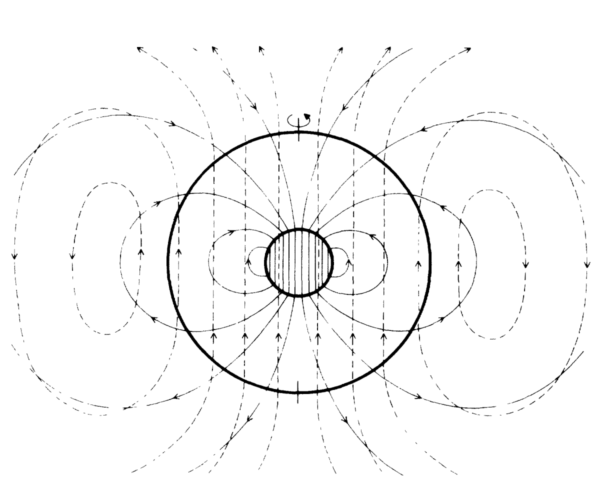

FIG. 1. Field lines of the components $\vec{B}^{\mathrm{I}}$ (full) and $\vec{B}^{\mathrm{II}}$ (stippled) of the magnetic field induced by the rotating mass shell.

The Electric Field

The perturbed electric field $\vec{E}_{(0)}+\vec{E}_{(1)}$ of the electromagnetic Thirring problem may be considered as the superposition of the linearized Reissner-Nordström field $\vec{E}_{(0)}+\vec{E}^{I}_{(1)}$ given by Eq. (3.26) and the additional perturbation $\vec{E}^{II}_{(1)}$ due to (a) those effective charges in Eq. (4.9) which arise from the Thirring gravitational potentials and (b) that contribution $\sigma^{II}_{(1)}$ to the first-order perturbation $\sigma_{(1)}$ of the charge density which is not contained in Eq. (3.27).

In order to determine $\sigma^{II}_{(1)}$ we assume that each element of the inner shell, identified by a range of angular coordinates θ, ϕ, carries a fixed charge which does not change if the shell is surrounded by the outer shell and if this outer shell is made to rotate; in other words, we take the inner shell to be an insulator and θ, ϕ to be material (Lagrangian) coordinates of that shell. Since the charge element contained in the cell $dxdydz$ at $t=0$ is $dq = J^4(\sqrt{-g})dxdydz$, or, with $J^4=:\hat{\sigma}\delta(r-a)$, $dq = \hat{\sigma}(\sqrt{-g})_{r=a}a^2\sin\theta d\theta d\phi$, the above condition means that $\hat{\sigma}(\sqrt{-g})_{r=a}$ must not change in the transition from the linearized Reissner-Nordström solution to the full solution with the outer rotating shell.

In the Reissner-Nordström solution we found [see Eq. (3.27)] that

$$\hat{\sigma}(\sqrt{-g})_{r=a}=q/4\pi a^2 .$$

In the presence of the rotating shell we have

$$\sqrt{-g} = (\sqrt{-g})_{\text{Reissner-Nordström}} + \tfrac{1}{2}\eta^{ij}(\delta g_{ij})_{\text{Thirring}} = 1+N/r-2U ,$$

and therefore

$$(\sqrt{-g})_{r=a} = 1+\frac{N}{a}+\frac{2kM}{R}(1+\tfrac{1}{15}a^2\omega^2Y_2)=:1+\chi ,$$

where

$$Y_2:=r^{-2}(x^2+y^2-2z^2) \tag{4.13}$$

is a surface harmonic of second degree. Thus $\hat{\sigma}$ in the presence of the outer shell is determined by $\hat{\sigma}(1+\chi)=q/4\pi a^2$, i.e.,

$$\hat{\sigma}=\frac{q}{4\pi a^2}(1-\chi)=\frac{q}{4\pi a^2}\left[1-\frac{N}{a}-\frac{2kM}{R}(1+\tfrac{1}{15}a^2\omega^2Y_2)\right] .$$

If we omit from this expression the Reissner-Nordström contribution, we finally get the desired charge density

$$\sigma^{II}_{(1)}=-\frac{kqM}{2\pi a^2R}(1+\tfrac{1}{15}a^2\omega^2Y_2) . \tag{4.14}$$

