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Threshold Behavior of Regge Trajectories in a Simple Field-Theoretic Model

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The threshold behavior of the trajectory function in $g\varphi^3$ field theory in the ladder approximation is reconsidered. A brief pinch analysis indicates how the elastic and inelastic thresholds arise. Working directly in the angular momentum plane, the "accumulation" of Regge poles near the "promoted" value $l=-\frac{1}{2}$ when $s\to 4m^2$ is demonstrated in a simple manner, starting from the homogeneous Bethe-Salpeter equation.

I. INTRODUCTION

The threshold behavior of Regge trajectories in potential scattering (e.g., off a Yukawa potential) has been studied in detail.[1] It is well known that as k^2 approaches the threshold value 0, an infinite number of Regge poles of the scattering amplitude approach the point $-\frac{1}{2}$ in the plane of the complex angular momentum l. A similar result has been obtained long ago[2] on the basis of general arguments (elastic unitarity) for the case of relativistic scattering. Further work along these lines has been carried out recently[3] in connection with the unitarization of Regge parameters. In the well-studied relativistic model of scalar-scalar scattering via the exchange of another scalar particle in the ladder approximation, this result has been confirmed by Polkinghorne[4] using the Mellin transform of the scattering amplitude. As part of their detailed program on the high-energy behavior of field-theoretic models, Cheng and Wu[5] have worked out carefully the asymptotic behavior ($t\to\infty$) at, near, and away from threshold ($s=4m^2$) of the box diagram and the three-rung (sixth-order) ladder diagram in this ($g\varphi^3$) theory. They have also obtained some results on the behavior of higher-order ladder diagrams, and have shown the correspondence between this model near the threshold and potential scattering (which they have also studied separately[6] in this context). Such work on this "accumulation" and the related "promotion" phenomenon is of considerable interest both experimentally and theoretically.[7]

The treatments of the relativistic case referred to above[4,5,7] are perturbation-theoretic in nature, and are carried out in terms of the Mellin-transformed amplitudes. [The advantages of the use of Mellin transforms in the study of the asymptotic behavior of certain types of Feynman diagrams have been adequately stated in the literature (e.g., Ref. 8).] Since the usual perturbation series (for, say, the trajectory function) diverges term by term at the threshold ($s=4m^2$), one has to go back and take the threshold limit carefully before finding the asymptotic ($t\to\infty$) behavior. To begin answering the question of whether the sum of the asymptotic expressions for the individual diagrams is the asymptotic behavior of the sum of the individual amplitudes, of course, a complete summation of terms of all orders in t^{-1} has to be done.[4] Near the threshold, this problem has not been solved with "full rigor" owing to the formidable difficulties encountered in the higher-order diagrams, as emphasized in Ref. 5.

An "alternative" method that has also been used extensively in the field is the study of the (analytically continued) partial-wave Bethe-Salpeter equation. Indeed, the original paper[9] on Regge behavior in field theory is based on this approach. A method of (stripwise) analytic continuation to the left half plane ($\mathrm{Re}\,l<-\frac{3}{2}$) of the integral equation has also been given,[10] and information on "secondary" Regge poles can be obtained[11,12] using certain operator identities and theorems[10] on the resolvent of an operator that is meromorphic in a parameter (l, in this case). At the point $s=0$, the O(4) symmetry of the integral equation enables one to deduce the existence of daughter sequences.[13] The ladder model referred to above can be explicitly studied in this context and results obtained[12,14] on the secondary-pole spectrum.[15]

It would thus appear to be natural and desirable to reconsider also the threshold behavior of the trajectories in $g\varphi^3$ theory, in the ladder approximation, directly in the complex l plane. In Sec. II, we give a brief pinch analysis to indicate how the various threshold singularities arise in the trajectory function. This analysis, already hinted at in Ref. 9, is merely carried out explicitly here. We then point out how the fact[16] that certain thresholds are absent in some of the trajectory functions can be derived quite simply in our approach; some statements are then made regarding the unequal-mass situation. In Sec. III, we begin directly with the eigenvalue equation concerned, the homogeneous (partial-wave) Bethe-Salpeter equation. It turns out that the "accumulation" phenomenon can be analyzed in a compact and straightforward manner in terms of this eigenvalue equation. After studying how the singularity of the kernel at $s=4m^2$ arises, we write the kernel as the sum of a separable (but "singular") part and a nondegenerate (but "regular") part (as is done, for instance, in Ref. 10 in a different context) and obtain an eigenvalue equation for the singularities of interest in the l plane. From this we easily find the position of the leading Regge pole when $s=4m^2$, to lowest order in the coupling constant[4,5]; we also get an implicit equation for the Regge poles that "accumulate" near $l=-\frac{1}{2}$ as $s\to 4m^2$. This equation is identical in form to that obtained in potential scattering[1] or from elastic unitarity.[2,3] The similarity with the potential-scattering result is of course well known.[2-5] We make a few remarks on the detailed behavior of the poles for s near $4m^2$, and conclude with some comments on the situation when the exchanged mass is zero, and on the question of the "accumulation" of poles when s approaches one of the inelastic thresholds.

