scattering amplitude increases the falloff rate.

[16] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), pp. 162, 167, 172, 176, or other standard texts on relativistic quantum mechanics.

[17] For this reason we limit our discussion of $F_2^{\mathrm{el}B}(\Delta^2)$ to the region $\Delta^2 \geq 8\ \mathrm{F}^{-2}$.

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# Meson Multiplicity for the Reaction $pp \to ppX$ in a Bremsstrahlung Model*

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The differential meson multiplicity as a function of the six invariant variables that determine the process $pp \to ppX$ is derived in a bremsstrahlung model, and contrasted with the corresponding prediction of the multiperipheral model.

Recent experiments[1] indicate a logarithmic dependence of charged-particle multiplicity on incident energy, and are consistent with a Poisson distribution for the number of events considered as a function of pairs of charged tracks produced in inelastic, high-energy $pp$ scattering. These features emerge from an approximate treatment of a soft-vector-meson (SVM) model previously introduced to provide qualitative descriptions of the nucleon electromagnetic form factors[2], elastic $p$-$p$[3] and $n$-$p$[4] scattering, and deep-inelastic $e$-$p$[5] scattering at high energies. It is the purpose of this note to sketch the derivation of these properties, while exhibiting a prediction for the differential multiplicity of charged pions as a function of the (six) invariants defined when both final protons are measured, as in a proposed NAL experiment.[6]

The essential idea of the model is that low-energy (relative to the nucleons) neutral vector mesons are copiously produced by a mechanism analogous to simple bremsstrahlung of soft photons. The model is thus a realization of the bremsstrahlung aspect of Feynman's view of inelastic hadron collisions at high energies.[7] This picture of inelasticity differs from that of the multiperipheral model[8] in that mesons are here supposed to be emitted from the legs of any graph, rather than from its interior. The observed $s$-independent transverse momentum cutoff of emitted pions arises in the bremsstrahlung model because the pions are decay products of soft vector mesons, whereas in the multiperipheral model it arises from the peripheral nature of each of the two-particle amplitudes along the multiperipheral chain. While the $\ln s$ dependence of the total multiplicity is a feature of both models, the particular angular and energy dependence of the differential multiplicity predicted below may serve to distinguish between them.

A theory of nucleons ($\psi, \bar{\psi}$) and massive neutral vector mesons ($A_\mu$) possesses an $S$ matrix given, in terms of the time-ordered generating functional

$$\mathfrak{z}\{j_\mu, \eta, \bar{\eta}\} = \left\langle \exp i \int (jA + \bar{\eta}\psi + \bar{\psi}\eta) \right\rangle,$$

by the Wick-ordered relation[9]

$$S = :\exp \int \left( A_{\mathrm{in}} \vec{\mathrm{K}} \frac{\delta}{\delta j} + \bar{\psi}_{\mathrm{in}} \vec{\mathrm{D}} \frac{\delta}{\delta \bar{\eta}} - \frac{\delta}{\delta \eta} \overleftarrow{\mathrm{D}} \psi_{\mathrm{in}} \right) : \mathfrak{z}\{j, \eta, \bar{\eta}\} \Big|_{j=\eta=\bar{\eta}=0} \langle S \rangle . \tag{1}$$

From Eq. (1) there easily follows an exact expression for the production probability of $n$ vector mesons in a $p$-$p$ collision,

$$P_n = \frac{1}{n!} \left( i \int \frac{\delta}{\delta j'} \overleftarrow{\mathrm{K}} \cdot \Delta^{(+)} \cdot \vec{\mathrm{K}} \frac{\delta}{\delta j} \right)^n M^*(P_1, P_2, P_3, P_4; j') M(P_1, P_2, P_3, P_4; j) \Big|_{j'=j=0} , \tag{2}$$

where, in the notation of Ref. 2, $K = \mu^2 - \partial^2$ and

$$M(P_1, P_2, P_3, P_4; j) = \int dx_1\, e^{iP_1x_1} \int dx_2\, e^{iP_2x_2} \int dx_3\, e^{-iP_3x_3} \int dx_4\, e^{-iP_4x_4} \exp\left(\tfrac{1}{2}i \int j\Delta_c j\right)$$
$$\times \exp\left(-\tfrac{1}{2}i \int \frac{\delta}{\delta A}\Delta_c \frac{\delta}{\delta A}\right)[G(\bar{x}_3, \bar{x}_1|A)G(\bar{x}_4, \bar{x}_2|A) - 3 \leftrightarrow 4]N^{-1}\exp(L[A]). \qquad (3)$$

