Test of K_{l3} Soft-Pion Theorem and Chiral SW(3) Model*

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An exact inequality involving the K_{l3} scalar form factor $D(t)$ and its derivatives is derived by utilizing the soft-pion theorem. From our analysis, the chiral SW(3) model of Gell-Mann, Oakes, and Renner and of Glashow and Weinberg appears to be inconsistent with the present experimental data. The linear extrapolation method for $D(t)$ is also shown to lead to an inconsistent result.

Li and Pagels[1] have used analyticity to bound K_{l3} form factors in the spacelike region by their magnitude in the timelike region. Subsequently, we have proved[2] a stronger and more general theorem in a series of recent papers, and applied the result to test the chiral SW(3) model[3] of Gell-Mann, Oakes, and Renner and of Glashow and Weinberg (hereafter referred to as GMORGW) for the K_{l3} problem. Let $D(t)$ be the K_{l3} scalar form factor given by

$$D(t)=(m_K{}^2-m_\pi{}^2)f_+(t)+tf_-(t) \tag{1}$$

in terms of the standard form factors[4] $f_\pm(t)$. $D(t)$ is a real analytic function of t with a cut on the real axis at $t_0\le t\le\infty$. If we define

$$\Delta(q^2)=\frac{i}{2}\int d^4x\, e^{iqx}\langle 0|(\partial_\mu V_\mu^{(4-i5)}(x),\, \partial_\nu V_\nu^{(4-i5)}(0))_+|0\rangle,$$

then the positivity of the spectral weight in the Kamefuchi-Umezawa-Lehmann-Källén (KULK) representation gives us the inequality[1]

$$I^2\equiv\frac{1}{\pi}\int_{t_0}^\infty dt\, k(t)|D(t)|^2\le\Delta(0), \tag{2}$$

where

$$k(t)=\frac{3}{64\pi}t^{-2}(t-t_0)^{1/2}(t-t_1)^{1/2}, \tag{3}$$

$$t_0=(m_K+m_\pi)^2,\quad t_1=(m_K-m_\pi)^2. \tag{4}$$

In the previous papers,[2] we have discussed and solved the problem of finding the best bounds for $D'(0)$ and $D''(0)$ when $D(0)$ is known and the inequality (2) holds.

To estimate $\Delta(0)$, let us set

$$A_{\alpha\beta}=i\int d^4x\langle 0|(\partial_\mu A_\mu^{(\alpha)}(x),\, \partial_\nu A_\nu^{(\beta)}(0))_+|0\rangle .$$

The KULK representation for $A_{\alpha\beta}$ enables us to write

$$A_{33}=\tfrac{1}{2}m_\pi{}^2f_\pi{}^2+\delta_\pi,\quad A_{44}=\tfrac{1}{2}m_K{}^2f_K{}^2+\delta_K,$$

where δ_π and δ_K are the non-negative contributions to A_{33} and A_{44}, respectively, from multiparticle intermediate states. If δ_π and δ_K are negligible or, more weakly, if we have[5] $\delta_\pi\ge\delta_K\ge 0$ and $A_{44}\ge A_{33}\ge 0$, then we can derive the inequality[2]

$$[\Delta(0)]^{1/2}\le\frac{1}{\sqrt{2}}(m_Kf_K-m_\pi f_\pi) \tag{5}$$

assuming the GMORGW model.

Using the experimental value $f_K/f_\pi f_+(0)=1.28$ with $f_+(0)\le 1$ we derived[2] an exact bound,

$$-0.008\le\Lambda_0\le 0.044, \tag{6}$$

where Λ_0 is defined by

$$\Lambda_0=m_\pi{}^2D'(0)/D(0)=\lambda_++m_\pi{}^2(m_K{}^2-m_\pi)^{-1}\xi. \tag{7}$$

The bound Eq. (6) is only marginally compatible at the lower end with the present world-averaged value[4]

$$\Lambda_0\approx -0.024\pm 0.02. \tag{8}$$

In the above discussion, we did not take into account the soft-pion theorem,[6]

$$\begin{aligned}D(\delta)/D(0)&=(f_K/f_\pi f_+(0))[1+O(m_\pi{}^2)]\\&\approx 1.28,\end{aligned} \tag{9}$$

where we adopt $\delta=m_K{}^2-m_\pi{}^2$ for the soft-pion point because of an SU(3) consideration.[7] Now, the question we want to answer is the following: What is the best bound for $D'(0)$ when we use the additional information contained in Eq. (9), as well as Eq. (2)? A partial answer has been given elsewhere.[2] Here we shall approach the problem in a general way and then compare the results we obtain with experiment.