II. THRESHOLDS OF THE TRAJECTORY FUNCTION

Let us first introduce some notation. A typical ladder diagram is shown in Fig. 1. The mass of the scattering particles is m, that of the exchanged particle is M. The reduced, on-shell partial-wave amplitude corresponding to the n-rung ladder diagram may be written as[9,12]

$$F_n(l,s,\lambda)=(M^2/\pi)2^{2l+1}(s-4m^2)^{-l-1}\lambda^n\int_0^\infty dx_1\int_{-\infty}^\infty dy_1\,p(x_1,y_1)Q_l(z_{01})$$
$$\times\int_0^\infty dx_2\int_{-\infty}^\infty dy_2\cdots\int_0^\infty dx_{n-1}\int_{-\infty}^\infty dy_{n-1}\,p(x_{n-1},y_{n-1})Q_l(z_{n-2,n-1})Q_l(z_{n-1,0}),\tag{1}$$

where λ is related to the coupling constant g of the m-M-m vertex by $\lambda=g^2/8M^2\pi^3$. The variables x and y refer to the relative momentum and energy, scaled out with respect to M. The product of the two propagators corresponding to the internal lines on the sides of the ladder is denoted by $p(x,y)$ and is given by

$$p(x,y)=\{[x^2+y^2+(4m^2-s)/4M^2]^2+sy^2/M^2\}^{-1}.\tag{2}$$

Finally, the cosines z_{01}, z_{12}, etc., are given by

$$z_{kj}=\frac{1+x_k^2+x_j^2+(y_k-y_j)^2}{2x_kx_j}\equiv\frac{\eta_{kj}}{2x_kx_j}\tag{3}$$

with (for equal masses m, m on the sides of the ladder) $x_0^2=(s-4m^2)/4M^2$, $y_0=0$.

The representation (1) for $F_n(l,s,\lambda)$ is valid, as is well known,[9] in the region $\mathrm{Re}\,l>-\frac{3}{2}$. The full partial-wave amplitude $F(l, s, \lambda)$ has a Regge pole in this region; its position is given by the solution of the implicit equation[9,12]

$$l+1-\lambda L(l, s, \lambda)=0\,, \tag{4}$$

where

$$L_n(l, s)=\int_0^\infty dx_1\int_{-\infty}^\infty dy_1\, p(x_1, y_1)\int_0^\infty dx_2\int_{-\infty}^\infty dy_2\, p(x_2, y_2)R_l(z_{12})\cdots\int_0^\infty dx_{n+1}\int_{-\infty}^\infty dy_{n+1}\, p(x_{n+1}, y_{n+1})R_l(z_{n,n+1})\,,$$
$$L_0(s)=\int_0^\infty dx\int_{-\infty}^\infty dy\, p(x, y)\,. \tag{5}$$

Here $R_l(z_{kj})$ stands for the "regular" part of the "kernel," i.e., $R_l(z)=Q_l(z)-(l+1)^{-1}$. Equation (4) may be solved iteratively for l, to give a solution of the form

$$l=\alpha(s, \lambda)=-1+\lambda\sum_{n=0}^\infty \lambda^n\Lambda_n(s)\,,$$

with

$$\Lambda_0=L_{0,0}\,,\qquad \Lambda_1=L_{0,0}L_{0,1}+L_{1,0}\,,\qquad \text{etc.},$$

where

$$L_{n,r}=\left[\frac{\partial^r L_n(l, s)}{\partial l^r}\right]_{l=-1}. \tag{6}$$

The coefficients in (5) involve integrals of the type

$$I_n(l, s)=\int_0^\infty dx_1\int_{-\infty}^\infty dy_1\, p(x_1, y_1)\int_0^\infty dx_2\int_{-\infty}^\infty dy_2\, p(x_2, y_2)Q_l(z_{12})\cdots\int_0^\infty dx_{n+1}\int_{-\infty}^\infty dy_{n+1}\, p(x_{n+1}, y_{n+1})Q_l(z_{n,n+1})\,. \tag{7}$$

For given values of s and x_k, the poles of $p(x_k, y_k)$ in the y_k plane are at $y_k=\pm y_k^+$ and $\pm y_k^-$, where

$$2y_k^\pm=\pm i\sqrt{s}/M+2i(x_k^2+m^2/M^2)^{1/2}\,. \tag{8}$$

We have taken s to be on the positive real axis, below $4m^2$; $\sqrt{s}$ then stands for $|s|^{1/2}$. As s approaches $4m^2$, the poles at y_k^- and $-y_k^-$ move towards the real y_k axis (the contour of integration in this variable) from opposite sides. They pinch the contour at $y_k=0$ when $x_k=0$ (end point in x_k) and $s=4m^2$. Each $L_n(l, s)$ is singular at $s=4m^2$, $\sum_{n=0}^\infty \lambda^n L_n(l, s)$ diverges, and the usual perturbation theory in powers of λ breaks down at this point.

