Comments and Addenda

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Comment on the Spontaneous Breakdown of Scale Invariance

Curtis G. Callan, Jr.*
Institute for Advanced Study, Princeton, New Jersey 08540
and
P. Carruthers†
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850
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In a recent paper, Bose and McGlinn concluded that there cannot be massive particles in the limit of spontaneously broken scale invariance. The assumptions on which their result is based are criticized.

Doubts have recently been expressed[1] about the existence of massive particles in the limit of spontaneously broken scale invariance. On the other hand, there exist simple scale-invariant Lagrangian models whose (approximate) solutions do contain massive particles. Since the models generally use approximate techniques (typically the tree-graph approximation) one can ask whether some general constraint has been violated in their solution. This question might seem especially pertinent once it is noted that what is involved here is the "self-stress theorem" which is a notorious source of dilemmas and paradoxes even in classical physics.

The intent of this paper is to explain why the basic assumption of Ref. 1 leading to the cited conclusion is not necessarily appropriate in the context of canonical field theory. At the same time we discuss the resolution of a related pseudoparadox concerning the mass spectrum in such theories. These remarks do not guarantee the existence of spontaneous breakdown of scale invariance in nature but, we believe, leave open this possibility. The present discussion should be supplemented by the papers of Crewther,[2] Fujii,[3] and Korthals Altes.[4] These authors have discussed some of the tricky and seemingly paradoxical aspects related to the spontaneous breakdown of scale and conformal invariance. Many other authors have studied models in which this phenomenon occurs. A partial list of such papers is given in Refs. 5–10.

The essential point we wish to make (already stated in Ref. 11 but not explained in detail there) is that the elegant form given[12] by Callan, Coleman, and Jackiw (called CCJ henceforth) for the conformal generators in terms of an "improved" energy-momentum tensor $\Theta_{\mu\nu}$ is inappropriate in the limit of *spontaneously broken* scale invariance. In the presence of massless scalar particles necessitated by the spontaneous breakdown of scale invariance the "improved" dilation generator D of CCJ no longer coincides in general with that given directly by Noether's theorem. Bose and McGlinn have derived the consequences of the conservation of the *improved* D (equivalently $\Theta^{\mu}{}_{\mu}=0$). From the above remarks we see that the latter situation need not coincide with conservation of the canonical D, which we shall argue is to be preferred in the case of conflict. The modifications of $\Theta_{\mu\nu}$ which allow one to recast the generators given directly by Noether's theorem involve the "dilaton" pole in this limit. In contrast to the case considered in Ref. 12, these pole terms *do* modify the Poincaré generators when the dilaton field is a member of an internal-symmetry group. Hence, although the CCJ formulation is especially simple for many purposes (e.g., the limit of scale invariance is characterized by $\Theta^{\mu}{}_{\mu}\to 0$) it can be hazardous to use their formulation *in* the symmetry limit.[13]

We recall that the improved energy-momentum tensor $\Theta_{\mu\nu}$ of CCJ differs from the conventional symmetric (Belinfante) tensor $T_{\mu\nu}$ by a term $\tau_{\mu\nu}$;

$$\Theta_{\mu\nu} = T_{\mu\nu} + \tau_{\mu\nu}\,, \tag{1}$$

where $\tau_{\mu\nu}$ is a sum of terms involving spin-zero fields,

$$\tau_{\mu\nu} = -\tfrac{1}{6}\sum_{\phi}(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\phi^2\,. \tag{2}$$

It is also possible and sometimes necessary[12] to allow each ϕ to be accompanied by a constant $\phi\to\phi + c$. However, we shall here assume that all the spin-zero fields in the theory belong to sets of irreducible representations of an internal-symmetry group G (e.g., $SU_3\times SU_3$) and that the extra term (2) is invariant under G. (Any other assumption would mean that τ_{00} introduces explicit symmetry breaking into Θ_{00} which was not already contained in the Lagrangian underlying the canonical construction of $T_{\mu\nu}$.) Hence we shall treat the case $c=0$.

We now define two sets of operators which are candidates for generators of Poincaré and scale transformations. In terms of the old-fashioned tensor $T_{\mu\nu}$ we have[14]

$$\begin{aligned} P_\mu &= \int d^3x\, T_{\mu 0}\,,\\ M_{\mu\nu} &= \int d^3x\,(x_\mu T_{\nu 0} - x_\nu T_{\mu 0})\,,\\ D &= \int d^3x\left[x^\mu T_{0\mu} - \sum_\phi l_\phi \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\,\phi\right], \end{aligned} \tag{3}$$

where l_ϕ is the dimension of the field ϕ. In terms of the improved tensor $\Theta_{\mu\nu}$,

$$\begin{aligned} P'_\mu &= \int d^3x\,\Theta_{\mu 0}\,,\\ M'_{\mu\nu} &= \int d^3x\,(x_\mu\Theta_{\nu 0} - x_\nu\Theta_{\mu 0})\,,\\ D' &= \int d^3x\, x^\mu\Theta_{\mu 0}\,. \end{aligned} \tag{4}$$

(Strictly speaking, the nonconserved dilation charge does not exist but is nonetheless useful since commutators $[D, O(x)]$ can be given a sensible meaning by a limiting process. This problem, which also occurs for the familiar axial-vector charge, is well understood[15] and will be ignored here.)