When the $c$-number source $j(x)$ vanishes, $M(P_1, P_2, P_3, P_4; 0)$ represents the $S$-matrix element for elastic $p$-$p$ scattering. It should be emphasized that (2) is an exact expression, with energy-momentum conservation enforced by integration over the configuration-space variables of (3). One now observes that all the SVM dependence is contained in the factors $\exp(i\sum_{l=1}^{4}\int f_l A)$, extracted from the product of the $G(A)$ of (3); here, the $l$ index denotes each of the four proton legs in the process

$$P_1 + P_2 \rightarrow P_3 + P_4 + \sum_{i=1}^{n} k_i\,; \qquad A_\mu(x) \equiv \int \Delta_c(x-y) j_\mu(y) d^4y;$$

$L[A]$ denotes the closed-fermion-loop functional while $N$ represents the normalizing vacuum-to-vacuum amplitude; $\sum_i k_i$ denotes the total four-momentum of the vector mesons; $G(\bar{x}, \bar{y}|A)$ is the nucleon propagator amputated on the nucleon coordinates; and

$$f_l^\mu(w) = gP_l^\mu \int_0^\infty d\xi\, \delta(w - x_l \pm \xi P_l),$$

with the $+(-)$ sign associated with the in- (out-) going proton momenta. Carrying through all the functional operations of (3), but displaying only those due to the soft linkages explicitly,

$$M(P_1, \ldots, P_4; j) = \exp\left(\tfrac{1}{2}i \int j\Delta_c j\right) \int dx_1\, e^{iP_1x_1} \cdots \int dx_4\, e^{-iP_4x_4} M_H(x_1, \ldots, x_4) \exp\left(i\sum_l \int f_l A + \tfrac{1}{2}i \sum_{l,m} \int f_l \Delta_c f_m\right), \qquad (4)$$

where $M_H(x_1, \ldots, x_4)$ denotes that function (of coordinate differences) which contains all structure except that due to the exchange of SVM's. By dropping all source dependence inside $M_H$, we shall henceforth be considering the effects of SVM's only, and assume that – as in the elastic case – this is a reasonable approximation at asymptotic energies.

The indicated functional operations of (2) may now be performed, and yield

$$P_n = \int dx_1\, e^{iP_1x_1} \cdots \int dx_4\, e^{-iP_4x_4} M_H(x_1, \ldots, x_4) \int dx_1'\, e^{-iP_1x_1'} \cdots \int dx_4'\, e^{iP_4x_4'}$$
$$\times M_H^*(x_1', \ldots, x_4') \frac{1}{n!}\left(i\sum_{l,m} \int f_l \Delta_{(+)} f_m'\right)^n \exp\left(\tfrac{1}{2}i \sum_{l,m} \int f_l \Delta_c f_m - \tfrac{1}{2}i \sum_{l,m} \int f_l' \Delta_c^* f_m'\right), \qquad (5)$$

where the primes denote dependence on the $x_l'$ coordinates. Aside from the restriction to the special soft exponential structure, Eq. (5) is still exact, with four-momentum conservation maintained.

In the elastic scattering case of Ref. 2, a most convenient approximation (which provides a simple, qualitative description[3] of the $p$-$p$ data for all large $s$ and essentially all $t$) was obtained by decoupling the soft exponential factors [which remain in (4) after setting $j = 0$] from the nonsoft $M_H$; this was accomplished by a "dipole" approximation, neglecting the $x_l$ dependence contained in $f_l(w)$. Since the $x_l$ coordinate differences provide the virtual-momentum cutoff in the integrals[10] over $\tilde{f}_l(k)$, neglect of the $x_l$ dependence requires the insertion of a momentum cutoff, as in Ref. 2. In the present inelastic case, adoption of the same dipole approximation will ruin the energy-momentum conservation, since it is precisely the $x_l$, $x_m'$ dependence inside $(i\sum_{l,m}\int f_l \Delta_{(+)} f_m')^n$ which, together with the explicit $P_1x_1, \ldots, P_4x_4'$ dependence of (5), provides the necessary factors of $\delta(P_1 + P_2 - P_3 - P_4 - \sum k)$.[11]

