An Absolute Calculation of $K^+ \to \pi^+\pi^0$ Decay

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Starting with the usual Cabibbo form of the current-current theory of weak interactions, an absolute rate for $K^+ \to \pi^+\pi^0$ is calculated. Accepting the criticism of Lee against the soft-pion analysis, we analyze here the soft-kaon limit. The extrapolation to the physical point is then performed using a recent suggestion of Okubo and Mathur.

The soft-pion analysis of $K \to 2\pi$ decays was first carried out by Suzuki,[1] who showed in particular that in this limit, the nonleptonic K decays obey the $\Delta I = \frac{1}{2}$ rule. Lee[2] has recently pointed out that this is not a general result and is true only in special models. To see how this arises note that in the soft-pion limit, one may or may not conserve 4-momenta of particles, although of course for the physical process this must be so. Without energy-momentum conservation but with fixed π and K masses, the matrix element for any $K \to 2\pi$ process can be described by a function of three scalars formed out of the 4-momenta. Even for the simplest case where the matrix elements for various $K \to 2\pi$ decays are assumed to be linear functions of these scalars, Lee has shown that the soft-pion constraints do not determine the matrix elements uniquely, and in particular the $K^+ \to \pi^+\pi^0$ amplitude may not vanish. The case when energy-momentum may be taken to be conserved in the soft-pion limit corresponds to the situation of fixed kinematics. In this case the matrix elements are not functions of momenta, but the current-algebra constraints force all $K \to 2\pi$ amplitudes to vanish. This last result can be understood simply as follows. With energy-momentum conservation for zero-mass pions [SU(2) $\times$SU(2) limit], one observes that in the soft-pion limit one must also have a massless kaon. It is simple to see that in the soft-pion limit one is then working with the larger SU(3)$\times$SU(3) symmetry. The vanishing of all $K \to 2\pi$ matrix elements is then a consequence of the well-known result[3] for the SU(3) subgroup. From the point of view of these considerations it is clear that an understanding of the $K \to 2\pi$ decay is very much an open problem. In particular, $K^+ \to \pi^+\pi^0$ may not vanish in the SU(2)$\times$SU(2) limit.

A study of $K^+ \to \pi^+\pi^0$ is of special interest also for other reasons. This process violates the $\Delta I = \frac{1}{2}$ rule. Despite many investigations, it is not clear whether this arises due to electromagnetic corrections to a strict $\Delta I = \frac{1}{2}$ weak nonleptonic interaction or due to intrinsic $\Delta I \neq \frac{1}{2}$ effects in the structure of the weak interaction. The main problem of course is that at the present time we do not have a reliable way to compute the $K \to 2\pi$ matrix elements, especially if the soft-pion approach leads nowhere. Furthermore, in order to test the effect of the Cabibbo angle in nonleptonic decays, one needs an absolute determination[4] of these matrix elements.

In this paper we present a calculation for the absolute determination of the $K^+ \to \pi^+\pi^0$ matrix element, based on the usual current-current picture of weak interactions. We neglect the electromagnetic corrections.[5] Our starting point is the observation that whereas the soft-pion limit in this process is intractable, the soft-kaon limit is not. First note that if energy-momentum is conserved in this limit, the kinematics does not force one to work in the SU(3)$\times$SU(3)-symmetric limit. However, this limit is unphysical, since one of the pions would acquire negative energy. But this may be suitably handled by crossing symmetry if we assume analyticity in the pion energy variable.[6] The question at this point is how good is the soft-kaon result.

For this purpose we use the technique recently suggested by Okubo and Mathur.[7] The central idea here is the suggestion that if SU(3)$\times$SU(3) is an approximate symmetry of nature, so that the Hamiltonian has one piece which is invariant under SU(3)$\times$SU(3) and another which violates this symmetry, characterized, say, by a parameter a, we may assume that matrix elements are smooth continuous functions of a. If we now know the matrix element for some special values of a, where for instance some known subgroups of SU(3)$\times$SU(3) are realized, we may use the hypothesis of maximal smoothness to compute the matrix elements for the "physical" value of a. The "physical" value corresponds to that value of a for which a broken-SU(3)$\times$SU(3) description, for instance, leads to the physical masses of the pion and the kaon. Thus for the matrix element of $K^+ \to \pi^+\pi^0$, knowing the soft-kaon limit [chimeral[7] SU(3)] and the well-known null result in the usual SU(3) limit, one can linearly extrapolate to the physical value of a. In the SU(3)×SU(3)-symmetry-breaking model of Gell-Mann,[8] recall that at $a=-1$, one realizes SU(2)×SU(2) symmetry with a Goldstone pion; at $a=0$, the usual SU(3) symmetry and at $a=2$, the chimeral SU(3) subgroup with a Goldstone kaon. Note that our assumption that matrix elements are gentle functions of a in the range $-1\leq a\leq 2$ is much stronger than the one implicit in the pion PCAC (partial conservation of axial-vector current) hypothesis. However, the combined success of the pion PCAC hypothesis and SU(3) perturbation theory, as well as the fact that the kaon PCAC hypothesis is not inconsistent with experiments, suggests strongly that the assumption of gentleness in the whole region $-1\leq a\leq 2$ may indeed be quite reasonable.