To that end, it is convenient to map the cut t plane into the interior of the unit circle, $|z|<1$, by the conformal transformation,

$$(t-t_0)^{1/2}=it_0{}^{1/2}(1+z)(1-z)^{-1}, \tag{10}$$

and set

$$F(z)\equiv D(t). \tag{11}$$

Then, $F(z)$ is a real analytic function of z inside the unit circle, $|z|<1$. Moreover, we define $\phi(z)$

by

$$\phi(z)=\exp\left[\frac{1}{4\pi}\int_0^{2\pi}d\theta(e^{i\theta}+z)(e^{i\theta}-z)^{-1}\ln w(\theta)\right], \tag{12}$$

$$w(\theta)=t_0|\cos\tfrac{1}{2}\theta||\sin\tfrac{1}{2}\theta|^{-3}k(t),\quad t=t_0(\sin\tfrac{1}{2}\theta)^{-2}. \tag{13}$$

Further, if we set

$$f(z)\equiv\phi(z)F(z), \tag{14}$$

then $f(z)$ is analytic in $|z|<1$ and Eq. (2) is simply rewritten as

$$I^2=(f,f)\leq\Delta(0), \tag{15}$$

where we have defined an inner product (g,f) for two analytic functions $g(z)$ and $f(z)$ by the formula

$$(g,f)=\frac{1}{2\pi}\int_0^{2\pi}d\theta\, g^*(e^{i\theta})f(e^{i\theta}). \tag{16}$$

All functions $f(z)$ which are analytic in $|z|<1$ and have finite norm $\|f\|=(f,f)^{1/2}$ form the Hilbert space[8] H^2. If g_n $(n=0,1,2,\ldots)$ forms an orthonormal set, i.e., if g_n satisfies

$$(g_n,g_m)=\delta_{nm}, \tag{17}$$

then we must have the Bessel inequality

$$\sum_{n=0}^{N}|(g_n,f)|^2\leq(f,f)=I^2 \tag{18}$$

for an arbitrary non-negative integer N.

For a given point $z=\lambda$ inside the unit circle (i.e., $|\lambda|<1$), let us set

$$\psi(z)=(1-|\lambda|^2)^{1/2}z^{N+1}(1-\lambda^*z)^{-1}, \tag{19}$$

which belongs to H^2. Then the $N+2$ functions $g_n(z)$ defined by

$$\begin{aligned}&g_n(z)\equiv z^n\quad(0\leq n\leq N),\\&g_{N+1}(z)\equiv\psi(z)\quad(n=N+1),\end{aligned} \tag{20}$$

are easily shown to form an orthonormal set in H^2. Moreover, the standard Cauchy theorem demands

$$(z^n,f)=\frac{1}{n!}f^{(n)}(0),$$

$$(\psi,f)=(1-|\lambda|^2)^{1/2}\lambda^{-(N+1)}\left[f(\lambda)-\sum_{n=0}^{N}\frac{\lambda^n}{n!}f^{(n)}(0)\right],$$

so that the Bessel inequality (18) is rewritten as

$$\sum_{n=0}^{N}\left|\frac{1}{n!}f^{(n)}(0)\right|^2+(1-|\lambda|^2)|\lambda|^{-2(N+1)}\left|f(\lambda)-\sum_{n=0}^{N}\frac{\lambda^n}{n!}f^{(n)}(0)\right|^2\leq I^2. \tag{21}$$

The equality in Eq. (21) is possible if and only if $f(z)$ is a linear combination of $\psi(z)$ and z^n $(0\leq n\leq N)$. If we omit the last term on the left-hand side of Eq. (21), then the expression reduces to the one discussed in previous papers.[2] In particular, the case $N=0$ reproduces the result given by Meiman.[9]

Now we let λ correspond to the soft-pion point, i.e.,

$$\lambda=(\beta_s-1)(\beta_s+1)^{-1},\quad \beta_s\equiv[1-(\delta/t_0)]^{1/2}. \tag{22}$$

