# Note on the Soft-Meson Limit and the Dispersion Relation

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Dispersion relations in the hard-meson limit and the soft-meson limit are considered. They take different forms, and a precise relation between the equal-time commutator term and the subtraction constant is given. We must be careful in adopting the form of the dispersion relation.

We consider here the dispersion relation for a process involving a meson in the hard- and soft-meson limits, and obtain different forms for these two limits. We have in mind a process such as nonleptonic hyperon decay or $K$ decay, but the discussion is a general one. To be definite we consider the following amplitude, typical of hyperon decay:

$$T \equiv \langle B'(p'), \pi(k)|A(0)|B(p)\rangle(2k_0)^{1/2}\left(\frac{p_0 p_0'}{m_1 m_2}\right)^{1/2} \tag{1}$$

with $A(0)$ some local operator. Four-momentum conservation is understood. The dispersion relation in the hard-meson case can most easily be obtained by introducing a spurion with momentum $q$, defining the usual $s$, $t$, and $u$ variables, and then taking the limit $q \to 0$:

$$\begin{aligned} s &= -(p+q)^2 = -(p'+k)^2 , \\ t &= -(p-p')^2 = -(q-k)^2 , \\ u &= -(p-k)^2 = -(p'-q)^2 , \\ s+t+u &= m_1^2 + m_2^2 - q^2 - k^2 , \end{aligned} \tag{2}$$

where $m_1$ and $m_2$ are the masses of the $B$ and $B'$. The physical amplitude is obtained setting $s = m_1^2$, $u = m_2^2$, and $t = m_\pi^2$. We follow the well-known procudure to disperse the amplitude $T$. After the reduction of $\pi$ in Eq. (1), the local property of $A$ allows us to write down the dispersion relation in $k_0$; then we take, for example, the rest frame of $B'$ and use the kinematical relation

$$k_0 = \frac{s + k^2 - m_2^2}{2m_2}$$

to transform to the variable $s$. In this way we arrive at the fixed-$t$ dispersion relation for each invariant amplitude:

$$T_i(s,t) = \frac{1}{\pi}\int_0^\infty \frac{A_i(s',t)ds'}{s'-s-i\epsilon} + \frac{1}{\pi}\int_0^\infty \frac{B_i(u',t)du'}{u'-u-i\epsilon} , \tag{3}$$

$$\begin{aligned} A_i(s,t) &= i\pi P_i \sum_{n,\text{int}} \left(\frac{p_0 p_0'}{m_1 m_2}\right)^{1/2} \\ &\quad\times \langle B'|j_\pi(0)|n\rangle\langle n|A(0)|B\rangle \delta(k_0 + p_0' - p_{n0}) , \\ B_i(u,t) &= i\pi P_i \sum_{n,\text{int}} \left(\frac{p_0 p_0'}{m_1 m_2}\right)^{1/2} \\ &\quad\times \langle B'|A(0)|n\rangle\langle n|j_\pi(0)|B\rangle \delta(k_0 - p_0 + p_{n0}) , \end{aligned} \tag{4}$$

where $j_\pi = (\Box + m^2)\phi$, $\phi$ being the pion field; $P_i$ is the projection operator which projects out the invariant amplitude $A_i$ or $B_i$; and $\sum_{n,\text{int}}$ means summation over internal degrees of freedom. These relations, being Lorentz-invariant, hold in an arbitrary frame. The limit $q \to 0$ can be taken without any change in Eq. (3), and it represents the dispersion relation for the amplitude in the case of a hard meson with squared mass $t$.

If we want to have the dispersion relation in the limit of the soft meson, the second limit $k_0 \to 0$ causes trouble in Eq. (4). In the rest frame of the meson, i.e., $\mathbf{k} = 0$, we have the following expressions for $p_0'$ and $p_{n0}$:

$$\begin{aligned} p_0' &= \frac{s' - k_0^2 - m_2^2}{2k_0} , \\ p_{n0} &= (p_0'^2 - m_2^2 + m_n^2)^{1/2} . \end{aligned}$$

It is then easy to see that the $\delta$ function in the $A$ term, for example, takes the form

$$\delta(k_0 + p_0' - p_{n0}) \sim \frac{m_2^2 - s'}{k_0}\delta(m_n^2 - s') \tag{5}$$

and similarly for the $B$-term $\delta$ function.[1,2] In order that the soft limit may be smooth, it is necessary that the matrix element $\langle B'|j_\pi|n\rangle$ should vanish like $k_0$ in this limit for any $n$, and this is assured in the case of partial conservation of axial-vector current (PCAC).