The formal structure of the commutators of P_μ, $M_{\mu\nu}$, D as well as the special conformal operator K_μ has been studied in the presence of scale breaking by several authors.[16–19] In particular, the limit of scale (and conformal) invariance may be characterized by the vanishing of the trace $\Theta^\mu{}_\mu$. However, the massless spin-zero fields which occur in this limit have the unfortunate consequence of making the "improved" operators P'_μ, $M'_{\mu\nu}$, D' not necessarily equal to the original P_μ, $M_{\mu\nu}$, D. (We shall always assume the latter to be correct, although even this has to be verified case by case and deserves further study.)

We now consider whether the quantities

$$\begin{aligned} \delta P_\mu &= P'_\mu - P_\mu\,,\\ \delta M_{\mu\nu} &= M'_{\mu\nu} - M_{\mu\nu}\,,\\ \delta D &= D' - D\,, \end{aligned} \tag{5}$$

vanish when some of the fields ϕ are massless.

The answer to this question is clearly delicate if one consults the original derivation,[12] which involves casting away surface terms. The point is illustrated by computing the energy change

$$\begin{aligned} \delta P_0 &= \int d^3x\,\tau_{00}\\ &= -\tfrac{1}{6}\sum_\phi\int d^3x\,\nabla^2(\phi^2)\\ &= -\tfrac{1}{6}\sum_\phi\int d\vec{s}\cdot\nabla(\phi^2)\,. \end{aligned} \tag{6}$$

If the fields $\phi\sim 1/r$ at large distances, δP_0 does vanish (as $1/R$, where R is the radius of the volume $V\to\infty$). However, when we have spontaneous breakdown of a continuous internal-symmetry group, some (but not all) fields go to a constant at large distances.[20] For such fields we have $\phi = \phi' + \phi_0$ (where ϕ_0 is the vacuum expectation value $\langle\phi\rangle_0$) and consequently long-range terms from $2\phi_0\phi'$, where $\phi'\sim 1/r$, give nonvanishing contributions to (6). For an ordinary massless field we expect $\phi^2 \sim 1/r^2$ to leave P_0 unchanged, hence the crucial role of our assumption that the fields in question belong to an internal-symmetry group. In this case $\delta P_0 \neq 0$ and is an operator quantity. The only way to avoid this conclusion is to subtract out the vacuum expectation values in $\tau_{\mu\nu}$ which is tantamount to introducing *explicit* breaking in the "improved" $\Theta_{\mu\nu}$. Such a procedure would seem unphysical as well as not useful.

Although a formal calculation indicates that $\Theta^\mu{}_\mu$ vanishes in the limit of scale invariance, more careful considerations show that matrix elements of $\Theta^\mu{}_\mu$ may survive in this limit due to the contribution of the massless Goldstone boson.[21] We regard this limit as that of a sequence of theories in which the dimensional constants are monotonically reduced to zero. The contribution of a scalar meson σ to the form factor occurring in matrix elements of $\Theta^\mu{}_\mu$ is of the form $[m_\sigma^2/(m_\sigma^2 - t)]F_\sigma G_{\sigma PP}$, where $G_{\sigma PP}$ is a coupling constant for particle P, and F_σ measures the strength of the matrix element $\langle 0|\Theta_{\mu\nu}|\sigma\rangle$. In models, what happens is very simple as $\Theta^\mu{}_\mu\to 0$: The vacuum is noninvariant (so that F_σ does not vanish) and the $t=0$ matrix element is smooth as $m_\sigma\to 0$. The nonuniformity of the ratio $m_\sigma^2/(m_\sigma^2 - t)$ is responsible for this phenomenon. This $t=0$ contribution is missed if one first sets $\Theta^{\mu}{}_{\mu}$ equal to zero. In this sense the discrepancy is a consequence of what one means by the (nonuniform) "limit" of scale invariance. This fact lies at the root of the following "puzzle": From the commutator

$$i[D,H]=H-\int d^3x\,\Theta^{\mu}{}_{\mu} \tag{7}$$

we might imagine that as $\Theta^{\mu}{}_{\mu}\to 0$, $i[D,H]=H$ so that $H'=e^{iD\ln\rho}He^{-iD\ln\rho}=\rho H$ is a symmetry transformation. Then the only consistent mass spectrum is either continuous or the point zero. The flaw in this argument is not that D fails to exist when $\dot{D}\neq 0$ (even $H\to H'$ can be represented as a series of well-defined commutators) but rather that the last term in (7) is never really zero when there are massless Goldstone particles coupling to $\Theta^{\mu}{}_{\mu}$. This is easily seen by taking single-particle matrix elements of (7) between states of equal momenta. Finally, we note that in the Goldstone limit the improved generator is not even diagonal; in particular the operator δP_0 has matrix elements between states A and $A\sigma$ where σ is the massless "dilaton". (Similar remarks hold for other operators.) These effects are due to the massless pole arising from the linear term in $\tau_{\mu\nu}$ coming from $(\sigma'+\sigma_0)^2$ (σ is the dilaton field having vacuum expectation value σ_0). This phenomenon is not new but was studied some time ago for the case of internal-symmetry groups by Goldstone, Salam, and Weinberg.[22]

In summary, the "improved" tensor $\Theta_{\mu\nu}$ allows a useful and elegant description of scale and conformal transformations and in particular a simple measure of violation of the symmetries associated with these transformations. However, considerable care is required when working "in the limit" of scale invariance (especially when the latter symmetry is spontaneously broken) because $\Theta_{\mu\nu}$ does not give equivalent results to $T_{\mu\nu}$ in this case. At present there is no reason to doubt the theoretical possibility of scale invariance being an underlying symmetry, broken spontaneously. However, its actual occurrence in nature is yet an open question.

Added note. A recent publication[23] by Castell has treated the problem of spontaneously broken scale invariance.

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