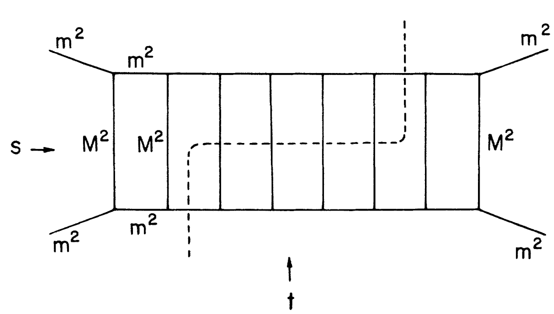

FIG. 1. A typical Feynman diagram in the set considered; m and M are the masses of the scattering and exchanged particles, respectively. The dashed line indicates a many-particle intermediate state in the s channel.

Let us now see how the inelastic thresholds at $s=(2m+nM)^2$ could arise. We continue past $s=4m^2$ by going around this point from above. The y_k contour must now be distorted, since the pole at $-y_k^-$ crosses the real axis from below and that at $+y_k^-$ crosses it from above, as we continue in s. Once the contour runs over complex values of y_k, there is the possibility that it may encounter the images of the singularities of $Q_l(z_{kj})$ at $z_{kj}=\pm 1$. Now the threshold at $s=(2m+nM)^2$ in the *scattering amplitude* arises from an intermediate state in the s channel in which two lines on opposite sides of the ladder, as well as the n rungs of the ladder between these two lines, are put on their mass shells (see Fig. 1). This singularity must therefore be reflected in $I_n(l, s)$ as follows: (i) The two "propagator" factors $p(x_1, y_1)$ and $p(x_{n+1}, y_{n+1})$ become singular, i.e., $y_1=y_1^-$ and $y_{n+1}=-y_{n+1}^-$ (or $y_1=-y_1^-$ and $y_{n+1}=y_{n+1}^-$). (ii) There occur end-point singularities in all the x_k integrals, i.e., $x_k=0$, $k=1, 2, \ldots, n+1$. (iii) $z_{n,n+1}=\pm 1$ pinches the y_{n+1} contour with the pole at $-y_{n+1}^-$; the resulting singularity pinches the y_n contour with $z_{n-1,n}=\pm 1$; and so on, all the cosines $z_{k,k+1}$ being equal to ± 1. (i), (ii), and (iii) give 2, $n+1$, and n conditions, respectively, a total of $2n+3$ constraints on the $2n+2$ integration variables x_k, y_k in $I_n(l, s)$. The result of eliminating all the variables will be a consistency condition on s, which will give the position of the singularity of $I_n(l, s)$. Using (i) and (ii) above, (iii) becomes

$$1+(y_{k+1}-y_k)^2=0, \quad 1\le k\le n \tag{9}$$

where

$$y_1=-y_{n+1}=i(2m-\sqrt{s}\,)/2M.$$

The only consistent sets of solution of (9) are $y_{k+1}=y_k+i$ and $y_{k+1}=y_k-i$, giving $y_{n+1}=y_1\pm ni$. Note that a solution of the type $y_{k-1}=y_{k+1}$ is not allowed since the y_k contour is then no longer pinched. Thus the singularities in s occur at $s=(2m+nM)^2$ and $s=(2m-nM)^2$, corresponding to the threshold and pseudothreshold, respectively. Proving that the pinches *do* occur as stated and that $s=(2m-nM)^2$ is not a singularity on the physical sheet involves a more detailed analysis of the integral $I_n(l, s)$, which we do not give here.

Before concluding this section, we may point out how some of the trajectories of the model may not have certain thresholds. In particular, the first daughter of the leading trajectory isolated in (4) does not have the two-particle (elastic) threshold at $s=4m^2$. This was first pointed out by Swift,[16] who also gave a Feynman-diagrammatic illustration of the reason for this peculiarity. We may show this result to be true very simply as follows: To isolate the particular Regge pole under consideration, the kernel $Q_l(z_{kj})$ of the Bethe-Salpeter equation concerned may be broken up into parts $Q_l^A(z_{kj})$ and $Q_l^S(z_{kj})$ that are, respectively, odd and even in y_k (and thus simultaneously in y_j).[17] The *odd* part may be analytically continued to the left of $\mathrm{Re}\,l=-\frac{3}{2}$, into the region $\mathrm{Re}\,l>-\frac{5}{2}$, simply by converting the x_k integrals to contour integrals,[12] each contour being a "hairpin" contour enclosing the positive real axis. This avoids the divergence at $\mathrm{Re}\,l=-\frac{3}{2}$ arising from the behavior of $Q_l(z_{kj})$ near $x_k=0$. (Such a simple method of continuation does not work for the *even* part of the kernel.) Once this is done, we may isolate the Regge pole of the full partial-wave amplitude that is generated by the pole of the Born term at $l=-2$, since the residue $-y_k y_j/x_k x_j$ of $Q_l^A(z_{kj})$ at $l=-2$ is separable (of finite rank). However, since the kernel is an odd function of y_k and is proportional to y_k near $y_k=0$, the singularity of $p(x_k, y_k)$ as $s\to 4m^2$ and $x_k\to 0$ is canceled, and there is no pinch of the y_k contour. The corresponding trajectory function does not have a threshold at $s=4m^2$.