Then Eq. (21), together with Eq. (15), gives our fundamental inequality:

$$\begin{aligned}&\sum_{n=0}^{N}|h_n|^2+(1-|\lambda|^2)|\lambda|^{-2(N+1)}\left|D(\delta)A\phi(\lambda)-\sum_{n=0}^{N}\lambda^n h_n\right|^2\leq A^2\Delta(0),\\&A=[\phi(0)]^{-1},\quad A^{-1}h_n=\frac{1}{n!}f^{(n)}(0).\end{aligned} \tag{23}$$

Note that h_n has the form[2]

$$h_n=\sum_{m=0}^{n}\gamma_{nm}t_0{}^mD^{(m)}(0), \tag{24}$$

with values of γ_{nm} given by $\gamma_{00}=1$, $\gamma_{11}=-4$, $\gamma_{10}=A\phi'(0)$, etc.

First, consider the case $N=1$ in Eq. (23). Then for $f_+(0)\leq1$ we find

$$0.0080\leq\Lambda_0\leq0.0186, \tag{25}$$

which greatly improves the previous bound Eq. (6). Our bound Eq. (25) is outside the experimental error for Λ_0 (see Eq. 8). Also, it is inconsistent with the value of $\Lambda_0=0.023$ which is computed from the soft-pion theorem on the basis of a naive linear extrapolation of $D(t)$ to the soft-pion point $t=\delta$. This fact clearly demonstrates the inadequacy[4,10] of the linear extrapolation procedure. It may be of interest that the bound Eq. (25) is marginally compatible at the upper end with the Dashen-Weinstein sum rule[7]:

$$\Lambda_0\approx\tfrac{1}{2}m_\pi{}^2(m_K{}^2-m_\pi{}^2)^{-1}[(f_K/f_\pi)-(f_\pi/f_K)]$$
$$\approx0.020.$$

For a comparison, if we had used the stronger estimate[11] for $\Delta(0)$,

$$[\Delta(0)]^{1/2}\approx1.01m_\pi f_\pi,\quad f_+(0)=0.85, \tag{26}$$

then we would have obtained an even stronger bound,

$$0.0126\leq\Lambda_0\leq0.0141. \tag{27}$$

Returning to the general case, let us next consider the case $N=2$. Defining the parameter ρ by

$$\rho = \tfrac{1}{2} m_\pi{}^4 D''(0)/D(0) = \tfrac{1}{2} m_\pi{}^4 f_+''(0) + m_\pi{}^2 (m_K{}^2 - m_\pi{}^2)^{-1}\lambda_- \xi , \quad (28)$$

we find from Eq. (23)

$$4.41\times 10^{-4} \le \rho \le 5.08\times 10^{-4} \quad \text{for} \quad \Lambda_0 = 0.008, \qquad 3.20\times 10^{-4} \le \rho \le 3.86\times 10^{-4} \quad \text{for} \quad \Lambda_0 = 0.018 . \quad (29)$$

If we use the experimental value of Chien *et al.*,[12]

$$\lambda_+ \approx 0.026 \pm 0.006 ,$$

$$\tfrac{1}{2} m_\pi{}^4 f_+''(0) = 0.0045 \pm 0.0015 ,$$

then our bound Eq. (29) enables us to estimate λ_-:

$$0.222 \le \lambda_- \le 0.226 \quad \text{for} \quad \Lambda_0 = 0.008 , \qquad 0.514 \le \lambda_- \le 0.522 \quad \text{for} \quad \Lambda_0 = 0.018 . \quad (30)$$

We believe that such a large value for λ_- is unlikely.

Finally, if we accept an error of 10% in the soft-pion theorem, the value $D(\delta)/D(0) \approx 1.15$ is possible, in which case Eq. (25) is replaced by the bound

$$0.0034 \le \Lambda_0 \le 0.0135 \quad (31)$$

for the range $f_+(0) \le 1.0$. However, Eq. (26) then admits no solution for real Λ_0.

In concluding this paper, our inequality may be in conflict with the experimental data. If this is the case, we have to abandon some of the assumptions we used in its derivation. For example, the KULK representation may need one subtraction, or the GMORGW model may be incorrect, or the soft-pion theorem may not be satisfied. Another intriguing possibility is that the Cabibbo theory for the weak interactions may need to be suitably modified.[13] However, in view of the experimental uncertainty, it is premature to speculate on these points.

Last, we simply remark that analogous techniques can be used for various other problems.[14]

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