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Transitions in Electromagnetic Fields of Arbitrary Intensity

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A nonperturbative method presented earlier for the interaction of electromagnetic radiation with bound quantum systems is refined and extended. A general T matrix is derived which is valid for arbitrarily high field intensity, subject to certain conditions on the frequency of the electromagnetic field. An example is done in detail for a $1s$-$2s$ transition in a hydrogen atom. It is shown that after a perturbative low-intensity region, the transition amplitude experiences oscillations which pass through zero as the field intensity increases. Qualitative arguments are also given to show why this is to be expected, and why it is inexplicable in perturbation theory. It is found that high-order processes are more important than low-order processes when the intensity is high, and in the hydrogen-atom example it is shown that the transition amplitude has peaks as a function of the order of the process. However, for sufficiently high orders, it is shown that there is an eventual exponential decline in the transition amplitude as the order of the process gets very large. Simple results are obtained for the low-intensity limit with any number of photons and for the high-intensity, large-photon-number limit. The last result should be useful, for example, in calculating optical transitions caused by intense microwave radiation.

I. INTRODUCTION

The purpose of this paper is to refine and extend considerably a general, nonperturbative, analytical method in electrodynamics.[1,2] The principal aim of the method is to provide a means of calculating transitions in quantum systems subjected to electromagnetic fields of arbitrarily high intensity. Within certain constraints involving the frequency of the field, this aim can be accomplished. The formalism is developed in this paper and applied to a simple example of a hydrogen-atom transition. Further applications will be presented in later papers.

The justification for the method is given in I.[1] In essence, it involves a unitary transformation (a momentum translation) which approximately removes the electromagnetic field from the problem when the energy of a single photon of the field is small as compared to characteristic transition energies of the bound system under consideration. The transformation is simply

$$\Psi(\vec{x}, t) = \exp(ie\vec{A}\cdot\vec{x})\Phi(\vec{x}, t), \tag{1}$$

where $\vec{A}$ is the vector potential of the electromagnetic field, taken to be an external plane-wave field. We set $\hbar = c = 1$ in this paper, and use a gauge with $A^0 = 0$ and $\vec{\nabla}\cdot\vec{A} = 0$. If, in Eq. (1), Φ is the wave function for the system with no external electromagnetic field, then Ψ is the approximate solution with the field present. Conditions for validity of the approximation are

$$\omega/E \ll 1 \tag{2a}$$

and

$$eaa_0(\omega/E) \ll 1, \tag{2b}$$

where ω is the frequency of the electromagnetic field, a is the amplitude of $\vec{A}$, E is a characteristic energy of the bound system, and a_0 is the "size" (e.g., the Bohr radius in the case of an atom) of the bound system.

One can also view the "momentum translation" approximation as a gauge transformation of the second kind, given by

$$A^\mu \to \bar{A}^\mu = A^\mu + \partial^\mu \Lambda\,, \quad \Psi \to \bar{\Psi} = e^{-ie\Lambda}\Psi\,,$$

with

$$\Lambda = \vec{A}\cdot\vec{x} = -A_\mu x^\mu\,.$$

The index notation of these expressions is relativistic, with a time-favoring real metric. In the new gauge we have

$$\bar{\vec{A}} = -(\vec{\nabla}A_i)x_i\,, \quad \bar{A}^0 = (\partial_t A_i)x_i = -\vec{\mathcal{E}}\cdot\vec{x}\,, \tag{3}$$

where $\vec{\mathcal{E}}$ is the electric field vector. Thus the transformed $\bar{A}^\mu$ involves only derivatives of the original A^μ potential. Throughout this paper we shall take the electromagnetic field to be a plane-wave field of wavelength long as compared to the size of the bound system. We shall use explicitly

$$\vec{A} = a\vec{\epsilon}\cos(\vec{k}\cdot\vec{x} - \omega t) \approx a\vec{\epsilon}\cos\omega t \tag{4}$$

with $\vec{\epsilon}$ the polarization vector and $\vec{k}$ the propagation vector of the field, where $|\vec{k}| = \omega$. Derivatives of $\vec{A}$ are of the order of magnitude of ω times $\vec{A}$. Thus, from Eq. (3), the order of magnitude of the components of $\bar{A}^\mu$ is given by ωa_0 times the magnitude of $\vec{A}$. The condition $\omega a_0 \ll 1$ is very well satisfied even when $\omega/E \approx 1$. When $\omega/E \ll 1$, as we require here, then ωa_0 is extremely small. Thus the effect of the gauge transformation is to replace the field $\vec{A}$ by a transformed field of much smaller amplitude, albeit at the expense of introducing $\bar{A}^0 \neq 0$.

The dimensionless product eaa_0 constitutes an intensity parameter for the electromagnetic field with respect to the bound system of radius a_0. The square of this parameter has a simple physical interpretation, since[3]

$$(\tfrac{1}{2}eaa_0)^2 = \rho\lambda\bar{\lambda}_e a_0 \equiv z\,, \tag{5}$$

where ρ is the density of photons; λ is the photon wavelength; and $\bar{\lambda}_e$ is the electron Compton wavelength, $\bar{\lambda}_e = 1/m$. Thus, the parameter z gives the number of photons contained within an interaction volume $\lambda\bar{\lambda}_e a_0$, characteristic of the radiation, the interaction length of the electron, and the size of the bound system. The parameter z is such that $z \ll 1$ for most situations. However, $z > 1$ is possible in certain environments, such as the focus of the output of a large mode-locked laser. It will be seen below that when $z \ll 1$, only lowest-order photon processes need be considered, whereas when $z \approx 1$, higher-order processes can be competitive with the lowest-order process. When $z \gg 1$, the lowest-order process can become relatively unimportant.