This conclusion is *not* valid when we consider unequal-mass scattering. The masses on the sides of the ladders are then m_1 and m_2, with $m_1^2+m_2^2=\Sigma$ and $m_1^2-m_2^2=\Delta$ (we may take $m_1>m_2$). The propagator factor $p(x, y)$ is no longer given by (2). It becomes

$$p(x,y)=\left[\left(x^2+y^2+\frac{2\Sigma-s}{4M^2}\right)^2+\frac{s}{M^2}\left(y-\frac{\Delta}{2iM\sqrt{s}}\right)^2\right]^{-1}, \tag{10}$$

which is not an even or odd function of y; the separation of Q_l into Q_l^A and Q_l^S no longer decouples the partial-wave Bethe-Salpeter equation into two parts. The three trajectories[11-13] that approach $l=-2$ in the zero-coupling limit have to be isolated together in this case, after an involved process of analytic continuation. In the weak-coupling limit, we get a cubic equation (first derived by Swift[16]) for the positions of these poles. In our notation we find[12]

$$(l+2)^3-\lambda(l+2)^2(c_{33}-2c_{12})+\lambda^2(l+2)(c_{12}{}^2-c_{11}c_{22}-2c_{13}c_{23}+2c_{12}c_{33})$$
$$+\lambda^3(2c_{12}c_{13}c_{23}-c_{11}c_{23}{}^2-c_{22}c_{13}{}^2+c_{11}c_{22}c_{33}-c_{12}{}^2c_{33})=0, \tag{11}$$

where c_{kj} stands for the "inner product"

$$\langle\varphi_k|\varphi_j\rangle=\langle\varphi_j|\varphi_k\rangle=\int_C dx\int_{-\infty}^{\infty}dy(e^{-i\pi l}/4\cos\pi l)p(x,y)\varphi_k(x,y)\varphi_j(x,y).$$

C is a hairpin contour enclosing the positive real axis in the clockwise sense. The "vectors" φ_k ($k=1, 2, 3$) are given by

$$\varphi_1=x^{l+1}, \quad \varphi_2=x^{l+1}(\tfrac{1}{2}+x^2+y^2)^{-l-1},$$
$$\varphi_3=i\sqrt{2}\,yx^{l+1}.$$

All the inner products in (11) stand for their values at $l=-2$. In some cases the integrals have first to be evaluated and the results explicitly continued analytically to $l=-2$. In any case, we see that two of the poles of the factor $p(x, y)$ pinch the y contour at $y=\Delta/2Mi\sqrt{s}$ when $x=0$ and $s=(m_1+m_2)^2$, i.e., at $y=-i(m_1-m_2)/2M$. (The y contour has already been distorted away from the real axis to avoid a pole of p as s crosses $4m_2{}^2$.) Thus the coefficients in the cubic equation become singular at the two-particle threshold $s=(m_1+m_2)^2$. We can show that at least two of the roots of the equation must have the two-particle threshold, even if $m_1 = m_2$, and that the third root must also have a branch point at $s=(m_1+m_2)^2$ *except* in the case $m_1 = m_2$, when its discontinuity across the associated cut vanishes identically. This root then decouples from the other two and becomes, as is well known,[16]

$$l+2=\pi\lambda M^2/2m^2 = g^2/16\pi^2 m^2,$$

which is independent of s. This is the daughter of the leading pole of the model. Its position of course becomes s-dependent in the higher orders in g^2.

III. "ACCUMULATION" NEAR $l=-\frac{1}{2}$

Having seen that the usual perturbation series in powers of g^2 is not valid as $s\to 4m^2$, let us turn to the homogeneous Bethe-Salpeter equation itself. This reads, in our notation,

$$\psi(x_1, y_1) = \lambda \int_0^\infty dx_2 \int_{-\infty}^\infty dy_2\, p(x_2, y_2) \times Q_l(z_{12})\,\psi(x_2, y_2)\,. \tag{12}$$

The kernel is square-integrable in $\mathrm{Re}\, l > -\frac{3}{2}$ (apart from the simple pole at $l=-1$), and the vanishing of its Fredholm denominator $D(l, s, \lambda)$ gives the positions of the Regge poles in this region.[9] For λ sufficiently small and s away from the threshold at $4m^2$, there is only one Regge pole in this region. near $l=-1$. As $s\to 4m^2$, the right-hand side in (12) approaches a singularity. By studying the way in which this singularity arises, we shall be able to write the kernel as the sum of a separable (but "singular") part and a nonseparable (but "regular") part, and then obtain an approximate equation for the eigenvalues of (12) near $s=4m^2$. This will be an equation of the form

$$l = O((\ln\nu)^0) + O((\ln\nu)^{-1}) + \cdots, \tag{13}$$

where $\nu = (4m^2 - s)/M^2$, and the coefficients of the powers of $(\ln\nu)^{-1}$ on the right-hand side involve λ, although the term $O((\ln\nu)^0)$ will turn out to be a constant, $-\frac{1}{2}$. We shall in fact obtain an infinite number of Regge poles that approach this point in the l plane as $\nu\to 0$. In addition, there will be a pole (the leading pole) to the right of the line $\mathrm{Re}\, l = -\frac{1}{2}$; we shall give the position of this pole to first order in λ, when $s=4m^2$.