To surmount this difficulty, while retaining the simple dipole approximation (and hoping for the same sort of accuracy in the reproduction of the experimental data as that of Refs. 3 and 4), we introduce the following artifice: Neglect all $x_l$, $x_m'$ dependence in every $f_l$, $f_m'$; but replace the $\delta(P_1 + P_2 - P_3 - P_4)$ which then occurs by $\delta(P_1 + P_2 - P_3 - P_4 - \langle \sum k \rangle)$, where $\langle \sum k \rangle$ denotes an average four-momentum carried off by the SVM's. In this approximate way of handling soft production, we now write

$$P_n \approx \frac{1}{n!}\left(i\sum_{l,m} \int f_l \Delta^{(+)} f_m\right)^n |\tilde{M}_H(P_1, P_2, P_3, P_4; \langle \textstyle\sum k \rangle)|^2 \exp\left[\tfrac{1}{2}i \sum_{l,m} \int f_l (\Delta_c - \Delta_c^*) f_m\right], \qquad (6)$$

where

$$\tilde{M}_H(P_1, \ldots, P_4; \langle \sum k\rangle) \equiv \delta(P_1+P_2-P_3-P_4-\langle \sum k\rangle)\mathfrak{M}_H(P_1,P_2,P_3,P_4),$$

with $\mathfrak{M}_H$ defined by

$$\int dx_1\, e^{iP_1x_1}\cdots \int dx_4\, e^{-iP_4x_4} M_H(x_1,\ldots,x_4) = \delta(P_1+P_2-P_3-P_4)\mathfrak{M}_H(P_1,\ldots,P_4).$$

The great advantage of this approximation, in which $\langle \sum k\rangle$ depends on $\langle n\rangle$ rather than $n$, is that all further manipulations are essentially the same as those of soft-photon physics. The probability for emitting all possible SVM's becomes

$$\sum_{n=0}^{\infty} P_n = |\tilde{M}_H|^2 \exp\left[\tfrac{1}{2}i\sum_{l,m}\int f_l(\Delta_c-\Delta_c^*+2\Delta^{(+)})f_m\right], \tag{7}$$

while the differential multiplicity[12] $\langle\nu\rangle$ (the SVM multiplicity given as a function of the four proton momenta) is simply

$$\langle\nu\rangle \equiv \frac{\sum_n nP_n}{\sum_n P_n} = i\sum_{l,m}\int f_l\Delta^{(+)}f_m. \tag{8}$$

A glance at the exponential factor of Eq. (7) shows that all the soft dependence cancels exactly, since

$$i\sum_{l,m}\int f_l\Delta_{(+)}f_m = -\tfrac{1}{2}i\sum_{l,m}\int f_l(\Delta_c-\Delta_c^*)f_m. \tag{9}$$

Thus in the sum $\sum_n P_n = |\tilde{M}_H|^2$ all the damping of every partial cross section is removed upon summing over all cross sections, as in the canonical infrared manner.

The distribution (6) is now Poisson,

$$P_n = \frac{1}{n!}\langle\nu\rangle^n e^{-\langle\nu\rangle}|\tilde{M}_H|^2, \tag{10}$$

while the evaluation of the integrals on the right-hand side of Eq. (9) may be performed exactly as in Ref. 2, with the result

$$\langle\nu\rangle = -2\gamma[F(t_{13})+F(t_{24})+F(u_{14})+F(u_{23}) - F(4m^2-s_{12})-F(4m^2-s_{34})], \tag{11}$$

where

$$t_{13} = -(P_1-P_3)^2,\quad t_{24} = -(P_2-P_4)^2,$$
$$u_{14} = -(P_1-P_4)^2,\quad u_{23} = -(P_2-P_3)^2,$$
$$s_{34} = -(P_3+P_4)^2,\quad s_{12} = -(P_1+P_2)^2 \equiv s,$$

and

$$\gamma \simeq (g^2/8\pi^2)\ln(1+\mu_c^2/\mu^2),$$

with $g$, $\mu$, and $\mu_c$ the coupling constant of SVM to proton, SVM mass, and momentum cutoff, respectively. $m$ is the proton mass; and $F(z)$ is a function which for negative $z$ may be reasonably approximated by $-\ln(1+0.4|z|)$, with $z$ in units of $\mathrm{GeV}^2$. Equation (11) represents the differential multiplicity of this bremsstrahlung model, given in terms of the six invariants formed from the experimentally measured proton momenta. It should be noted that this prediction is independent of the model used to relate $\langle\sum k\rangle$ to $\langle n\rangle$; but this will not be true of the over-all multiplicity $\langle n\rangle$, which does depend (weakly, as shown below) on $\langle\sum k\rangle$.