It is desired to apply the momentum-translation approximation of I to electromagnetic environments with arbitrarily large values of eaa_0. From Eq. (2b), it can be seen that this is possible if ω/E is sufficiently small – thus satisfying (2a) as well. Subject to the restrictions (2a) and (2b), the momentum-translation approximation is applicable to the case of arbitrary intensity, but the results given in I for the example of $1s$-$2s$ transitions in hydrogen were in the form of an infinite series in z with radius of convergence $z < (\frac{3}{4})^2$. Hence, an infinite-series result of this sort is useless for the case of arbitrarily high intensity unless an analytic continuation can be found. The results of such an analytic continuation for the $1s$-$2s$ problem in hydrogen were given in II.[2]

In this paper the momentum-translation method of I is recast in such a way that closed-form analytical results can be achieved without the appearance of a power series in the intensity parameter. The formalism is established in Sec. II. Section III contains the application of this formalism to the $1s$-$2s$ transition in hydrogen. An examination of the behavior of the transition amplitude in the hydrogen-atom example for a wide range of values of field intensity, and analytical approximations for the limiting cases of small intensity and of large photon number combined with high intensity, are given in Sec. IV. Finally, Sec. V is a discussion of some of the more interesting qualitative features which emerge. For instance, it is possible to explain qualitatively why peaks appear in the transition amplitude as a function of intensity. That is, although it is obvious that the transition probability should increase as field intensity increases from zero, it is also possible to understand why, after a certain intensity is reached, further increase of intensity causes the transition probability to decline.

II. T MATRIX

The S matrix which gives the probability amplitude for a transition from a state Φ_i at $t = -\infty$ to a state Φ_f at $t = +\infty$ is

$$(S-1)_{fi} = -i\int_{-\infty}^{\infty} dt(\Phi_f, H'\Psi_i)\,, \tag{6}$$

where H' is the interaction Hamiltonian which causes the transition, the Φ wave functions are solutions for the case where $H' = 0$, and the Ψ wave functions are solutions in the presence of the full Hamiltonian $H = H_0 + H'$. The Ψ_i which appears in Eq. (6) will be replaced by the approximation of Eq. (1).

A crucial stage in the calculation is the introduction of the approximation

$$(\Phi_f, H' e^{ie\vec{A}\cdot\vec{x}}\Phi_i) \approx (E_i - E_f)(\Phi_f, e^{ie\vec{A}\cdot\vec{x}}\Phi_i) \tag{7}$$

which is developed in I. The presumption is made throughout that H_0 is time-independent, so that the Φ functions represent stationary states with well-defined energy eigenvalues. The approximation in (7) follows from neglecting the second term on the right-hand side in the commutator expression

$$[H_0, e^{ie\vec{A}\cdot\vec{x}}] = -H'e^{ie\vec{A}\cdot\vec{x}} + e^{ie\vec{A}\cdot\vec{x}}[H_I - e(\partial_t\vec{A})\cdot\vec{x}], \tag{8}$$

where H_I is given by Eq. (12) of I. The condition required in I to justify the neglect of the last term in Eq. (8) above is not the most straightforward and not always appropriate [see Eq. (32) of I]. A more appropriate condition is simply to require that the magnitude of $H_I - e(\partial_t\vec{A})\cdot\vec{x}$ be small as compared to the magnitude of H'. The dominant term in $H_I - e(\partial_t\vec{A})\cdot\vec{x}$ is $em^{-1}(\partial_i A_j)x_j p_i$. First we will compare this with the usual $em^{-1}\vec{A}\cdot\vec{p}$ term of H'. The order-of-magnitude effect of taking a derivative of $\vec{A}$ is to multiply it by ω. In a bound system, the coordinate $\vec{x}$ is confined to a region of the magnitude of a_0. The ratio of the order of magnitude of $em^{-1}(\partial_i A_j)x_j p_i$ to that of $em^{-1}\vec{A}\cdot\vec{p}$ is simply ωa_0. As before, we know that $\omega a_0 \ll 1$. Hence, the second term on the right-hand side of (8) is extremely small as compared to the first term. The only possibility we have not accounted for here is when $em^{-1}\vec{A}\cdot\vec{p}$ is not the dominant term of H'; that is, when

$$|e^2A^2/2m| \geq |e\vec{A}\cdot\vec{p}/m| .$$

This will happen when $eaa_0 \geq 1$. In these circumstances, Eq. (32) of I is the appropriate constraint to use, and it gives a condition even less stringent than $\omega a_0 \ll 1$. Hence Eq. (7) is an excellent approximation in all circumstances.

Equation (6) can now be written

$$(S-1)_{fi} = -i(E_i - E_f)\left(\phi_f, \int_{-\infty}^{\infty} dt \exp[i(E_f - E_i)t + iea\vec{x}\cdot\vec{\epsilon}\cos\omega t]\phi_i\right), \tag{9}$$

where Eq. (4) and $\Phi(\vec{x}, t) = \phi(\vec{x})e^{-iEt}$ have been used. The integral over time in Eq. (9) will yield an energy-conserving δ function immediately if the relation

$$e^{iz\cos\theta} = \sum_{N=-\infty}^{\infty} i^N e^{iN\theta} J_N(z) \tag{10}$$

is used. The time integral then becomes

$$\int_{-\infty}^{\infty} dt \exp[i(E_f - E_i)t + iea\vec{x}\cdot\vec{\epsilon}\cos\omega t] = 2\pi \sum_{N=-\infty}^{\infty} i^N J_N(ea\vec{x}\cdot\vec{\epsilon})\delta(E_f - E_i + N\omega) . \tag{11}$$

With the replacement $N \to -N$, and the relations $J_{-N} = (-)^N J_N$, $(-i)^{-N} = i^N$, the right-hand side of Eq. (11) can equally well be written as

$$2\pi \sum_{N=-\infty}^{\infty} i^N J_N(ea\vec{x}\cdot\vec{\epsilon})\delta(E_f - E_i - N\omega) .$$

We shall simply select that form of Eq. (11) which leads to a positive value of N upon application of the δ function. For a process corresponding to emission or absorption of N photons of energy ω, we have

$$(S-1)_{fi}^{(N)} = -2\pi i\delta(E_f - E_i \pm N\omega)i^N(E_i - E_f) \times(\phi_f, J_N(ea\vec{x}\cdot\vec{\epsilon})\phi_i) . \tag{12}$$

As conventionally defined, the T matrix is then given by

$$T_{fi}^{(N)} = i^N(E_i - E_f)(\phi_f, J_N(ea\vec{x}\cdot\vec{\epsilon})\phi_i) . \tag{13}$$