Defining the operator $C$, which is local and smooth, we thus write for $k_0 \sim 0$:

$$\langle B'|j_\pi(0)|n\rangle \sim ik_0\langle B'|C(0)|n\rangle . \tag{6}$$

Here it is perhaps helpful to note that we are looking for the singularities in $k^2$ or in $k_0$ in our frame and not the singularities in $m_\pi$. We can see this most easily if we adopt the usual definition of the off-mass-shell continuation.[3] After the substitution of Eq. (6), we may have terms singular in $m_\pi$ as a result, but this does not cause trouble as long as $m_\pi$ remains finite.

Introducing Eq. (6) in Eq. (4) and Eq. (3) and assuming that the limit $k_0 \to 0$ can be taken inside the integral, we obtain

$$T_i(s,0)=\frac{1}{\pi}\int_0^\infty \frac{(s'-m_2^2)A_i'(s',0)ds'}{s'-s-i\epsilon}+\frac{1}{\pi}\int_0^\infty\frac{(u'-m_1^2)B_i'(u',0)du'}{u'-u-i\epsilon}$$

with

$$A_i'(s',0)=\pi P_i\sum_{n,\text{int}}\left(\frac{p_0p_0'}{m_1m_2}\right)^{1/2}\times\langle B'|C(0)|n\rangle\langle n|A(0)|B\rangle\delta(m_n^2-s'),$$
$$B_i'(u',0)=\pi P_i\sum_{n,\text{int}}\left(\frac{p_0p_0'}{m_1m_2}\right)^{1/2}\times\langle B'|A(0)|n\rangle\langle n|C(0)|B\rangle\delta(m_n^2-u'). \tag{7}$$

Subtraction at $s=m_2^2$, $u=m_1^2$, gives the simple form

$$T_i(s,0)=T_i(m_2^2,m_1^2)+\frac{s-m_2^2}{\pi}\int_0^\infty\frac{A_i'(s',0)ds'}{s'-s-i\epsilon}+\frac{u-m_1^2}{\pi}\int_0^\infty\frac{B_i'(u',0)du'}{u'-u-i\epsilon}. \tag{8}$$

Note the peculiar interchange of $m_1$ and $m_2$ in the subtraction term. Equation (8) represents the once-subtracted dispersion relation in the soft-meson limit. For comparison's sake, the twice-subtracted dispersion relation subtracted at a certain point $s=s_0$ for the hard-meson limit ($t\to 0$) takes the following form:

$$T_i(s,0)=T_i(s_0,0)+\frac{s-s_0}{\pi}\int_0^\infty\frac{A_i(s',0)ds'}{(s'-s-i\epsilon)(s'-s_0)}+\frac{u-u_0}{\pi}\int_0^\infty\frac{B_i(u',0)du'}{(u'-u-i\epsilon)(u'-u_0)} \tag{9}$$

with $s_0$ and $u_0$ related by Eq. (2). We observe that the dispersion relations take different forms for different limits. The most crucial difference is the convergence properties of Eqs. (8) and (9) if we assume the same high-energy behavior of $A_i$, $B_i$ as of $A_i', B_i'$, which is reasonable. The soft-meson limit when $B$ and $B'$ are on the mass shell can be calculated in the usual manner without using dispersion relations, and it turns out that we arrive at precisely Eq. (8) with $s=m_1^2$, $u=m_2^2$, and

$$T_i=-iP_i\langle B'|[C,A(0)]|B\rangle$$

with

$$C\equiv\int d^3x\,C(x,0). \tag{10}$$

Thus we conclude that the equal-time commutator is just the subtraction constant of the dispersion relation in the soft-meson limit, the subtraction point being specified as $s=m_2^2$, $u=m_1^2$. This is not a trivial fact, because *we cannot use Eq. (9) with $T_i(s,0)$ identified with the equal-time commutator,* as has commonly been done.[4]

In the conventional approach using current algebra, PCAC, and pole dominance,[5] the important factor in front of the δ function (5) was overlooked, and therefore if one applied Eq. (8) on the mass shell ($s=m_1^2$, $u=m_2^2$), adopting PCAC, one arrived at a situation quite different from the conventional approach for both $s$- and $p$-wave decays.[2] It seems that the simple octet-baryon pole-dominance model yields rather poor results for both waves.

Finally we remark that the argument given above does not, of course, apply to the hard-meson technique originated by Schnitzer and Weinberg,[6] because, using the Ward identity and pole dominance or effective Lagrangians only, it does not take the soft limit.

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[1]Other factors, such as $P_i$, do not display singularities when $k_0\to 0$.

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[3]See, for instance, S. Weinberg, Phys. Rev. Letters 17, 616 (1966).

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[5]See, for instance, Y. Hara, Y. Nambu, and J. Schechter, Phys. Rev. Letters 16, 380 (1966); L. S. Brown and C. M. Sommerfield, *ibid.* 16, 751 (1966).

[6]H. Schnitzer and S. Weinberg, Phys. Rev. 164, 1828 (1967). See also, for instance, R. Arnowitt, M. H. Friedman, and P. Nath, *ibid.* 174, 1999 (1968).