It may be somewhat more convenient to express (11) in terms of a different set of independent variables; $t_{13}$, $t_{24}$, $s_{12}$, $s_{34}$, $\sum k_0^{(\mathrm{lab})}$, and $M^2 = -(\sum k)^2$; with the latter two quantities representing the measured lab energy and effective $(\mathrm{mass})^2$ of emitted SVM's. One finds

$$\langle\nu\rangle = -2\gamma[F(t_{13})+F(t_{24})+F(4m^2-s_{34}+M^2-t_{13}-2m\sum k_0^{(\mathrm{lab})}) + F(4m^2-s_{12}+2m\sum k_0^{(\mathrm{lab})}-t_{24})-F(4m^2-s_{12})-F(4m^2-s_{34})] \tag{12}$$

with the help of the relation $t_{13}+t_{24}+u_{23}+u_{14}+s_{12}+s_{34} = 8m^2+M^2$. To simplify Eq. (12) further it is necessary to assume that most of the events measured correspond to the assumptions of the model, and to replace $\sum k$ by its average value $\langle\sum k\rangle$. In particular, we assume that $\langle\sum \vec{k}_i^{(\mathrm{c.m.})}\rangle = 0$ and that $M = \bar{M} = \langle\sum k_0^{(\mathrm{c.m.})}\rangle$. Under these conditions we have, for the differential multiplicity for those events in which the momentum transfers remain fixed as $s\to\infty$,

$$\langle\nu\rangle \to -2\gamma[F(t_{13})+F(t_{24})], \tag{13}$$

where

$$t_{24} \approx t_{13} = 2m^2 - 2E_1E_3 + 2P_1P_3\cos\theta_{13}.$$

Particle 3 is the fast outgoing proton and particle 4 the slow outgoing proton; particle 1 is the incoming proton; and all quantities refer to the lab coordinate system.

This relation is easier to compare with experiment than Eq. (11) and it also displays in a clear way the difference between a bremsstrahlung model and multiperipheral models[8] for the differential multiplicity. Equation (13) shows rapid variation of $\langle\nu\rangle$ [like $s(1-\cos\theta_{13})$] near $\theta_{13}=0$ and is independent of $s$ as $s\to\infty$ with $t_{13}$ and $t_{24}$ fixed. On the other hand, in both the Amati-Fubini-Stanghellini model and the Chew-Pignotti Regge multiperipheral model the differential multiplicity has the asymptotic form

$$a \ln s + b,$$

where $a$ is the same coefficient that multiplies the $\ln s$ growth of the total multiplicity.

The nature of the prediction for the differential multiplicity in the simplified form of Eq. (13) is illustrated by Fig. 1, in which we plot $\langle\nu\rangle$ as a function of $\theta_{13}^{(\text{lab})}$ and $\theta_{13}^{(\text{c.m.})}$ for various fixed values of incident lab energy $E_1$. We have taken $\eta = E_3/E_1 = \frac{1}{2}$ in plotting $\langle\nu\rangle$, although the curves are rather insensitive to the value used. Note that the simplified form (13) applies to events (presumably a significant fraction of the total) for which $\vec{p}_4^{(\text{c.m.})} \sim -\vec{p}_3^{(\text{c.m.})}$ and $-t_{13} \sim -t_{24} \ll s_{12}, s_{34}$. In other cases the general form (11) or (12) must be used. We also note that, as a consequence of the rapid motion of the c.m. at high energies, it may be somewhat easier to see the effect in a colliding-beam experiment than in a stationary-target experiment.