We now turn to the soft-kaon evaluation of the matrix element for $K^+\to\pi^+\pi^0$. Define

$$A^{+}_{+0}=-i(8k_{10}k_{20}p_0V^3)^{1/2}\langle\pi^+(k_1)\pi^0(k_2)|H_w(0)|K^+(p)\rangle\,, \tag{1}$$

where H_w is the weak nonleptonic Hamiltonian in the Cabibbo form

$$H_w(x)=(G/\sqrt{2})\tfrac{1}{2}[J_\mu^{1+i2}(x)J_\mu^{4-i5}(x)+J_\mu^{4-i5}(x)J_\mu^{1+i2}(x)]\sin\theta\cos\theta\,, \tag{2}$$

with

$$J_\mu=(V+A)_\mu\,. \tag{3}$$

Note only parity-violating terms in (2) contribute to (1). Using the kaon PCAC $\partial_\mu A_\mu^{4+i5}(x)=f_Km_K^2\times K^{4+i5}(x)$, we obtain in the soft-kaon limit

$$A^{+}_{+0}(p\to 0)=-\frac{1}{f_K}(4k_{10}k_{20}V^2)^{1/2}\int d^4x\,e^{ipx}\langle\pi^+(k_1)\pi^0(k_2)|[A_0^{4+i5}(x),H_w(0)]\delta(x_0)|0\rangle\,. \tag{4}$$

The equal-time commutator can be evaluated using the standard SU(3)×SU(3) algebra. Since the two pions are in an $I=2$ state, we may rewrite (4) after some simple manipulations with Clebsch-Gordan coefficients:

$$A^{+}_{+0}(p\to 0)=\frac{1}{f_K}(4k_{10}k_{20}V^2)^{1/2}\frac{G}{\sqrt{2}}\frac{1}{2}\sin\theta\cos\theta\sqrt{2}[\langle\pi^+\pi^-|J_\mu^3(0)J_\mu^3(0)|0\rangle-\langle\pi^0\pi^0|J_\mu^3(0)J_\mu^3(0)|0\rangle]\,. \tag{5}$$

In the soft-kaon limit, since $k_1=-k_2=k$ (say) on using crossing symmetry and assuming analyticity in the complex pion-energy plane, we obtain from Eq. (5)

$$A^{+}_{+0}(p\to 0)=\frac{1}{f_K}(2k_0V)\frac{G}{2}\sin\theta\cos\theta[\langle\pi^+(k)|V_\mu^3(0)V_\mu^3(0)+A_\mu^3(0)A_\mu^3(0)|\pi^+(k)\rangle-\langle\pi^0(k)|V_\mu^3(0)V_\mu^3(0)+A_\mu^3(0)A_\mu^3(0)|\pi^0(k)\rangle]\,. \tag{6}$$

At this stage, we note that the matrix element in Eq. (6) is simply related to the one that appears in the hard-pion analysis of the $\pi^+-\pi^0$ mass-difference problem.[9] Defining

$$2k_0V[\langle\pi^+(k)|V_\mu^3(0)V_\nu^3(0)+A_\mu^3(0)A_\nu^3(0)|\pi^+(k)\rangle-(\pi^+\to\pi^0)]=\delta_{\mu\nu}A+\frac{k_\mu k_\nu}{m_\pi^2}B\,, \tag{7}$$

where A and B are constants, Eq. (6) reduces to

$$A^{+}_{+0}(p\to 0)=\frac{1}{f_K}\frac{G}{2}\sin\theta\cos\theta(4A-B)\,. \tag{8}$$

