Fixed and Regge Poles in the Virasoro Model

W. L. Kennedy and C. H. Oh
Physics Department, University of Otago, Dunedin, New Zealand
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We give closed-form expressions both for the residues $\beta_p(s)$ of the Regge poles at $l=\alpha(s)-p$ and for the residues $\overline{\beta}_p(s)$ of the fixed poles at $l=-p-1$, $p=0,1,2,\ldots$. We note that $\beta_p(s)$ should vanish for p odd integral if the Regge trajectories are spaced by two units of angular momentum.

The structure of the Virasoro model[1] in the complex angular momentum (l) plane has been analyzed by Argyres and Lam.[2] They have followed a method used by Fivel and Mitter[3] in their l-plane analysis for the Veneziano amplitude. The method exhibits clearly the analytic structure of the partial wave $A(s,l)$, but does not allow one to calculate the residues $\beta_p(s)$ of the Regge poles easily. In this short note, we give an expression for the residue $\beta_p(s)$ of fixed poles as well as moving poles. Our approach is similar to that used by Drago and Matsuda,[4] who have also performed the l-plane analysis of the Veneziano amplitude. In Ref. 2, the residue $\beta_p(s)$ is not given for arbitrary p.

The Virasoro model[1] shares many features with that of Veneziano, but differs in some other respects. In particular, it is completely symmetric in s, t, and u, and its Regge trajectories are spaced by two units of angular momentum apart. Thus in its partial-wave projection we should find an infinite family of Regge poles with parallel trajectories spaced by two units. The result given in Eq. (10) of Ref. 2, which states that the partial wave $A^+(s,l)$ has moving poles in l at

$$\mathrm{Re}\,l=\mathrm{Re}\,\alpha(s)-k,\quad k=0,1,2,\ldots \tag{1}$$

is therefore not quite correct. In fact, we show below that $\beta_1(s)=0$, and it is expected that the poles at odd-integer values of k will be spurious.

We follow the same notation as Ref. 2 (note that this reference omits an over-all negative constant). Consider the η^0-η^0 scattering in the Virasoro model. The partial-wave projection is given by[2]

$$A^+(s,l)=\frac{\gamma}{2aq^2\Gamma(2\mu-1)\Gamma(1-\gamma)}\times\sum_{n=0}^{\infty}\frac{\Gamma(n+1-\gamma)\Gamma(n+2\mu)}{n!\,\Gamma(n+2\mu+\gamma)}\,Q_l\left(\frac{n+\mu}{aq^2}\right). \tag{2}$$

The odd signature vanishes identically. The series is summed by using the integral representation of $Q_l(z)$,[5]

$$Q_l(z)=\frac{\pi^{1/2}2^{-l-1}}{\Gamma(l+\frac{3}{2})}\int_0^\infty dx\,x^l\,e^{-xz}(2x)^{-l-1}M_{0,\,l+1/2}(2x),$$

to give

$$A(s,l)=G\int_0^\infty dx\,x^l\,e^{-x}\,F(x), \tag{3}$$

where

$$G=\frac{\gamma(2\mu-1)\pi^{1/2}2^{-l-2}}{aq^2\Gamma(2\mu+\gamma)\Gamma(l+\frac{3}{2})} \tag{4}$$

and

$$F(x)=e^{-(1-b/2)(x/aq^2)}F(1-\gamma,2\mu;2\mu+\gamma;e^{-x/aq^2})\times(2x)^{-l-1}M_{0,\,l+1/2}(2x). \tag{5}$$

Here $F(a,b;c;z)$ is the hypergeometric function and $M_{0,\,l+1/2}(2x)$ is the Whittaker function. $M_{0,\,l+1/2}(2x)/\Gamma(l+\frac{3}{2})$ is analytic in l. Using the linear transformation formula,[6]

$$F(a,b;c;z)=(1-z)^{c-a-b}F(c-a,c-b;c;z),$$

in Eq. (5), we then have

$$A(s,l)=g\int_0^\infty dx\,x^{l+2\gamma-1}\,e^{-x}f(x), \tag{6}$$

where

$$g=\gamma(2\mu-1)\pi^{1/2}2^{-l-2}(aq^2)^{-2\gamma}/\Gamma(2\mu+\gamma) \tag{7}$$

and

$$f(aq^2y)=e^{-(1-b/2)y}\left(\frac{1-e^{-y}}{y}\right)^{2\gamma-1}\times F(2\mu+2\gamma-1,\gamma;2\mu+\gamma;e^{-y})\times(aq^2y/2)^{-l-1/2}I_{l+1/2}(aq^2y). \tag{8}$$