The above results hold when the only electromagnetic field present is the intense field, defined by the potential $\vec{A}$. However, in most situations the energy difference between initial and final states of the bound system cannot be matched by any integral number of photons of energy ω. The δ function in (12) would then cause the transition probability to vanish. Under these circumstances, transitions can occur only if another electromagnetic field appears in an emission process, or is present in an absorption process. We will take this field to be a low-intensity, plane-wave field of frequency $\hat{\omega}$, described by the vector potential $\hat{\vec{A}}$. This additional field can be accommodated in the formalism by the substitution

$$e\vec{A}\cdot\vec{x} \to e\vec{A}\cdot\vec{x} + e\hat{\vec{A}}\cdot\vec{x}$$

in Eq. (7), where we take

$$\hat{\vec{A}} = \hat{a}\hat{\vec{\epsilon}}\cos(\hat{\omega}t + \alpha) ,$$

and α is a displacement in phase between the $\vec{A}$ and $\hat{\vec{A}}$ fields. $\hat{\vec{A}}$ is to be retained to first order only, and so instead of employing Eq. (10) for the $\hat{\vec{A}}$ field we use instead the expansion

$$\exp[ie\hat{a}\vec{x}\cdot\hat{\vec{\epsilon}}\cos(\hat{\omega}t+\alpha)] \approx 1 + \tfrac{1}{2}ie\hat{a}\vec{x}\cdot\hat{\vec{\epsilon}}\,e^{i(\hat{\omega}t+\alpha)} + \tfrac{1}{2}ie\hat{a}\vec{x}\cdot\hat{\vec{\epsilon}}\,e^{-i(\hat{\omega}t+\alpha)} . \tag{14}$$

The first term on the right-hand side of Eq. (14) leads to the same result as if no $\hat{\vec{A}}$ were present, and has thus already been considered. We address ourselves only to the remaining two terms. The analog of Eq. (11) is now

$$\int_{-\infty}^{\infty} dt \exp[i(E_f - E_i)t + iea\vec{x}\cdot\vec{\epsilon}\cos\omega t + ie\hat{a}\vec{x}\cdot\hat{\vec{\epsilon}}\cos(\hat{\omega}t+\alpha)]$$
$$= \pi i e\hat{a}\vec{x}\cdot\hat{\vec{\epsilon}}\, e^{i\alpha}\sum i^N J_N(ea\vec{x}\cdot\vec{\epsilon})\delta(E_f - E_i \pm N\omega + \hat{\omega}) + \pi i e\hat{a}\vec{x}\cdot\hat{\vec{\epsilon}}\, e^{-i\alpha}\sum i^N J_N(ea\vec{x}\cdot\vec{\epsilon})\delta(E_f - E_i \pm N\omega - \hat{\omega}) .$$

Hence for a process involving absorption or emission of N photons of energy ω, and absorption or emission of one photon of energy $\hat{\omega}$,

$$(S-1)_{fi}^{(N,1)} = -2\pi i\delta(E_f - E_i \pm N\omega \pm \hat{\omega})\tfrac{1}{2} i^{N+1}(E_i - E_f) \times e^{\pm i\alpha}(e\hat{a})(\phi_f, \vec{x}\cdot\hat{\vec{\epsilon}} J_N(ea\vec{x}\cdot\vec{\epsilon})\phi_i), \qquad (15)$$

which yields the T matrix

$$T_{fi}^{(N,1)} = \tfrac{1}{2} i^{N+1}(E_i - E_f)\, e^{\pm i\alpha}(e\hat{a})(\phi_f, \vec{x}\cdot\hat{\vec{\epsilon}} J_N(ea\vec{x}\cdot\vec{\epsilon})\phi_i). \qquad (16)$$

III. APPLICATION TO HYDROGEN ATOM

The object now is to calculate the T matrices of Eqs. (13) and (16) when ϕ_i, ϕ_f are solutions of the hydrogen-atom problem. A constraint on the matrix elements follows immediately from parity considerations. With P as the parity operator, we have $P\phi = (-)^l\phi$ and $PJ_N(ea\vec{x}\cdot\vec{\epsilon})P^{-1} = (-)^N J_N(ea\vec{x}\cdot\vec{\epsilon})$, so that

$$(\phi_f, J_N\phi_i) = (-)^{N+l_f+l_i}(\phi_f, J_N\phi_i) \qquad (17)$$

and

$$(\phi_f, \vec{x}\cdot\vec{\epsilon} J_N\phi_i) = (-)^{N+1+l_f+l_i}(\phi_f, \vec{x}\cdot\vec{\epsilon} J_N\phi_i), \qquad (18)$$

where l is the angular momentum quantum number.

A. Problem with Intense Field Only

First we consider the single-field problem given by Eq. (13). The integral representation

$$J_N(ea\vec{x}\cdot\vec{\epsilon}) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \exp(-iN\theta + iea\vec{x}\cdot\vec{\epsilon}\sin\theta) \qquad (19)$$

gives rise to a matrix element whose angular and radial parts can be separated conveniently. The matrix element is $(\phi_f, e^{iea\vec{x}\cdot\vec{\epsilon}\sin\theta}\phi_i)$, which from Eq. (38) of I is

$$(\phi_f, e^{iea\vec{x}\cdot\vec{\epsilon}\sin\theta}\phi_i) = \sum_{l=0}^{\infty}(2l+1)i^l[(2l_i+1)(2l_f+1)]^{1/2} \times (-)^{m_i}\begin{pmatrix} l & l_i & l_f \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & l_i & l_f \\ 0 & m_i & -m_i\end{pmatrix} \times \int_0^{\infty} r^2 dr\, R_f^*(r)R_i(r)j_l(ear\sin\theta). \qquad (20)$$

The properties of the Wigner 3-j coefficients give

$$l + l_i + l_f = \text{even integer}, \quad |l_i - l_f| \le l \le l_i + l_f .$$

Since, from Eq. (17), $N + l_i + l_f$ must also be an even integer, we see that N and the index l always have the same parity.

If either $l_i = 0$ or $l_f = 0$, the sum over l in (20) reduces to a single term. We shall calculate the special case of transitions between the 1s and 2s states in hydrogen, so that $l_i = l_f = 0$. Equation (20) simplifies to

$$(\phi_f, e^{iea\vec{x}\cdot\vec{\epsilon}\sin\theta}\phi_i) = \int_0^{\infty} r^2 dr\, R_{2s}(r)R_{1s}(r)j_0(ear\sin\theta).$$

This integral can be evaluated to give

$$(\phi_f, e^{iea\vec{x}\cdot\vec{\epsilon}\sin\theta}\phi_i) = \tfrac{8}{3}\sqrt{2}\,(\tfrac{2}{3})^3 y^2\sin^2\theta(1+y^2\sin^2\theta)^{-3},$$

where we have introduced the definition

$$y \equiv \tfrac{2}{3} eaa_0 . \qquad (21)$$

The energy difference between the 1s and 2s levels is

$$E_i - E_f = \pm\frac{3}{8}\frac{1}{ma_0^2},$$

where the ambiguous sign arises from the fact that we have not specified which state is initial and which final. The T matrix of Eq. (13) is now

$$T_{fi}^{(N)} = \pm i^N \frac{1}{2\pi}\sqrt{2}\left(\frac{2}{3}\right)^3 \frac{y^2}{ma_0^2} \times \int_0^{2\pi} d\theta\, e^{-iN\theta}\sin^2\theta(1+y^2\sin^2\theta)^{-3}. \qquad (22)$$