We now show how the $\ln s$ growth of total multiplicity comes about in the bremsstrahlung model despite the lack of explicit $s$ dependence in Eq. (13). The total multiplicity, obtained by summing over all proton coordinates, is defined by

$$\langle n\rangle = \frac{\int (dP_3)\int (dP_4)\sum_n nP_n}{\int (dP_3)\int (dP_4)\sum_n P_n}, \tag{14}$$

where $\int (dP_3)\int (dP_4)$ denotes integration over the phase space of the final protons. With the approximations leading to Eq. (6), this becomes

$$\langle n\rangle = \frac{\int (dP_3)\int (dP_4)\langle\nu\rangle |\tilde{M}_H(P_1,\ldots,P_4;\langle\sum k\rangle)|^2}{\int (dP_3)\int (dP_4)|\tilde{M}_H(P_1,\ldots,P_4;\langle\sum k\rangle)|^2}, \tag{15}$$

with $\langle\nu\rangle$ given by Eq. (11). Because of the symmetry of the final protons, the numerator of Eq. (15) may be replaced by

$$-2\gamma\int (dP_3)\int (dP_4)[2F(t_{13})+2F(u_{23})-F(4m^2-s_{12}) -F(4m^2-s_{34})]|\tilde{M}_H|^2. \tag{16}$$

If $\sum k$ is again replaced by its average value, as after Eq. (12), then in the limit $s\to\infty$ Eq. (16) may

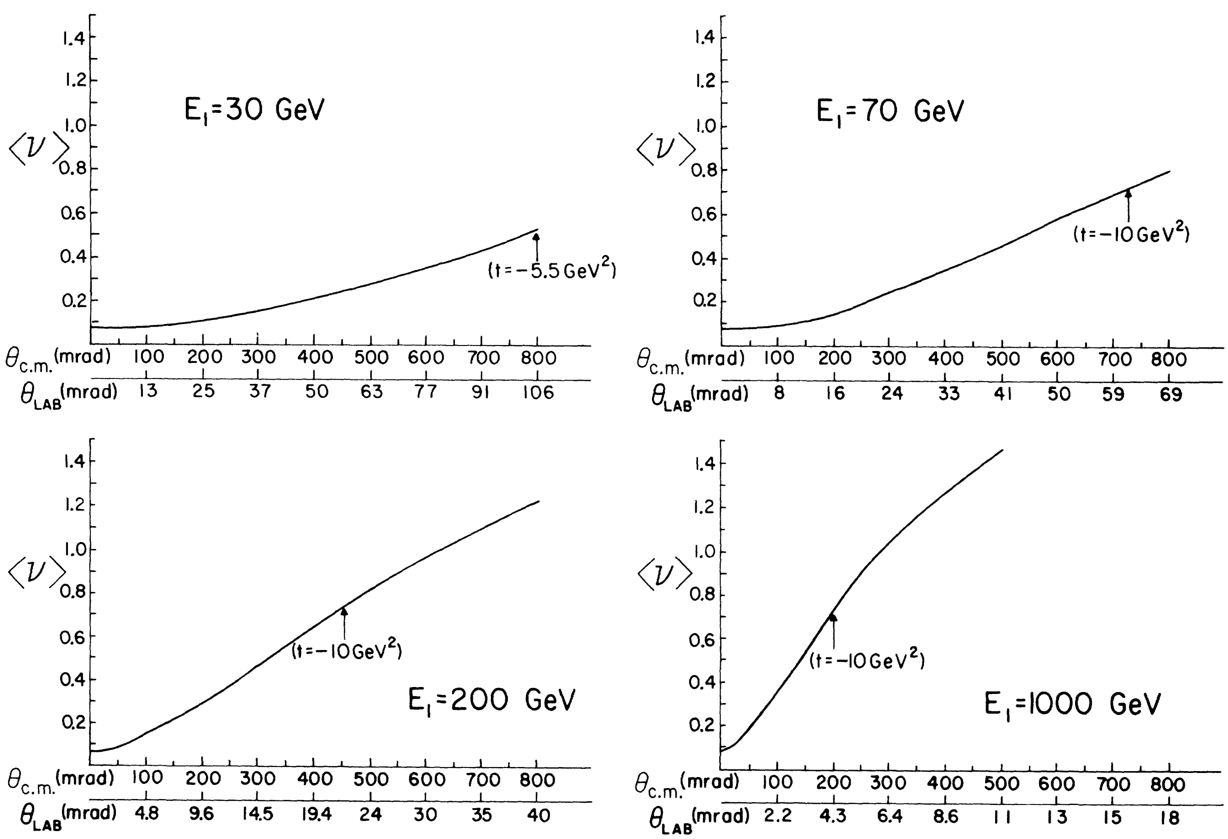


FIG. 1. Differential multiplicity $\langle\nu\rangle$ as a function of angle for various values of incident lab energy $E_1$ for the simplified kinematics of Eq. (13).

be written as

$$-4\gamma \int (dP_3) \int (dP_4)[F(t_{13}) + F(4m^2 - \eta s - t_{13}) - F(-s) + \ln\eta]\,|\tilde{M}_H|^2, \qquad (17)$$

where $\eta \equiv (s_{34}/s)^{1/2}$ is the average elasticity.