To obtain $4A-B$ we proceed as follows. To order e^2, the pion mass difference is given by

$$m_{\pi^+}{}^2-m_{\pi^0}{}^2=-\frac{e^2}{4\pi}2k_0\,\mathrm{Re}\int\frac{d^4q}{q^2-i\epsilon}T_{\mu\mu}(k,q)\,, \tag{9}$$

where

$$T_{\mu\nu}(k,q)=\int d^4x\,e^{iqx}[\langle\pi^+(k)|T(V_\mu^{\mathrm{em}}(x)V_\nu^{\mathrm{em}}(0))|\pi^+(k)\rangle-(\pi^+\to\pi^0)] \tag{10}$$

and, as usual, by covariance,

$$T_{\mu\nu}(k,q)=(q_\mu q_\nu-\delta_{\mu\nu}q^2)T_1(q^2,\nu)+[(k\cdot q)^2\delta_{\mu\nu}+q^2k_\mu k_\nu-(k\cdot q)(k_\mu q_\nu+k_\nu q_\mu)]T_2(q^2,\nu)\,, \tag{11}$$

with $\nu=-k\cdot q/m_\pi$. Recall that the logarithmically divergent part in Eq. (9) is given by the first nonvanishing term in the Bjorken[10] expansion. Using the Lee-Weinberg-Zumino field algebra,[11] one obtains in the Bjorken limit

$$T_{\mu\nu}\xrightarrow[\text{Bjorken}]{}\frac{i}{q^2}\frac{1}{f_\pi^2}\frac{1}{2k_0V}\delta_{\mu i}\delta_{\nu j}\left(\delta_{ij}A+\frac{k_ik_j}{m_\pi^2}B\right)+O\left(\frac{1}{q^4}\right), \tag{12}$$

where f_π is the π_{l2} decay constant. From Eqs. (11) and (12), one obtains in the Bjorken limit

$$q^2 T_1 \xrightarrow[\text{Bjorken}]{} \frac{i}{f_\pi^2}\frac{1}{2k_0 V}\left(-\frac{A}{q^2}+\frac{\nu^2 B}{q^4}\right)+\cdots,$$

$$q^2 T_2 \xrightarrow[\text{Bjorken}]{} \frac{i}{f_\pi^2}\frac{1}{2k_0 V}\frac{B}{q^2 m_\pi^2}+\cdots,$$

so that

$$T_{\mu\mu} \xrightarrow[\text{Bjorken}]{} \frac{i}{f_\pi^2}\frac{1}{2k_0 V}\left[\frac{3A}{q^2}-\frac{B}{q^4}(\nu^2+q^2)\right]+\cdots, \quad (13)$$

where the terms contained in the $\cdots$ lead to finite results for the mass difference. Substituting Eq. (13) in Eq. (9), we obtain for the divergent term in the mass difference

$$(m_{\pi^+}{}^2 - m_{\pi^0}{}^2)_{\text{div}} = \frac{3\alpha}{32\pi}\frac{1}{f_\pi^2}(4A-B)\ln\Lambda^2. \quad (14)$$

Thus the constant $4A-B$ appears in the coefficient of the divergent term (14). This coefficient has been evaluated using hard-pion techniques by Gerstein *et al.*,[12] who get

$$(m_{\pi^+}{}^2 - m_{\pi^0}{}^2)_{\text{div}} = \frac{3\alpha}{32\pi} m_\pi^2(1+\delta^2)\ln\Lambda^2, \quad (15)$$

where δ is the usual parameter which appears[13] in hard-pion calculations. Comparing Eqs. (14) and (15), one obtains[14] from Eq. (7)

$$A^+_{+0}(p\to 0) = \frac{G}{2}\sin\theta\cos\theta\frac{f_\pi^2}{f_K} m_\pi^2(1+\delta^2) \quad \text{(soft kaon)}. \quad (16)$$

We now return to the problem of extrapolation discussed before. Note also that in the current-current model (2), CP conservation implies that in the SU(3) limit[3,15]

$$A^+_{+0} = 0 \quad [\text{SU(3)}]. \quad (17)$$

The constraints (16) and (17) and the hypothesis of maximal smoothness discussed before simply lead to the result

$$A^+_{+0} = \frac{G}{2}\sin\theta\cos\theta(m_\pi^2 - m_K^2)\frac{f_\pi^2}{f_K}(1+\delta^2). \quad (18)$$

Numerically[16]

$$|A^+_{+0}| \simeq (2.0\text{–}2.5)\times 10^{-7} m_\pi, \quad (19)$$

to be compared with the experimental result[17] $1.3\times 10^{-7} m_\pi$. In conclusion we wish to make the following observations:

(1) Note first that our technique leads to an absolute determination of the decay amplitude. Numerically our result is somewhat larger than the experimental value. It is however remarkable that we do obtain a suppression of the $\Delta I \neq \frac{1}{2}$ effects in our approach starting from the weak Hamiltonian in the standard Cabibbo form.