Since in the Thirring metric (2.1) $\Gamma=0$, the effective charge density in Eq. (4.9) is simply $(1/2\pi)\times\vec{E}_{(0)}\cdot\nabla U$ [with U taken from Eq. (2.6)]. Adding to it $\sigma^{II}_{(1)}$ we get the total charge density which generates $\vec{E}^{II}_{(1)}$; it is given by

$$\frac{-kqM}{2\pi a^2R}\times\begin{cases}0 \quad (0\le r<a)\\ (1+\frac{1}{15}a^2\omega^2Y_2)\delta(r-a) \quad (\text{“at” } r=a)\\ (2a^2\omega^2/15r)Y_2 \quad (a<r\le R)\\ -(a^2R/r^4)[1+\frac{1}{5}(\omega^2R^4/r^2)Y_2] \quad (R<r) .\end{cases} \tag{4.15}$$

The determination of the electrostatic field corresponding to this charge distribution is a standard potential problem. Using the identity

$$\Delta(r^{-n}Y_l)=(n+l)(n-l-1)r^{-n-2}Y_l ,$$

valid for any surface harmonic of degree l, one obtains immediately one particular integral of Poisson's equation for the source (4.15) in each of the regions $a<r<R$ and $R<r$. Since these, like the source functions, contain only monopole and quadrupole terms, one is led to the following ansatz for the potential ϕ belonging to the field $\vec{E}^{II}_{(1)}$:

$$\phi=\begin{cases}A+Br^2Y_2 \quad (0\le r<a)\\ C+r^{-1}D+(rE+r^2F+r^{-3}G)Y_2 \quad (a<r<R)\\ r^{-2}H+(r^{-4}K+r^{-3}L)Y_2 \quad (R<r) .\end{cases} \tag{4.16}$$

Here A, B, C, D, F, G, H, K, L are constants to be determined from the boundary conditions which require ϕ to be continuous everywhere and $\partial\phi/\partial r$ to be continuous at $r=R$ and to have a jump discontinuity at $r=a$ corresponding to the surface charge distribution $\sigma^{II}_{(1)}$. The constants

$$H=-kMq , \quad E=\frac{H\omega^2}{15R} , \quad D=\frac{2H}{R} , \quad K=\tfrac{1}{15}H\omega^2R^4 \tag{4.17}$$

occur in the two particular integrals mentioned above. Straightforward computation yields the others,

$$\begin{aligned}&A=\frac{2H}{aR}\left(1-\frac{a}{2R}\right) , \quad C=-\frac{H}{R^2} ,\\ &F=-\frac{H\omega^2}{15R^2} , \quad L=-\tfrac{1}{15}H\omega^2R^3\left[1-\tfrac{1}{5}\left(\frac{a}{R}\right)^4\right] ,\\ &G=\frac{H\omega^2a^4}{75R} , \quad B=F\left(1-\frac{6R}{5a}\right) .\end{aligned} \tag{4.18}$$

We could now write down the field $\vec{E}^{II}_{(1)}=-\nabla\phi$ everywhere, but we restrict our attention to its values in the interior of the charged shell as perhaps the most interesting. There the Reissner-Nordström field vanishes and we obtain, from Eqs. (4.16), (4.17), and (4.18),

$$\vec{E} = \vec{E}^{II}_{(1)} = \frac{4kM}{25aR} q\omega^2 \left(1 - \frac{5a}{6R}\right)(x, y, -2z) \quad (r < a). \tag{4.19}$$

This gravitationally induced quadrupole field is sketched, together with the effective charge distribution (4.15) for the field $\vec{E}^{II}_{(1)}$ everywhere, in Fig. 2.

The two invariants of the total field F_{ij} (with dual $*F_{ij}$) inside the charged shell can be obtained from Eqs. (3.26), (4.11), (4.12), and (4.19); they are, to second order in k,

$$\tfrac{1}{2}F_{ij}F^{ij} = \left(\frac{kM}{R}\frac{q\omega}{a}\right)^2 \left[\tfrac{64}{81}\left(1 - \frac{3a}{4R}\right) - \tfrac{16}{625}\omega^2\left(1 - \frac{5a}{6R}\right)^2 (x^2 + y^2 + 4z^2)\right], \tag{4.20}$$

$$-\tfrac{1}{4}F_{ij}{*F^{ij}} = \frac{64}{225}\left(\frac{kM}{R}\frac{q\omega}{a}\right)^2 \omega z\left(1 - \frac{5a}{R}\right)\left(1 - \frac{3a}{4R}\right) \quad (r < a). \tag{4.21}$$

This field is therefore neither purely electric nor purely magnetic for any observer (the bivector F_{ij} is not simple), except at $z = 0$; and for any observer $|\vec{B}| > |\vec{E}|$.