Using the identity[18]

$$Q_l(z) = c_l (2z)^{-l-1} F(z^{-2}), \tag{14}$$

where $c_l = B(\frac{1}{2}, l+1)$ and $F(z^{-2})$ stands for

$${}_2F_1(\tfrac{1}{2}(l+1), \tfrac{1}{2}(l+2); \tfrac{1}{2}(2l+3); 1/z^2)\,,$$

(12) becomes

$$\psi(x_1, y_1) = \lambda c_l \int_0^\infty dx_2 \int_{-\infty}^\infty dy_2\, p(x_2, y_2) \left(\frac{x_1 x_2}{\eta_{12}}\right)^{l+1} \times F(z^{-2})\psi(x_2 y_2)\,. \tag{15}$$

Distorting the y_2 contour to pick up the contribution from the pole of $p(x_2, y_2)$ at $y_2 = y_2^-$, the right-hand side of Eq. (15) becomes the sum of two terms: the pole contribution, plus an integral over the distorted contour.[19] It is the former that must become singular at $s=4m^2$. This term is

$$\pi c_l \frac{M}{\sqrt{s}} \int_0^\infty dx_2 \frac{(x_1x_2)^{l+1}\left[\eta_{12}{}^{-l-1}F(z_{12}{}^{-2})\psi(x_2, y_2)\right]_{(y_2=y_2^-)}}{(x_2{}^2 + m^2/M^2)^{1/2}\left[2(x_2{}^2+m^2/M^2)^{1/2} - (\sqrt{s}/M)\right]^{1/2}}\,. \tag{16}$$

When $s=4m^2$, the integrand in (16) behaves like $x_2{}^{2l}$ near the origin. The integral therefore diverges for $\mathrm{Re}\, l \leq -\frac{1}{2}$ because of the lower limit of integration. We may expect the function defined by the integral (in the region where it converges) to be singular at $l=-\frac{1}{2}$, *when* $s=4m^2$. This singularity comes from the behavior of the integrand in (16) in the region near $y_2=0$, $x_2=0$. When $y_2=0$, $x_2=0$, the quantity $\eta_{12}{}^{-l-1}F(z_{12}{}^{-2})$ reduces to $(1+x_1{}^2+y_1{}^2)^{-l-1}$. In a symmetric form, we could write instead of this the expression $[(1+x_1{}^2+y_1{}^2)(1+x_2{}^2+y_2{}^2)]^{-l-1}$, which also reduces to the same thing. This latter expression has the advantage that the *large* x_2, y_2 behavior of the original expression is not altered (thus preventing the introduction of spurious divergences at the upper limits of integration). This last expression is thus a logical candidate for the separable part of the kernel.

Let us write (15) symbolically as

$$|\psi\rangle = \lambda K_l |\psi\rangle\,, \tag{17}$$

where K_l is an integral operator with the kernel

$$K_l(x_1, y_1; x_2, y_2) = c_l (x_1x_2)^{l+1} \eta_{12}{}^{-l-1} F(z_{12}{}^{-2})\,,$$

and the variables are in the region $0\leq x<\infty$, $-\infty<y<\infty$. The measure of integration is $p(x, y)$. We define the operator H_l by

$$H_l = K_l - |f\rangle\langle f|\,,$$

where

$$f(x, y) = (c_l)^{1/2} x^{l+1} (1+x^2+y^2)^{-l-1}\,. \tag{18}$$

Then the following formal manipulations may be carried out: Eq. (17) becomes

$$|\psi\rangle = \lambda\langle f\,|\,\psi\rangle\,(1-\lambda H_l)^{-1}|f\rangle ,$$

which gives the eigenvalue condition

$$1 = \lambda\langle f\,|(1-\lambda H_l)^{-1}|f\rangle . \tag{19}$$

Keeping λ within the circle of convergence of the Neumann series for the resolvent $(1-\lambda H_l)^{-1}$, we may write

$$1 = \lambda\langle f\,|\,f\rangle + \lambda\sum_{n=1}^{\infty}\lambda^n\langle f\,|\,(H_l)^n\,|f\rangle . \tag{20}$$