The integral in (22) can be rewritten in a convenient fashion as

$$\int_0^{2\pi} d\theta\, e^{-iN\theta} y^2\sin^2\theta(1+y^2\sin^2\theta)^{-3} = \int_0^{2\pi} d\theta\, e^{-iN\theta}(1+y^2\sin^2\theta)^{-2} - \int_0^{2\pi} d\theta\, e^{-iN\theta}(1+y^2\sin^2\theta)^{-3}. \qquad (23)$$

Integrals of the type on the right-hand side of (23) are evaluated in Appendix A. The integrals vanish unless N is even, which is consistent with Eq. (17) with $l_i = l_f = 0$. With Eqs. (A7) and (A8) from Appendix A inserted in Eq. (23), and (23) substituted into (22), the T matrix is found to be

$$T_{fi}^{(N)} = \pm i^N \frac{\sqrt{2}}{3^3}\frac{1}{ma_0^2}\frac{1}{B^{5/2}}\left(\frac{y}{B^{1/2}+1}\right)^N \times [B^2 + NB^{3/2} - (N^2-2)B - 3NB^{1/2} - 3], \qquad (24)$$

with B defined as $B=1+y^2$. This result corresponds to Eq. (46) of I with the observations that $y=\frac{2}{3}eaa_0$ and that the N used here was denoted $2n$ in I. The T matrix in I is given as a power series in y^2 with a radius of convergence $y^2<1$. Equation (24) makes clear the reason for this radius of convergence. It arises from the essential singularity at $B=0$, i.e., at $y^2=-1$. Because $y^2=-1$ is in an unphysical region, Eq. (24) gives the analytical continuation of the results of I to arbitrary real, positive values of y^2 – i.e., Eq. (24) holds for any intensity, no matter how large, as long as the conditions (2a), $\omega/E\ll 1$, and (2b), $y\,\omega/E\ll 1$, are satisfied.

The fact that (24) does exactly correspond to Eq. (46) of I can be verified either by development of Eq. (24) in a power series in y^2, or by performing the analytical sum of the result of I. The latter procedure is accomplished in Appendix B.

B. Combined Intense and Weak Fields

In most physical problems an integral number of photons from the intense field will not be resonant with the transition energy of the bound system, so an evaluation of Eq. (16) for hydrogen is more interesting than (13). For the $l_i=l_f=0$ case, we note that Eq. (18) requires N to be odd. We again employ Eq. (19) and find that

$$(\phi_f,\vec{x}\cdot\hat{\vec{\epsilon}}J_N(ea\vec{x}\cdot\vec{\epsilon})\phi_i)=\frac{1}{2\pi}\int_0^{2\pi}d\theta\,e^{-iN\theta}(\phi_f,\vec{x}\cdot\hat{\vec{\epsilon}}\,e^{iea\vec{x}\cdot\vec{\epsilon}\sin\theta}\phi_i)\,, \tag{25}$$

where the matrix element on the right-hand side can be written

$$(\phi_f,\vec{x}\cdot\hat{\vec{\epsilon}}\,e^{iea\vec{x}\cdot\vec{\epsilon}\sin\theta}\phi_i)\approx i\vec{\epsilon}\cdot\hat{\vec{\epsilon}}\int_0^\infty r^3\,dr\,R_{2s}(r)R_{1s}(r)j_1(ear\sin\theta)$$

as shown in the Appendix of I and in Eq. (49) of I. With hydrogenic wave functions substituted, the result is

$$(\phi_f,\vec{x}\cdot\hat{\vec{\epsilon}}\,e^{iea\vec{x}\cdot\vec{\epsilon}\sin\theta}\phi_i)\approx i\vec{\epsilon}\cdot\hat{\vec{\epsilon}}a_0 16\sqrt{2}\,(\tfrac{2}{3})^4 y\sin\theta\times[\tfrac{2}{3}(1+y^2\sin^2\theta)^{-3}-(1+y^2\sin^2\theta)^{-4}]. \tag{26}$$

Equation (16) together with Eqs. (25) and (26) gives

$$T_{fi}^{(N,1)}=\pm e^{\pm i\alpha}i^{N+2}(E_i-E_f)\vec{\epsilon}\cdot\hat{\vec{\epsilon}}\,\hat{y}\,16\sqrt{2}\left(\frac{2}{3}\right)^3\frac{1}{4\pi}\times\int_0^{2\pi}d\theta\,e^{-iN\theta}\left(\frac{2}{3}\frac{y\sin\theta}{(1+y^2\sin^2\theta)^3}-\frac{y\sin\theta}{(1+y^2\sin^2\theta)^4}\right) \tag{27}$$

with $\hat{y}$ defined in a fashion analogous to y as $\hat{y}\equiv\frac{2}{3}e\hat{a}a_0$. The integrals of Appendix A can be employed again. The combinations which occur follow from the fact that

$$e^{-iN\theta}\sin\theta=(2i)^{-1}(e^{-i(N-1)\theta}-e^{-i(N+1)\theta})\,,$$

along with the knowledge that N is odd here. Appendix A gives results where the exponents in the integrals are of the form $e^{\pm i2j\theta}$. Equation (27) together with (A8) and (A9) yields the result

$$T_{fi}^{(N,1)}=\pm e^{\pm i\alpha}i^{N+1}\frac{1}{\sqrt{2}3^3}\frac{\vec{\epsilon}\cdot\hat{\vec{\epsilon}}\hat{y}}{ma_0^2}\frac{1}{B^{7/2}}\left(\frac{y}{B^{1/2}+1}\right)^N\times[NB^{5/2}+N^2B^2-N(N^2-7)B^{3/2}-6(N^2-2)B-15NB^{1/2}-15]. \tag{28}$$

This is the result that was employed in II. It corresponds to Eq. (54) of I in the sense of being an analytical continuation of that earlier result. This can be demonstrated either by an expansion of (28) in a power series in y^2 or by performing a closed-form sum of Eq. (54) of I following the method developed in Appendix B.