The physical meaning of F_{ij} everywhere can, of course, be elucidated in a coordinate-independent manner by studying the motion of test charges, but we shall not enter into this.

V. DISCUSSION

We now discuss briefly the fact that, whereas at first sight our magnetic field $\vec{B}^1$ seems to be in accordance with Machian expectation ("Mach-positive"), it is in fact Mach-negative or, at best, Mach-neutral. However, we also show how a Mach-positive result can be extracted from our results. Finally, we make some remarks on the perturbation of the electric field.

The Magnetic Field

We know from the solution of the Thirring problem that the inertial frame S' inside the large shell rotates, relative to infinity and in first order, with the Thirring angular velocity $-\vec{\Omega}$. Relative to S', therefore, the inner shell rotates with angular velocity $+\vec{\Omega}$, and we thus expect to find in S' a magnetic dipole field of moment $\frac{1}{3}qa^2\vec{\Omega}$ and an electric Coulomb field of strength qr^{-2}. But when one transforms such a field, by a rotation of coordinates, to the frame S which is at rest relative to infinity and in which all our calculations are made, one does *not even approximately* get the field which we found. This is most easily seen on transforming the expected field in S' at a typical point of the equatorial plane by a local Lorentz transformation to S.

It is possible that our Machian expectation was too indirect. A version of Mach's principle applicable to situations involving electromagnetic charges should perhaps be formulated without mention of *fields* and only in terms of directly observable entities like mass, charge, and motion. One reasonable formulation might be as follows: "All that should matter in the determination of the motion of a charged particle are the masses, charges, and relative motions of all the bodies in the universe." But even in this sense our solution of the electromagnetic Thirring problem appears to be Mach-negative. Consider a charged shell rotating at angular velocity $-\vec{\omega}$ within a stationary shell representing the whole universe. Then there will be, at least locally, a magnetic dipole field of moment $-\frac{1}{3}qa^2\vec{\omega}$ and an electric Coulomb field of strength qr^{-2}, relative to the stationary universe. The initial motion of any test magnetic monopole released from rest will be entirely determined by the magnetic field. A Mach-equivalent situation would be one with the central shell at rest, the "universe" rotating around it at velocity $\vec{\omega}$, and the test monopole released *from rest with respect to the uni-*

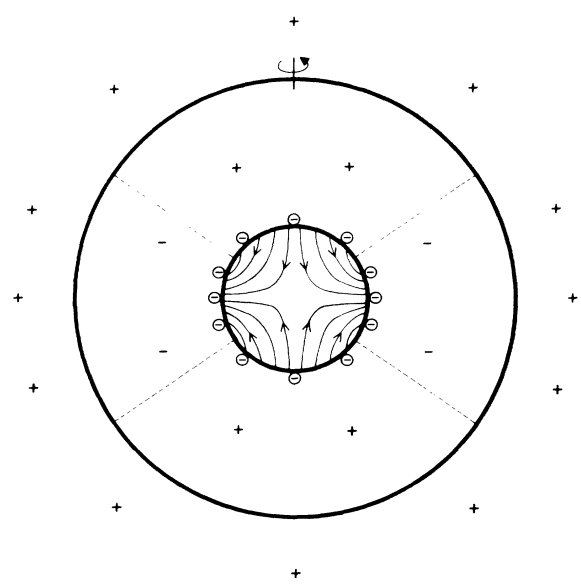

FIG. 2. Effective charge distribution for the electric field $\vec{E}^{II}_{(1)}$ induced by the rotating mass shell, and field lines of $\vec{E}^{II}_{(1)}$ within the small shell. The rotation-independent Coulomb-like field is not included here.

verse. If the outer shell is, in fact, just a shell and not the whole universe, and if the remaining universe is, in fact, at rest beyond it, we expect from Thirring's solution that the gravitationally *effective* angular speed of the universe relative to the inner shell is $\vec{\Omega}$. Thus test monopoles released from rest relative to a frame rotating with angular velocity $\vec{\Omega}$ around the charged sphere in the electromagnetic Thirring problem should experience the same accelerations relative to this frame as they would when released from rest at corresponding points in an inertial frame in the field of the same charged shell rotating at angular speed $-\vec{\Omega}$.