A typical term of the infinite sum on the right-hand side in Eq. (20) is *not* singular at $l=-\frac{1}{2}$ when $s=4m^2$, because $H_l(x_1, y_1; x_2, y_2)$ vanishes when $x_2, y_2=0$ or $x_1, y_1=0$ or both. We have in fact subtracted out the singular part in defining H_l. H_l is square-integrable in the neighborhood of $l=-\frac{1}{2}$ even if $s=4m^2$, and the second term on the right-hand side in (20) is regular in the neighborhood of $s=4m^2$ and $l=-\frac{1}{2}$, for λ sufficiently small.[20] We may expand it (near $s=4m^2$) in the form

$$\sum_{n=1}^{\infty}\lambda^n\langle f\,|\,(H_l)^n\,|f\rangle = c_l\sum_{n=0}^{\infty}h_n\nu^n . \tag{21}$$

The first term on the right-hand side in Eq. (20) is

$$\lambda\langle f\,|f\rangle = \lambda c_l\int_0^{\infty}dx\int_{-\infty}^{\infty}dy\,\frac{x^{2l+2}(1+x^2+y^2)^{-2l-2}}{[x^2+y^2+(4m^2-s)/4M^2]^2+sy^2/M^2} . \tag{22}$$

The integral in (22) is divergent for $\mathrm{Re}\,l\leq-\frac{1}{2}$ when $s=4m^2$. Imposing the condition (20) (keeping λ small) will then force l to approach a particular value $(-\frac{1}{2})$ as $s\to 4m^2$. To see how this comes about, let us again isolate the contribution to (22) from the pole of the integrand at $y=y^-$. This will pinch the contour in y with the pole at $y=-y^-$ when $x=0$, $s=4m^2$. We write the pole contribution as $\lambda\langle f\,|f\rangle_S$. The rest of $\lambda\langle f\,|f\rangle$ will be regular at $l=-\frac{1}{2}$ even for $s=4m^2$.[21]

With a change of variables, we find

$$\langle f\,|f\rangle_S = \pi c_l\frac{M}{\sqrt{s}}\times\int_0^{\infty}dz\,\frac{z^{l+1/2}[z+(4m/M)^{1/2}]^{l+1/2}[z+(2m-\sqrt{s})/M]^{-1}}{\{2+(\sqrt{s}/M)[z+(2m-\sqrt{s})/M]\}^{2l+2}} . \tag{23}$$

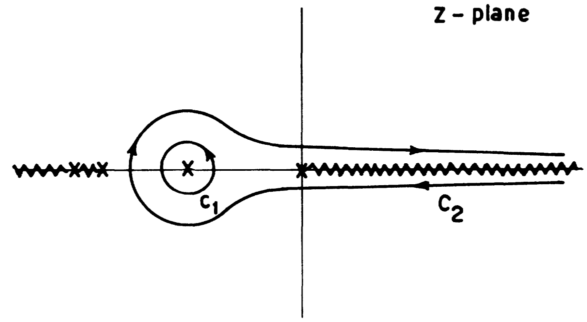

FIG. 2. The distorted contour of integration for the integral in Eq. (23). C_1 encircles the pole of the integrand at $z=-(2m-\sqrt{s})/M$. The branch cut of $z^{l+\frac{1}{2}}$ runs along the positive real axis.

For $s<4m^2$, Eq. (23) converges in the region $\mathrm{Re}\,l>-\frac{3}{2}$. When $s=4m^2$, it converges in the region $\mathrm{Re}\,l>-\frac{1}{2}$.[22] The form (23) makes the mechanism clear. Let us write (23) as a contour integral, the contour being the familiar hairpin contour around the branch cut of $z^{l+1/2}$ from 0 to ∞. This contour may then be distorted to the contour of Fig. 2. The integral over C_2 is not singular at $s=4m^2$. Its value exactly at $s=4m^2$ can in fact be found by putting $s=4m^2$ in the integrand and reverting to a line integral along the positive real axis. We find that, in the neighborhood of $\nu=0$, the integral over C_2 may be written as

$$\frac{\pi c_l M}{4m(l+\frac{1}{2})}\,{}_2F_1\!\left(-l-\tfrac{1}{2},\, l+\tfrac{1}{2};\, l+\tfrac{3}{2};\, 1-\frac{M^2}{4m^2}\right)[1+O(\nu)] . \tag{24}$$

The integral over C_1 becomes singular at $s=4m^2$, for then the contour is pinched between the pole it encircles and the branch point at $z=0$. The integral may be evaluated trivially to give

$$-(\pi c_l M/4m)\Gamma(l+\tfrac{1}{2})\Gamma(-l+\tfrac{1}{2})\nu^{l+1/2}[1+O(\nu)] . \tag{25}$$