In discussing the physical implications of Eq. (28), we shall consider small values of N (like 1, 3, 5, etc.) as well as large N. Since we must have $\omega/(E_{2s}-E_{1s})\ll 1$, a small value of N implies either that we are considering induced emission from the $2s$ state with most of the energy carried off by a photon of energy $\hat{\omega}=E_{2s}-E_{1s}-N\omega$, or else we are considering excitation of the $1s$ level in the presence of an $\hat{\vec{A}}$ field as well as an $\vec{A}$ field, where the energy of an $\hat{\vec{A}}$ photon is as stated above. If N is large then we can consider as well processes where, for instance, $N\omega$ exceeds $E_{2s}-E_{1s}$, and an $\hat{\omega}$ photon is emitted into the final state of a $1s\to 2s$ excitation.

IV. GENERAL BEHAVIOR AND LIMITING CASES FOR THE HYDROGEN PROBLEM

A. Intensity Dependence in $N=1$ Case

Some of the general properties we wish to emphasize can be illustrated in simple fashion by the $N=1$ case of Eq. (28). We shall deal only with that part of (28) which refers to the intense field. For this purpose, we introduce a "reduced" transition amplitude:

$$\mathcal{T}_N=\frac{1}{8B^{7/2}}\left(\frac{y}{B^{1/2}+1}\right)^N[-NB^{5/2}-N^2B^2+N(N^2-7)B^{3/2}+6(N^2-2)B+15NB^{1/2}+15]. \tag{29}$$

The coefficient and the sign of (29) are chosen to be consistent with the notation of II. With $N=1$, Eq. (29) reduces to

$$\mathcal{T}_1 = (y/8B^{7/2})(-B^2 - 6B + 15) . \qquad (30)$$

Equation (30) has a zero at $y=0$ and at two other values of y which follow from the solution of the quadratic in Eq. (30). One solution gives a negative (unphysical) result. The other solution corresponds to a zero in $\mathcal{T}_1$ at

$$y = 2[(\tfrac{3}{2})^{1/2} - 1]^{1/2} = 0.948 .$$

Extrema can be found from

$$\frac{\partial}{\partial y}\mathcal{T}_1 = \frac{1}{8B^{9/2}}(2B^3 + 21B^2 - 120B + 105) = 0 . \qquad (31)$$

The cubic in Eq. (31) has one negative (unphysical) real root and two positive real roots in B corresponding to $y=0.341$ and $y=1.475$. Since $\mathcal{T}_1$ clearly approaches zero as B approaches infinity, the qualitative picture is now complete. $\mathcal{T}_1$ increases from zero at $y=0$, reaches a maximum at $y=0.341$, passes through a zero at $y=0.948$, has a minimum at $y=1.475$, and then approaches zero asymptotically as $y \to \infty$. All this is illustrated by the $N=1$ curve in Fig. 1, which gives the results of a numerical computation of Eq. (30). Figure 1 also shows results for other N values derived from Eq. (29).

The over-all conclusion to be drawn from the foregoing is that the transition amplitude obtained here is profoundly different from anything that perturbation theory would predict. A perturbative transition amplitude would start at zero for $y=0$, but would have no extremum and certainly no other zero values as intensity increased. Perturbation theory predicts what seems intuitively reasonable – as intensity increases, so does the transition probability. Yet we see here that there are intensity regions beyond each of the extrema where an increase in intensity actually leads to a decrease of transition probability. At this point we limit ourselves to pointing out that higher-order processes become important in those regions where

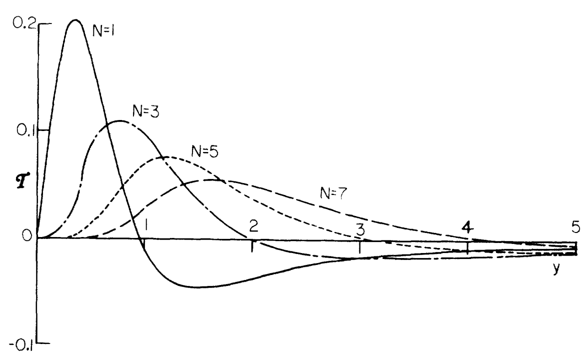

FIG. 1. "Reduced" transition amplitude $\mathcal{T}$ as a function of the intensity parameter y for several values of the photon multiplicity N.

the $N=1$ process is declining, so that other channels compete in the transition. In Sec. V of this paper this issue will be discussed in a more substantial way.

B. Low-Intensity Limit

The low-intensity case is described by $y \to 0$ or $B \to 1$. Equation (29) then reduces to

$$\mathcal{T}_N \to \tfrac{1}{8}(\tfrac{1}{2}y)^N (N+3)(N+1)^2 . \qquad (32)$$

Equation (32) shows the y^N behavior characteristic of perturbation theory. This behavior is evident in Fig. 1, where the $N=1$ curve starts linearly, the $N=3$ curve starts as a cubic, etc. However, for all of its simplicity, Eq. (32) has a great deal of content, especially for large values of N. In any but a first-order perturbation calculation, sums over intermediate states occur which are notoriously difficult to carry out. Equation (32) gives a simple, closed-form approximation for such intermediate-state summations for any order N for the $1s$-$2s$ transition in hydrogen.[4] Since we must have $y \ll 1$ for Eq. (32) to be valid, the transition amplitude diminishes as N increases, but the factor cubic in N tends to give greater importance to high-order processes than one might expect.

C. High-Order, High-Intensity Limit

The case where N and $B^{1/2}$ are both large is particularly interesting to analyze. One reason is that it is a domain completely inaccessible to perturbation theory. Another reason is that the results for this case can be analyzed in some detail, and seem to be reasonably accurate over a very broad range of values.