But this expectation turns out to be true only for particles released (a) on the rotation axis or (b) anywhere *inside* the inner shell. Consider a typical point *between* the shells, say on the equatorial plane at distance r from the center, and let a magnetic monopole be released there with equatorial velocity $-\Omega r$ in the sense of the rotation of the inner shell. At the point in question, the predominant magnetic field $\vec{B}^{I}$ is $-qa^2\Omega/3r^3$ in the direction of the rotation axis and the predominant electric field $\vec{E}_{(0)}$ is q/r^2 in the radial direction. A magnetic monopole released at equatorial velocity $-\Omega r$ therefore experiences an acceleration $-(q\Omega/r) \times[(a^2/3r^2)-1]$ in the axial direction, and this is not even close to the acceleration $-q\Omega a^2/3r^3$ expected on Machian grounds.

The question remains why the exact magnetic dipole field, as expected on Machian grounds for the dragged frame S', should in fact arise in the "wrong" frame, S. It is possible that in order to obtain a true Mach equivalent of the system consisting of a small rotating charged shell and a concentric spherical fraction of the universe, it is necessary to represent that fractional universe by a mass shell rotating around the charged shell and, *in addition*, by a sink at infinity[21] (for the electric field lines) which rotates at the Thirring fractional angular velocity corresponding to the mass shell. Still, even if it may lack true Machian interest (i.e., be "Mach-neutral"), the situation discussed by us is the one relevant to possible experimental checks of the general-relativistic influence of rotating matter on the field of a charged shell. One could, however, compromise with a realistic experiment consisting of a *pair* of concentric spheres of unequal radii, carrying equal and opposite electric charges,[22] and then the need for a sink at infinity would not arise. Furthermore, without additional labor we can deduce from our results that in the double-shell experiment the *right* dipolar magnetic field would arise in the *right* reference frame S'. For in the linear approximation electromagnetic solutions can be superimposed, and thus the magnetic field due to the two charged shells in the frame S is the resultant of their two separate magnetic fields, while the electric field lacks a zeroth-order component except between the shells; however, as is clear from the standard Lorentz transformation of the electromagnetic field, only a zeroth-order electric field in S can cause the first-order magnetic field in S' to differ from that in S, and thus our assertion is established.

The Electric Field

It is clear from our derivation that the field (4.19) is due to the quadrupolar gravitational field inside the rotating shell. This field can be understood as the Newtonian gravitational field due to the mass-quadrupole moment which the rotating shell possesses by virtue of the special-relativistic mass increase of its moving parts, as indicated already by Thirring[1] and elaborated by Bass and Pirani.[11] In fact, it is easy to see that if, following Bass and Pirani, we modify the rest-mass distribution of the rotating shell so that its relative mass (with respect to the inertial frame at infinity) is independent of position, then the electric field inside the charged shell vanishes. Also, it is clear that the charged shell will be intrinsically spherical, as judged by the total metric to first order in k, if the rotating shell has no mass-quadrupole moment; otherwise it will be an oblate spheroid and therefore would have an electric-quadrupole field in its interior according to ordinary electrostatics. Should one, therefore, conclude that the field (4.19) is of no real interest since it can be understood without general relativity? We think not, for the combination of special relativity (mass increase), Newtonian gravitation theory, and Maxwellian electrodynamics used in the above argument, though plausible, is not a consistent theory, whereas the Einstein-Maxwell theory is consistent; it is therefore useful to have one more (physically interpretable) solution of the latter theory, irrespective of whether or not some of its features can be made plausible in an apparently elementary way.