Therefore, the eigenvalue condition (20) becomes, using (21), (24), and (25),

$$\begin{aligned}1 = {}&\pi\lambda c_l(M/4m)(l+\tfrac{1}{2})^{-1}\,{}_2F_1(-l-\tfrac{1}{2},\, l+\tfrac{1}{2};\, l+\tfrac{3}{2};\, 1-M^2/4m^2)[1+O(\nu)]\\ &-\pi\lambda c_l(M/4m)\Gamma(l+\tfrac{1}{2})\Gamma(-l+\tfrac{1}{2})\nu^{l+1/2}[1+O(\nu)]+\lambda c_l\sum_{n=0}^{\infty}h_n'\nu^n ,\end{aligned} \tag{26}$$

where the last term on the right-hand side represents

$$\lambda\left[\sum_{n=1}^{\infty}\lambda^n\langle f\,|\,(H_l)^n\,|f\rangle+\langle f\,|f\rangle-\langle f\,|f\rangle_S\right] .$$

The right-hand side in (26) is of course not singular at $l=-\frac{1}{2}$ for $\nu\neq 0$; the first two lines add up to $\lambda\langle f|f\rangle_s$, for which this property has been explicitly demonstrated.

Let us analyze Eq. (26). If we ask for solutions that are to the right of the line $\mathrm{Re}\, l=-\frac{1}{2}$, these are given, exactly at $s=4m^2$, by

$$1=\pi\lambda c_l(M/4m)(l+\tfrac{1}{2})^{-1}{}_2F_1(-l-\tfrac{1}{2},\, l+\tfrac{1}{2};\, l+\tfrac{3}{2};\, 1-M^2/4m^2)+\lambda c_l h_0',$$

where

$$c_l h_0'=\left[\sum_{n=1}^{\infty}\lambda^n\langle f|(H_l)^n|f\rangle+\langle f|f\rangle-\langle f|f\rangle_s\right]_{s=4m^2}.$$

The solution to the above equation is

$$l=-\tfrac{1}{2}+g^2/(32\pi mM)+O(g^4), \tag{27}$$

which gives the approximate position[4,5] of the leading Regge pole at $s=4m^2$.

Equation (26) also has other solutions, located near $l=-\frac{1}{2}$. The equation may be written in the form

$$C(\xi)\nu^{\xi}[1+(\text{terms proportional to positive powers of }\nu)]=1, \tag{28}$$

where $\xi=l+\frac{1}{2}$, and

$$C(\xi)=B(1-\xi,1+\xi)\left[{}_2F_1(-\xi,\,\xi;\,1+\xi;\,1-M^2/4m^2)-\frac{4m\xi\Gamma(1+\xi)}{\lambda\pi^{3/2}M\Gamma(\frac{1}{2}+\xi)}+\frac{4m\xi h_0'}{\pi M}\right]^{-1}.$$

The point to note is that $C(0)=1$. Equation (28) is solved by an expansion for ξ in inverse powers of $\ln\nu$. The parts involving powers of ν higher than ξ will lead to terms $\sim\nu^{\epsilon}(\ln\nu)^{-n}$, $\epsilon>0$. These may be neglected compared to terms proportional to $(\ln\nu)^{-n}$. As $\nu\to 0$, therefore, the equation to be solved is of the form

$$C(\xi)\nu^{\xi}=1, \tag{29}$$

with the property $C(0)=1$. This is precisely of the form found in potential scattering.[1] We are concerned here with the "0" type of Regge poles,[23] i.e., those that approach $\xi=0$.[24] The method of solution of Eq. (29) is familiar.[1] One writes $C(\xi)=\sum_n\gamma_n\xi^n$, $\gamma_0=1$, and $\nu^{\xi}=\exp(\xi\ln\nu+2\pi n_1 i\xi)$, where the integer n_1 must be taken to be zero on the physical sheet in s. The solution sought is of the type

$$\xi=\sum_{n=0}^{\infty}\delta_n(\ln\nu)^{-n},$$

where δ_0 must obviously be zero. As $s\to 4m^2$ on the real axis from the left, $\nu\to 0+$ and $\ln\nu=\ln|\nu|$. For $s>4m^2$, $\ln\nu=\ln|\nu|-i\pi$ just *above* the real axis. We find the following expansions for the positions of the Regge poles in the two cases:

(1) below threshold,

$$\begin{aligned}\mathrm{Im}\, l&=2\pi n(\ln|\nu|)^{-1}-2\pi n\gamma_1(\ln|\nu|)^{-2}\\&\quad+2\pi n\gamma_1^2(\ln|\nu|)^{-3}+\cdots,\end{aligned} \tag{30}$$

$$\mathrm{Re}\, l=-\tfrac{1}{2}+4\pi^2n^2(\gamma_2-\tfrac{1}{2}\gamma_1^2)(\ln|\nu|)^{-3}+\cdots;$$

(2) above threshold,

$$\begin{aligned}\mathrm{Im}\, l&=2\pi n(\ln|\nu|)^{-1}-2\pi n\gamma_1(\ln|\nu|)^{-2}\\&\quad+2\pi n(\gamma_1^2-\pi^2)(\ln|\nu|)^{-3}+\cdots,\\ \mathrm{Re}\, l&=-\tfrac{1}{2}-2\pi^2 n(\ln|\nu|)^{-2}+4\pi^2 n\\&\quad\times(\gamma_1+n\gamma_2-\tfrac{1}{2}n\gamma_1^2)(\ln|\nu|)^{-3}+\cdots.\end{aligned} \tag{31}$$