An essential step in this case involves the factor in the T matrix which is raised to the power N, that is,

$$\left(\frac{y}{B^{1/2}+1}\right)^N = \left(\frac{B^{1/2}-1}{B^{1/2}+1}\right)^{N/2} = \left(\frac{1-B^{-1/2}}{1+B^{-1/2}}\right)^{N/2} . \qquad (33)$$

Since we wish to consider both N and $B^{1/2}$ large, we set

$$N = \beta B^{1/2} . \qquad (34)$$

We can then write (33) in the suggestive form

$$\left(\frac{y}{B^{1/2}+1}\right)^N = \left(1 - \frac{\beta/2}{N/2}\right)^{N/2} \left(1 + \frac{\beta/2}{N/2}\right)^{-N/2} .$$

Hence, when we take the large-N limit, we have

$$\left(1 - \frac{\beta/2}{N/2}\right)^{N/2} \left(1 + \frac{\beta/2}{N/2}\right)^{-N/2} \underset{N\to\infty}{\sim} e^{-\beta} = e^{\ N/B^{1/2}} .$$

Thus we shall employ the approximation

$$\left(\frac{y}{B^{1/2}+1}\right)^N \approx e^{-N/B^{1/2}} \tag{35}$$

in Eq. (29). If in the square bracket in (29) we count N and $B^{1/2}$ as quantities of the same order, we find that the highest combined order is six (e.g., as in $NB^{5/2}$ or N^2B^2 or $N^3B^{3/2}$). Thus we achieve the result

$$\mathcal{T}_N \approx (8B^{7/2})^{-1} e^{-N/B^{1/2}} (-NB^{5/2} - N^2B^2 + N^3B^{3/2}) .$$

It is convenient to express this in terms of the parameter β of (34), whence we obtain

$$\mathcal{T}_N \approx (8B^{1/2})^{-1} e^{-\beta}(\beta^3 - \beta^2 - \beta) . \tag{36}$$

Equation (36) leads immediately to the existence of roots at $\beta = 0, \frac{1}{2}(1 \pm \sqrt{5})$. The only one of these roots which is physical and consistent with the restrictions that N and $B^{1/2}$ are large is

$$\beta = \tfrac{1}{2}(1 + \sqrt{5}) = 1.618 . \tag{37}$$

Extrema as a function of β can also be found readily from Eq. (36). We find that

$$\frac{\partial}{\partial\beta}[e^{-\beta}(\beta^3 - \beta^2 - \beta)] = e^{-\beta}(-\beta^3 + 4\beta^2 - \beta - 1) ,$$

so that extrema are located at the solutions of the cubic

$$\beta^3 - 4\beta^2 + \beta + 1 = 0 . \tag{38}$$

The solutions of (38) are $\beta = -0.38$, 0.73, 3.65. The first of these solutions is unphysical. Hence extrema occur when

$$\beta = 0.73, \ 3.65 . \tag{39}$$

Figure 1 shows that Eqs. (37) and (39) give reasonably accurate predictions for those zeros and extrema shown in the figure which involve relatively large values of N and $B^{1/2}$.

The predictions we obtain here are more striking when the reduced transition amplitude is plotted as a function of N, as in Fig. 2 (where we make the convenient presumption that N is a continuous variable, even though only odd integer values of N are physical). Just as perturbation theory is unambiguous about predicting increasing transition probabilities with increasing intensity, so also perturbation theory is quite clear in forecasting that low-order allowed processes dominate higher-order allowed processes. Figure 2 shows such behavior only for the $y = 0.1$ case. For the higher intensities shown in the figure, some process more complex than the simplest becomes dominant. The features of one zero and two extrema indicated by Eqs. (37) and (39) are clearly shown by the curves in Fig. 2 for $y = 5$ and $y = 10$, and the explicit predictions of (37) and (39) are very accurately borne out by these two curves.

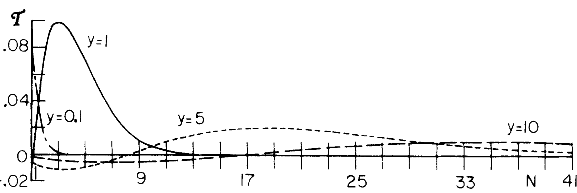

FIG. 2. "Reduced" transition amplitude $\mathcal{T}$ as a function of photon multiplicity N for several values of the intensity parameter y. The multiplicity N is regarded as a continuous parameter for convenience. Only odd-integer values of N are physical.

V. DISCUSSION

The example of the $1s$-$2s$ transition in a hydrogen atom presented here is an analytically simple case intended primarily to illustrate novel qualitative features which are obscure or totally absent in a perturbation treatment. An important point is that perturbation theory has served for a long time as a reliable guide to the intuition. When intense electromagnetic fields are present, we now see that the old intuitions are totally misleading. Some of the behavior we have discussed is specific to the $1s$-$2s$ transition in hydrogen, but most of the novel features we have emphasized are of general applicability.

A major new feature which is quite general is the relative importance of high-order transitions. The appearance of peaks in the transition probability for certain values of N when intensity is high is not a universal characteristic; but a trend towards "flatness" of the transition probability as a function of N at high intensity is true in general. Thus the notion of low-order dominance fails entirely at high field intensity. However, as photon multiplicity gets large as compared to y, there is an eventual exponential decline in transition amplitude, as exemplified by the $\exp(-N/B^{1/2})$ factor (or $e^{-\beta}$) in Eq. (36).

Other general nonperturbative characteristics appear in the intensity dependence of the transition amplitude. One general feature is the eventual trend towards zero of the transition amplitude as the intensity becomes very large. This effect can be thought of as a depletion effect, since if one calculates the projection of the momentum-translation wave function onto the corresponding unperturbed wave function, the projection oscillates in time but gets progressively smaller on the average as $y \to \infty$.

Another aspect of the intensity dependence which appears to be universal is the occurrence of a peak in the transition amplitude as a function of intensity. It is possible to see in a qualitative way why this should happen. Consider the matrix element which appears on the right-hand side of Eq. (7). If there is no applied electromagnetic field ($\vec{A}=0$), then

$$(\Phi_f, \Phi_i) = 0$$

since we consider nontrivial transitions. If the field is weak, we can expand the exponential containing the field and obtain

$$(\Phi_f, e^{ie\vec{A}\cdot\vec{x}}\Phi_i) \approx (\Phi_f, \Phi_i) + (\Phi_f, ie\vec{A}\cdot\vec{x}\Phi_i), \qquad (40)$$