To go further, an assumption is required about the behavior of $\tilde{M}_H$. There are indications[13] that the primary continues to carry off a sizeable fraction of the energy as $s \to \infty$ (perhaps $\eta \sim 0.5$). This suggests that in the c.m. computation of the integrals of Eq. (17), the $\langle \sum k \rangle$ dependence inside $\tilde{M}_H$ may be dropped in the large-$s$ limit. We now assume this is true and find

$$\langle n \rangle \approx \frac{-4\gamma \int_0^{\eta s} d|t|[F(t) + F(-\eta s - t) - F(-s) + \ln\eta]\,d\sigma_H/dt}{\int_0^{\eta s} d|t|\,d\sigma_H/dt}, \qquad (18)$$

where $d\sigma_H/dt$ denotes a fictitious differential cross section for elastic $p$-$p$ scattering *without* the latter's experimentally observed damping at large momentum transfer (or, at least, without that portion of it ascribable to soft, virtual, neutral vector mesons). No assumptions are needed for $d\sigma_H/dt$ except the reasonable requirement that it varies sufficiently slowly and does not fall off too rapidly with increasing $|t|$, in order to extract the leading $s$-dependence

$$\langle n \rangle \sim 4\gamma \ln s + \cdots, \qquad (19)$$

where use has been made of the property $F(-s) \sim -\ln s + \cdots$. Equation (19) would be trivially true if $d\sigma_H/dt$ were represented by the Born approximation corresponding to single $\pi^0$ exchange between two protons, which is independent of $t$. The additive constant to the $\ln s$ term of (19) cannot be determined without specifying $d\sigma_H/dt$.

It is interesting to relate the constant $\gamma$ in Eq. (19) to the coefficient of the $\ln s$ term in the experimentally observed charged-pion multiplicity.[1] Since $\langle n \rangle$ is the SVM multiplicity, we must multiply by two to get the charged-pion multiplicity. With $g^2/4\pi \cong 1$ and $\mu_c^2 \cong \mu^2$, one finds $8\gamma \cong 0.9$, which is to be compared with the experimental value of 0.7. It must be noted that the value of $\gamma$ required here is an order of magnitude smaller than the value $\gamma \cong 2.3$ used to fit the elastic $pp$ data. Difficulties of this kind, in which effective elastic couplings are much stronger than inelastic ones, have appeared in other calculations.[14] It is possible to understand this effect qualitatively within the context of the SVM model, although the arguments are somewhat involved and are not necessary for the present computation. As in the elastic situations,[2-4] it is wisest to treat $\gamma$ as a parameter to be determined from the data.

The probability for producing $n$ SVM's when the final protons are not measured may be written, from (10), as

$$\mathcal{P}_n = \frac{\int (dP_3) \int (dP_4)(1/n!)\langle\nu\rangle^n e^{-\langle\nu\rangle}|\tilde{M}_H|^2}{\int (dP_3) \int (dP_4)|\tilde{M}_H|^2}, \qquad (20)$$

and it is natural to ask what form (20) takes when the same neglect of $\langle \sum k \rangle$ is made as $s \to \infty$. For $d\sigma_H/dt$ independent of $t$, Eq. (20) reduces to

$$\mathcal{P}_n \approx (1/n!)(4\gamma \ln s + \cdots)^n s^{-4\gamma} C(\gamma), \qquad (21)$$

with $C(\gamma) = (5/2\eta)^{4\gamma} B(1 - 4\gamma, 1 - 4\gamma)$, assuming $4\gamma < 1$. Thus one finds an effective Poisson distribution for the probability of events considered as a function of pairs of charged tracks, in agreement with the data.