In conclusion, we remark that the fields $\vec{B}^{I}$ and $\vec{B}^{II}$ are unchanged if, instead of allowing the inner shell to affect the metric, one treats it as a test shell. The field $\vec{E}^{I}_{(1)}$, however, arises only from the non-test character of the inner shell, and the field $\vec{E}^{II}_{(1)}$ is affected by it.

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[1]H. Thirring, Z. Physik 19, 33 (1918); *ibid.* 22, 29(E) (1921).

[2]The ratio of the drag angular velocity to the angular velocity of a star of radius r and mass M is of the order of kM/rc^2, and thus of the order of unity when r is near the Schwarzschild radius $2kM/c^2$.

[3]See, for example, H. Weyl, Naturwiss. 12, 197 (1924).

[4]Actually, not quite: There was a difficulty with the centrifugal field. This has been resolved in Ref. 13.

[5]W. Rindler, *Essential Relativity* (Van Nostrand-Reinhold, New York, 1969), p. 16.

[6]J. Ehlers and W. Rindler, Phys. Letters 32A, 257 (1970). Erratum: In the formula for $\vec{B}$ and in the third line of the final column of text, replace $\vec{\omega}'$ by $-\vec{\omega}'$. Also, the assertion that $\vec{B}$ is Machian is now modified. (Note: The notation differs somewhat from that of the present paper.)

[7]J. M. Cohen, Phys. Rev. 148, 1264 (1966).

[8]Equivalent equations were derived, by somewhat more devious calculations, by W. Alexandrow, Ann. Physik 65, 675 (1921); Z. Physik 22, 593 (1921).

[9]See, for example, R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill, New York, 1965), Sec. 13-1.

[10]See, for example, L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley, Reading, Mass., 1951), Sec. 11-10.

[11]L. Bass and F. A. E. Pirani, Phil. Mag. 46, 850 (1955).

[12]H. Hönl and A.-W. Maue, Z. Physik 144, 152 (1956).

[13]J. Ehlers (in preparation).

[14]See Ref. 10, Sec. 5-6.

[15]See V. Fock, *The Theory of Space, Time, and Gravitation* (Pergamon, New York, 1959), p. 132.

[16]See Ref. 15, Eq. (58.01).

[17]If such continuation were our *only* purpose, it could more easily be accomplished in isotropic coordinates; for these, see Ref. 18, Eq. (29).

[18]See J. C. Graves and D. R. Brill, Phys. Rev. 120, 1507 (1960).

[19]A convenient method for finding the stress-energy tensor of a thin shell from a discontinuous metric has been given by A. Papapetrou and A. Hamoui, Ann. Inst. H. Poincaré A9, 179 (1968).

[20]K.-D. Hofmann, Z. Physik 166, 567 (1962).

[21]We thank Professor Dieter Brill for discussion of this point.

[22]In an early version of our problem, L. I. Schiff [Proc. Nat. Acad. Sci. U. S. 25, 391 (1939)] considered two such *test* shells at rest in an inertial frame. An observer orbiting this system detects no electromagnetic field, while a stationary observer sees a magnetic field if the spheres rotate. This local asymmetry was ascribed along Machian lines to the presence of the distant cosmos. A calculation for the (vanishing) magnetic field was then carried out in a *rotating* frame, and the currents which in that frame formally cancel the currents caused by the shells were identified with those induced by the rotating cosmos. Though interesting, this work was, of course, not an application of general relativity, and made no testable predictions.

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Reversible Transformations of a Charged Black Hole*

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A formula is derived for the mass of a black hole as a function of its "irreducible mass," its angular momentum, and its charge. It is shown that 50% of the mass of an extreme charged black hole can be converted into energy as contrasted with 29% for an extreme rotating black hole.

The mass m of a rotating black hole can be increased and (Penrose[1]) decreased by the addition of a particle and so can its angular momentum L; but (Christodoulou[2]) there is no way whatsoever to decrease the irreducible mass m_{ir} in the equation

$$E^2 - p^2 = m^2 = m_{ir}^2 + L^2/4m_{ir}^2 \qquad (1)$$

for the mass of a black hole. The concept of reversible (m_{ir} unchanged) and irreversible transformations (m_{ir} increases), which was introduced and exploited by one of us to obtain this result, is extended here to the case where the object also has charge, to yield the following four conclusions:

(1) The rest mass of a black hole is given in