where the first term is zero and the second term is the usual first-order dipole matrix element. Clearly, the right-hand side of (40) can never exhibit extrema as a function of $|\vec{A}|$ (i.e., as a function of y). However, the left-hand side of (40) has an oscillatory factor which yields a zero result when $y=0$, and could conceivably lead to an extremum when y is such that in some average sense $e\vec{A}\cdot\vec{x}=\pi/2$. We might then expect a zero result again when $e\vec{A}\cdot\vec{x}=\pi$. These statements do not have a well-defined meaning, since $e^{ie\vec{A}\cdot\vec{x}}$ occurs within a matrix element which implies an integration over the $\vec{x}$ variable. However, a qualitative view is still possible. If we consider the $N=1$ case in the $1s$-$2s$ transition, the integrand of the radial integral involved in the matrix element has a form as shown in the equation following Eq. (25). The factor $r^3R_{2s}R_{1s}$ goes to zero at $r=0$ and $r=\infty$ and achieves its largest values between about $r=a_0$ and $r=4a_0$. As a crude approximation, we shall set $|\vec{x}|=r=2.5a_0$. If we substitute $|\vec{x}|=2.5a_0$ in $e\vec{A}\cdot\vec{x}=\pi/2$, we find $y=\frac{2}{3}eaa_0\approx\pi/7.5\approx0.4$. This value is consistent with a large contribution from the $j_1(ear\sin\theta)$ function in the radial integral, so we should not be surprised then to find that $y\approx0.4$ is a result which roughly approximates the location of the peak in $\mathcal{T}_1$ as a function of y already found to be at $y=0.341$. The essential point we are trying to establish is that in contrast to the case of ordinary intensities where perturbation theory would be valid, and where $|\vec{A}|$ is sufficiently small that the phase of $e^{ie\vec{A}\cdot\vec{x}}$ is a monotonically increasing function of the intensity, for high intensities $e^{ie\vec{A}\cdot\vec{x}}$ can exhibit its oscillatory properties which can then introduce features such as peaks and zeros in the transition amplitude as a function of intensity.

The large-N, large-y (large-$B^{1/2}$) results obtained are of very simple form and should have practical application to a type of problem to which very little attention has been paid – a problem in which multiphoton quantum transitions are caused by electromagnetic fields which can normally be considered entirely classically. To give a very specific example, consider microwave radiation of 1-cm wavelength. Each "photon" of such a field carries only about 10^{-4} eV of energy, and so something of the order of 10^4 such photons are required to cause a transition in the optical domain. When y is smaller than about 10^3, the probability of such transitions becomes very small because of the $e^{-N/y}$ factor. A value $y=10^3$ corresponds to an intensity of a 1-cm microwave field of about 10^{11} W/cm^2. Smaller values of y might still give detectable results. A value $y=10^2$ corresponds to 10^9 W/cm^2.

APPENDIX A: EVALUATION OF $\int_0^{2\pi}\frac{d\theta\, e^{i2j\theta}}{(\beta+\alpha\sin^2\theta)^m}$

The parameters α, β are taken to be real and positive, m is a positive integer, and j is either a positive or a negative integer. Results will be required for arbitrary α, β, j (subject to the constraints already listed), and for $m=1, 2, 3, 4$.

First consider the case $m=1$. We introduce the transformation

$$z=e^{2i\theta}$$

which leads to

$$d\theta=dz/(2iz),$$

$$e^{i2j\theta}=z^j,$$

$$\beta+\alpha\sin^2\theta=-\frac{\alpha}{4z}\left[z^2-\frac{4}{\alpha}\left(\beta+\frac{\alpha}{2}\right)z+1\right].$$

We require here that j is a positive integer. If j is negative, the appropriate transformation is $z=e^{-2i\theta}$, which leads to the same final result,

$$\int_0^{2\pi}\frac{d\theta\, e^{i2j\theta}}{\beta+\alpha\sin^2\theta}=\frac{2i}{\alpha}\int_C\frac{dz\, z^j}{(z-z_1)(z-z_2)}, \qquad (A1)$$

where the contour C is a circle of unit radius centered at the origin, followed in a counterclockwise sense through an angle of 4π, and z_1, z_2 are the solutions of

$$z^2-2(1+2\alpha^{-1}\beta)z+1=0, \qquad (A2)$$

i.e.,

$$z_{1,2}=(1+2\alpha^{-1}\beta)\pm[(1+2\alpha^{-1}\beta)^2-1]^{1/2}. \qquad (A3)$$

The parameters α, β are such that z_1 and z_2 are real and positive. Since from (A2) we know that $z_1z_2=1$, then $z_1>1$, $z_2<1$, where z_1, z_2 correspond to upper and lower signs, respectively, in (A3). Therefore, z_1 is outside the contour and z_2 is inside. Only the residue at z_2 contributes. The result is

$$\int_0^{2\pi} \frac{d\theta\, e^{i2j\theta}}{\beta+\alpha\sin^2\theta} = \frac{2i}{\alpha} 4\pi i \frac{z_2{}^j}{z_2 - z_1} = 2\pi \frac{1}{(\alpha\beta+\beta^2)^{1/2}} \left\{ \left(1+\frac{2\beta}{\alpha}\right) - \left[\left(1+\frac{2\beta}{\alpha}\right)^2 - 1\right]^{1/2} \right\}^j . \tag{A4}$$

As employed in the main text, the parameters in the integrals are such that $\beta = 1$, $\alpha = y^2$, $N = 2j$. With the notation $B = 1+\alpha = 1+y^2$, we find that z_2 can be rewritten conveniently as

$$z_2 = \left(1+\frac{2\beta}{\alpha}\right) - \left[\left(1+\frac{2\beta}{\alpha}\right)^2 - 1\right]^{1/2} = \frac{B^{1/2}-1}{B^{1/2}+1} .$$

Another convenient form for z_2 is

$$z_2 = \frac{y^2}{(B^{1/2}+1)^2} .$$

We thus find for the $m=1$ integral

$$\int_0^{2\pi} \frac{d\theta\, e^{i2j\theta}}{\beta+\alpha\sin^2\theta} = \frac{2\pi}{B^{1/2}} \left(\frac{y}{B^{1/2}+1}\right)^{2j} . \tag{A5}$$

Results for $m = 2, 3, 4$ can be obtained either by repeated differentiation of (A4) with respect to β, or by direct contour integration with the same transformation that led to Eq. (A1). If the $m=2$ integral is done by contour integration, the result analogous to (A1) is

$$\int_0^{2\pi} \frac{d\theta\, e^{i2j\theta}}{(\beta+\alpha\sin^2\theta)^2} = \frac{1}{2i}\left(-\frac{4}{\alpha}\right)^2 \int_c \frac{dz\, z^{j+1}}{(z-z_1)^2(z-z_2)^2} . \tag{A6}$$