One final comment concerning a detail of the model may be appropriate. There is no *a priori* reason why the cutoff $\mu_c$ of these inelastic reactions need be chosen the same as that of the elastic reaction of Ref. 2, and, in fact, the smaller $\gamma$ value required in the present inelastic calculation suggests that a smaller $\mu_c$ should be used. (The only firm requirement is the precise cancellation of terms which would become infrared-divergent, were $\mu$ allowed to vanish.) If $\gamma_{\rm inel} < \gamma_{\rm el}$, there remains a residual soft damping in $\sum_n P_n$, and the computation of $\langle n \rangle$ must be reexamined. It turns out that if $4(\gamma_{\rm el} - \gamma_{\rm inel}) > 1$, these approximate forms produce the desired $\ln s$ dependence of $\langle n \rangle$ only if the $s \to \infty$ limit is taken before the value of $\gamma_{\rm el} - \gamma_{\rm inel}$ is allowed to exceed $\frac{1}{4}$. While such an approximation-dependent, sequential limiting procedure is then crucial in obtaining certain properties of these inclusive reactions, we expect that it should not be significant for the differential multiplicity, with kinematical dependence given by (11) or (13). It is this aspect of the bremsstrahlung model which we hope will be tested shortly.

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[11]Direct summation of the SVM diagrams by neglecting all $k^2$ dependence in nucleon propagators makes explicit the relation between the exact energy-momentum-conserving $\delta$ function and the coordinate dependence in $(i\sum_{l,m}\int f_l \Delta_{(+)} f'_m)^n$. Explicitly summing over all emitted SVM's gives

$$P_n \propto \frac{1}{n!}\left[\frac{g^2}{(2\pi)^3}\right]^n \prod_{i=1}^{n} \int \frac{d^3k_i}{2\omega_i}\left(\frac{P_{3\mu}}{P_3\cdot k_i}-\frac{P_{1\mu}}{P_1\cdot k_i}+\frac{P_{4\mu}}{P_4\cdot k_i}-\frac{P_{2\mu}}{P_2\cdot k_i}\right)^2 \times |M_{\text{el}}|^2\,\delta^4\left(P_1+P_2-P_3-P_4-\sum_{j=1}^{n}k_j\right),$$

where $M_{\text{el}}$ is the amplitude for $P_1+P_2\to P_3+P_4$, including all soft exchanges, and the $\delta$ function comes from the phase-space factors for the emitted SVM's. Writing an exponential representation for the $\delta$ function puts the expression in a factorized form in which an $e^{ik_i\cdot x}$ is associated with each $k_i$ factor. The replacement $e^{ik_i\cdot x}\to 1$ required to decouple the SVM emission thus ruins the energy-momentum conservation. The subsequent (molecular-field type of) approximation may be simply defined starting from this explicit form for $P_n$.

[12]Let $i$ represent the six independent variables in $p_1p_2\to p_3p_4X$. $P_{n(i)}$ is the probability of producing $n$ vector mesons when $p_3$ and $p_4$ go into the $i$th phase-space bin. $P_{(i)}=\sum_{n=0}^{\infty}P_{n(i)}$ is the total (inclusive) probability for $p_3$ and $p_4$ to go into the $i$th phase-space bin. Thus the average differential multiplicity has the experimental meaning $\langle\nu_i\rangle=\langle n_i\rangle/P_{(i)}$, where $\langle n_i\rangle$ is the observed average vector-meson multiplicity in the $i$th phase-space bin and $\sum_{\text{all } i}P_{(i)}=1$.

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# Nonanalytic Behavior of the $\Sigma$ Term in $\pi$-$N$ Scattering*

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The $\Sigma$ term in the $\pi N$ scattering amplitude is commonly evaluated at squared nucleon momentum transfer $t=2\mu^2$, where $\mu$ is the pion mass. Because of the nonanalytic nature of perturbations about the chiral SU(2)×SU(2) limit, $\Sigma(2\mu^2)$ differs from $\Sigma(0)$ by a term linear in $\mu$. We calculate the difference term exactly to $O(\mu)$ and find

$$\Sigma(2\mu^2)-\Sigma(0)=\frac{3}{8\pi}\left(\frac{g_A}{2F_\pi}\right)^2\frac{\mu}{F_\pi}+O(\mu^2\ln\mu^2)\,.$$

This represents a 14-MeV correction to the value of $\mu^2F_\pi\Sigma(2\mu^2)$.

It was recently observed by Li and Pagels[1] that, as a consequence of long-range forces introduced as the pion mass $\mu$ vanishes in the chiral limit, many amplitudes are not analytic about zero in $\mu^2$ and naive expansion in the symmetry-breaking parameter fails. We find this occurs for the nucleon matrix element of the current-algebra $\Sigma$ term, which we define (crossing one nucleon) by