(2) We do not consider the numerical discrepancy of our result with the experimental value as serious, in view of the approximations involved in deriving Eq. (18). Aside from possible uncertainties in the extrapolation discussed above, the result in Eq. (15) also involves several approximations. In particular, this result has been obtained using a low-energy description, which has probably been unjustifiably extrapolated[12] to high-energy virtual processes.

(3) Based on our result, we observe that the Cabibbo-angle suppression is indeed required in Eq. (18), in contrast to the conclusion of Sakurai.[18]

Applications of this method to the other nonleptonic decays will be discussed elsewhere.

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Comments on the Unitarity Bound in $K_L^0 \to \mu^+\mu^-$†

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Two alternatives to reconcile the present disagreement between experiment and the "simple" unitarity bound in $K_L^0 \to \mu^+\mu^-$ are considered. These alternatives presume that *CP* violation does not play the dominant role (via Reϵ) in the resolution of this puzzle. They are (1) the introduction of new muonic interactions, and (2) the existence of a new pseudoscalar meson. The first alternative can already be eliminated by present experimental evidence. The second alternative must satisfy a number of restrictions.

I. INTRODUCTION

Recently, the result of an experimental search for the decay

$$K_L^0 \to \mu^+\mu^- \tag{1}$$

was reported by Clark *et al.*[1] They observed no clear evidence for reaction (1), and set an upper bound to this branching ratio. It is

$$R_L^{\rm exp}(\mu^+\mu^-) \leq 1.8\times10^{-9} \text{ (90\% confidence limit)}. \tag{2}$$

This upper bound is significantly *below* the "simple" unitarity lower bound[2,3] calculated from assuming (a) unitarity and *CPT* invariance, (b) *CP* invariance, and (c) dominance of the unitarity sum for reaction (1) by the two-photon state.[4] (See Fig. 1.) Taking the branching ratio for[5]

$$K_L^0 \to \gamma\gamma \tag{3}$$

to be

$$R_L^{\rm exp}(\gamma\gamma) \simeq 5\times10^{-4}, \tag{4}$$

the "simple" unitarity bound[2,3] gives

$$R_L^{\rm cal}(\mu^+\mu^-) \geq 6\times10^{-9}. \tag{5}$$

Dimensional estimates on the validity of (c) have shown that other intermediate states can contribute no more than 10% in the unitarity sum.[3] A detailed estimate of the $\pi\pi\gamma$ contribution[6] and a model estimate of the $\pi\pi\pi$ contribution[7] both support this conclusion.

The role of (b) in deriving the inequality (5) has also been examined. Two cases have been studied:

(i) Neglecting Reϵ,[8] the real part of the *CP*-violating parameter in the neutral kaon system, it was shown that a lower bound still holds.[9-11] This lower bound may be 18% lower than (5) if the decay

$$K_L^0 \to (\gamma\gamma)_{CP=+1}, \tag{6}$$

which violates *CP*, dominates reaction (3). Conversely, the experimental limit given by (2) was used by Farrar and Treiman[9] to set an upper bound[12] on the presence of reaction (6).

(ii) Retaining Reϵ but assuming (c), a triangle inequality was derived by Christ and Lee[7] relating the decay rate for

$$K_S^0 \to \mu^+\mu^- \tag{7}$$

to the decay rates for reactions (1) and (3). In this case, the branching ratio for reaction (7) is constrained [using (2)] to be within

$$10^{-5} \geq R_S^{\rm CL}(\mu^+\mu^-) \geq 5\times10^{-7}. \tag{8}$$

This range is roughly 6 orders of magnitude above the corresponding "simple" unitarity bound for the reaction. It also requires the presence of significant *CP* violation either in reaction (7) or in reaction (3).[13]