To evaluate the residue in (A6) at the z_2 pole it is necessary to expand $z^{j+1}/(z-z_1)^2$ in a Laurent series in $(z-z_2)$ and select the term linear in $(z-z_2)$. The final result is

$$\int_0^{2\pi} \frac{d\theta\, e^{i2j\theta}}{(\beta+\alpha\sin^2\theta)^2} = \frac{\pi}{B^{3/2}} \left(\frac{y}{B^{1/2}+1}\right)^{2j} (B + 2jB^{1/2} + 1) . \tag{A7}$$

Results for $m = 3$ and 4 are

$$\int_0^{2\pi} \frac{d\theta\, e^{i2j\theta}}{(\beta+\alpha\sin^2\theta)^3} = \frac{\pi}{B^{5/2}} \left(\frac{y}{B^{1/2}+1}\right)^{2j} \tfrac{1}{4}\{3B^2 + 3(2j)B^{3/2} + [(2j)^2+2]B + 3(2j)B^{1/2} + 3\} , \tag{A8}$$

$$\int_0^{2\pi} \frac{d\theta\, e^{i2j\theta}}{(\beta+\alpha\sin^2\theta)^4} = \frac{\pi}{B^{7/2}} \left(\frac{y}{B^{1/2}+1}\right)^{2j} \tfrac{1}{24}\{15B^3 + 15(2j)B^{5/2} + 3[2(2j)^2+3]B^2 + 2j[(2j)^2+14]B^{3/2} + 3[2(2j)^2+3]B + 15(2j)B^{1/2} + 15\} . \tag{A9}$$

In all of the above final results, we have set $\beta = 1$, $\alpha = y^2$, $B = 1+y^2$.

APPENDIX B: ANALYTIC SUM OF T-MATRIX SERIES

The T matrix for $1s$-$2s$ transitions in hydrogen in the presence of an intense field is given by Eq. (24). A result for the same problem was given in the form of an infinite series in Eq. (46) of I. When the equation from I is written in the notation used here, it has the form

$$T_{fi}^{(N)} = i^N \frac{\sqrt{2}}{3^3} \frac{1}{m a_0{}^2} \left(\frac{y}{2}\right)^N \sum_{k=0}^{\infty} (N+2k)(N+2k+2) \times \binom{N+2k}{k} (-)^k \left(\frac{y}{2}\right)^{2k} . \tag{B1}$$

We wish to accomplish the summation in (B1) in closed form. The fact that the sum is a power series in $(-y^2/4)^k$ is suggestive of the Bessel-function series.[5]

Denote the factor $(y/2)^N$ times the sum in (B1) by $\mathcal{T}$, that is,

$$\mathcal{T} = \left(\frac{y}{2}\right)^N \sum_{k=0}^{\infty} (N+2k)(N+2k+2) \frac{(N+2k)!}{k!(N+k)!} \left(-\frac{y^2}{4}\right)^k . \tag{B2}$$

This is to be compared with

$$J_N(z) = \left(\frac{z}{2}\right)^N \sum_{k=0}^{\infty} \frac{1}{k!(N+k)!} \left(-\frac{z^2}{4}\right)^k . \tag{B3}$$

We rewrite the factors which distinguish (B2) from (B3) by noting that $N+2k = (N+2k+1) - 1$, and so

$$(N+2k)(N+2k)! = (N+2k+1)! - (N+2k)! .$$

In a similar way we have

$$(N+2k)(N+2k+2)(N+2k)! = (N+2k+2)! - (N+2k+1)! - (N+2k)! . \tag{B4}$$

Corresponding to the three terms in (B4), we will evaluate (B2) in three parts denoted $\mathcal{T}_A$, $\mathcal{T}_B$, and $\mathcal{T}_C$. The first part is

$$\mathcal{T}_A = \left(\frac{y}{2}\right)^N \sum_{k=0}^{\infty} \frac{(N+2k+2)!}{k!(N+k)!} \left(-\frac{y^2}{4}\right)^k . \tag{B5}$$

We now substitute the integral representation of $(N+2k+2)!$, given by

$$(N+2k+2)! = \int_0^\infty dt\, e^{-t} t^{N+2k+2} .$$

The order of integration and summation in (B5) can be reversed to yield

$$\mathcal{T}_A = \int_0^\infty dt\, e^{-t} t^2 \left(\frac{yt}{2}\right)^N \sum_{k=0}^{\infty} \frac{1}{k!(N+k)!} \left(-\frac{y^2t^2}{4}\right)^k$$

$$= \int_0^\infty dt\, e^{-t} t^2 J_N(yt) . \tag{B6}$$

In like fashion, we find

$$\mathcal{T}_B = -\int_0^\infty dt\, e^{-t} t J_N(yt) \tag{B7}$$

and

$$\mathcal{T}_C = -\int_0^\infty dt\, e^{-t} J_N(yt) , \tag{B8}$$

with

$$\mathcal{T} = \mathcal{T}_A + \mathcal{T}_B + \mathcal{T}_C . \tag{B9}$$

Equations (B6) through (B9) can be combined to give

$$\mathcal{T} = \left[\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial}{\partial\alpha} - 1\right) \int_0^\infty dt\, e^{-\alpha t} J_N(yt)\right]_{\alpha=1} . \tag{B10}$$

The integral required for (B10) is

$$\int_0^\infty dt\, e^{-\alpha t} J_N(yt) = \frac{[(\alpha^2+y^2)^{1/2} - \alpha]^N}{y^N(\alpha^2+y^2)^{1/2}} .$$

Carrying out the operations indicated in (B10) leads to the result

$$\mathcal{T} = \left(\frac{B^{1/2}-1}{B^{1/2}+1}\right)^{N/2} \frac{1}{B^{5/2}} [-B^2 - NB^{3/2} + (N^2-2)B + 3NB^{1/2} + 3] , \tag{B11}$$

where $B = 1 + y^2$. Equation (B11) substituted into (B1) is equivalent to Eq. (24). The ambiguous sign in (24) arises from the fact that (24) is applicable to both emission and absorption processes, as distinct from (B1) which is for absorption.

The series given in I for combined intense and weak fields can be summed in closed form exactly the same way as above, to give results identical to those in the present paper.

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[1] H. R. Reiss, Phys. Rev. A 1, 803 (1970). This paper will be referred to in the text as I.

[2] H. R. Reiss, Phys. Rev. Letters 25, 1149 (1970). This paper will be referred to in the text as II. Results presented in this reference are among the topics derived in the present work.

[3] Equation (5) is stated in I with λ_e instead of $\bar{\lambda}_e$.

[4] This matter will be discussed more fully and in more generality in a later paper.

[5] I wish to thank Dr. William M. Frank for pointing this out.