With the aid of Taylor's theorem, in the form

$$f(x)=\sum_{k=0}^{N}\frac{f^{(k)}(0)}{k!}x^k+R_{N+1}(x),$$

where $R_{N+1}(x)$ is the remainder term, we obtain

$$A(s,l)=g\left(\sum_{k=0}^{N}\frac{f^{(k)}(0)}{k!}\Gamma(l+2\gamma+k)+\int_0^\infty dx\,x^{l+2\gamma-1}e^{-x}R_{N+1}(x)\right). \qquad (9)$$

Hence the residue at the pole $l=\alpha(s)-p$ is given by

$$\beta_p(s)=g\frac{1}{p!}\left[\left(\frac{d}{dx}\right)^p[f(x)e^{-x}]\right]_{x=0}. \qquad (10)$$

At $l=\alpha(s)$, we easily get

$$\beta_0(s)=\frac{\gamma(aq^2)^{\alpha(s)}\Gamma(\frac{1}{2}+\frac{1}{2}\alpha(s))}{4\Gamma(2\mu-1)\Gamma(\frac{3}{2}+\alpha(s))}, \qquad (11)$$

which agrees with Eq. (11) of Ref. 2.

We calculate, from Eq. (10),

$$\beta_1(s)=0.$$

At $l=\alpha(s)-2$, the next residue is

$$\beta_2(s)=\tfrac{1}{2}g\left\{\frac{\Gamma(2\mu+\gamma)\Gamma(1-2\gamma)}{\Gamma(2\mu)\Gamma(1-\gamma)\Gamma(-2\gamma+\frac{1}{2})}+\frac{(aq^2)^{-2}}{\Gamma(-2\gamma-\frac{1}{2})}\left(F(2\mu+2\gamma-1,\gamma;2\mu+\gamma;1)\{\mu^2+[\mu+\tfrac{1}{3}+\tfrac{1}{2}(\gamma-1)](2\gamma-1)\}\right.\right.$$
$$+F(2\mu+2\gamma,\gamma+1;2\mu+\gamma+1;1)\frac{\gamma(2\mu+2\gamma-1)(2\gamma+2\mu)}{2\mu+\gamma}$$
$$\left.\left.+F(2\mu+2\gamma+1,\gamma+2;2\mu+\gamma+2;1)\frac{\gamma(\gamma+1)(2\mu+2\gamma-1)(2\gamma+2\mu)}{(2\mu+\gamma)(2\mu+\gamma+1)}\right)\right\}. \qquad (12)$$

The positivity condition for the residue requires

$$-\beta_2(s)\geq 0$$

since an over-all minus sign has been dropped. We give the result for the residue of the first recurrence on the trajectory, i.e., $\gamma=-1$. We then have

$$-\beta_2(s)=-[2(1-\mu)-\tfrac{4}{3}a^2q^4]/4\Gamma(2\mu-1)\geq 0,$$

evaluated for $4aq^2=2-b-4am^2$, where m is the η^0 mass.

We now derive the expression for the residues of the fixed poles by the same method as above. From Eqs. (3), (4), and (5), we get

$$A(s,l)=G\left[\sum_{k=0}^{N}\frac{F^{(k)}(0)}{k!}\Gamma(l+k+1)+\int_0^\infty dx\,x^l e^{-x}\bar{R}_{N+1}(x)\right], \qquad (13)$$

where $\bar{R}_{N+1}(x)$ is the remainder term of the Taylor expansion for $F(x)$, defined by Eq. (5). The residues at $l=-p-1$, $p=0,1,2,\ldots$, are given by

$$\bar{\beta}_p(s)=G\frac{1}{p!}\left[\left(\frac{d}{dx}\right)^p[F(x)e^{-x}]\right]_{x=0}.$$

At $l=-1$, we have

$$\bar{\beta}_0(s)=\frac{\gamma(2\mu-1)\Gamma(2\gamma-1)}{2aq^2\Gamma(\gamma)\Gamma(2\mu+2\gamma-1)},$$

which agrees with Eq. (13) of Ref. 2. $\bar{\beta}_1(s)$ vanishes identically. It appears likely that $\bar{\beta}_p(s)$ vanishes for each odd-integral value of p.

We note that the analytic structure of the partial-wave projection of the Virasoro formula resembles closely that of its Lorentz amplitude[7] and also its Khuri amplitude.[8] It is seen that only when $s=0$ are the Toller poles spaced by two units; the same is true for the spacing of the Khuri poles when $q^2=0$. Since the relation between Regge poles and Toller poles, and that between Regge poles and Khuri poles have been established,[9,10] the similarity is not unexpected. Mandelstam[11] has pointed out one disadvantage of the Virasoro amplitude, which is that the Regge residues have poles at the negative wrong-signature integers, which is not allowed by unitarity. That this is so can be seen from the expressions (11) and (12). However, in dual-resonance models, unitarity is completely neglected.

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