In (30) and (31), $n=0,\pm1,\pm2,\ldots$, with $2\pi|n|<<|\ln|\nu||$. We have used the fact that γ_1, γ_2, etc., are real [because $C(\xi)$ is a real analytic function of ξ]. The solution corresponding to $n=0$ is not a fixed pole at $l=-\frac{1}{2}$ (our model has no such pole). It must be understood as a possible Regge pole which approaches $l=-\frac{1}{2}$ according to $l=-\frac{1}{2}+O(\nu^{\epsilon}/\ln\nu)$, $\epsilon>0$. We have not considered such solutions of (28).

Equations (30) and (31) exhibit several features that obtain in potential scattering.[1] As $\nu\to 0$, n can take on larger and larger values, and more and more poles approach $l=-\frac{1}{2}$. Below threshold, $\mathrm{Re}\, l$ begins to deviate from the value $-\frac{1}{2}$ only in third order in $(\ln|\nu|)^{-1}$. It may be shown from (30) and (31) that the pole trajectories osculate the line $\mathrm{Re}\, l=-\frac{1}{2}$. To find out whether the poles osculate this line from the left (as in potential scattering) or from the right for s below threshold, we have to know if $\gamma_2-\frac{1}{2}\gamma_1^2 \lesseqgtr 0$, i.e., if $[d^2\ln C(\xi)/d\xi^2]_{\xi=0}\lesseqgtr 0$. This would require an involved calculation which we do not carry out here.[25-27] Above threshold, since $2\pi n(\ln|\nu|)^{-1}$ and $-2\pi n(\ln|\nu|)^{-2}$ have the same sign (that of $-n$), Regge poles approach $l=-\frac{1}{2}$ from above to the right of $\mathrm{Re}\, l=-\frac{1}{2}$ and from below to the left of that line as $\nu\to 0$.

We conclude with a few general remarks. The "accumulation" of Regge poles near $l=-\frac{1}{2}$ as $s\to 4m^2$ has been explained by Gribov and Pomeranchuk[28] as follows: An infinite number of Landau curves of the scattering amplitude $F(s,t)$ have the line $s=4m^2$ as their common asymptote; as $t\to\infty$ (the region where the s-channel Regge poles dominate), an infinite number of singularities of $F(s,t)$ approach $s=4m^2$. We note that if the exchanged mass M is equal to zero in our set of diagrams, all the Landau curves become degenerate (all the t-channel thresholds merge) and we do not expect any "accumulation" to occur. Nor does it,[29] as a study of the Regge-pole spectrum of this model has shown. It does not occur in nonrelativistic Coulomb scattering either,[30] as expected.

We have seen that inelastic thresholds at $s=(2m+nM)^2$ are also present in the trajectory function. Once again, Gribov and Pomeranchuk[28] have given arguments supporting the "accumulation" of Regge poles near the line $\mathrm{Re}\,l=-\frac{1}{2}(3N-5)$ as s approaches an N-particle threshold. However, our set of diagrams does not treat three- and more-particle intermediate states very realistically. Specifically, the amplitudes for the set of Feynman diagrams considered have no Landau curves with $s=(2m+nM)^2$, $n\geq 1$, as an asymptote; all the Landau curves of the model have as their only asymptote in s the two-particle threshold line $s=4m^2$. Therefore we do not expect the Regge trajectories generated by *this* set of diagrams to accumulate near $l=-\frac{1}{2}(3N-5)$ as s approaches the N-particle threshold at $[2m+(N-2)M]^2$, where $N\geq 3$.

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[17] This separation into two equations is, of course, possible because the "measure" of integration $p(x,y)$ is an even function of y, and the kernel is a function of $(y_k-y_j)^2$, so that if $\psi(x,y)$ is an eigenfunction, so is $\psi(x,-y)$; the eigenfunctions can therefore be classified as even or odd in y. Q_l^S and Q_l^A "generate" trajectories of even and odd time parity, respectively.

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[19] This step of course presumes that the eigenfunction $\psi(x,y)$ can be analytically continued off the real axis in y, without encountering any singularities, to the region where the contour lies. The analyticity required can be established, but we do not concern ourselves with this point any further since this part of the analysis is only of heuristic value: It serves to help us arrive at a suitable choice for the separable, "singular," part of the kernel.

[20] The radius of convergence of the series on the right in (20) does not shrink to zero as $s\to 4m^2$, since the Fredholm denominator corresponding to the (square-integrable) kernel H_l approaches unity as $\lambda\to 0$.

[21] This would not be true if the factor $(1+x^2+y^2)^{-2l-2}$ were absent in the integrand, because of the infinite ranges of integration.

[22] With a further change of variables, to $z'=z/[z+(2m-\sqrt{s})/M+2M/\sqrt{s}]$, the integral in (23) may be brought to a standard form (Ref. 18) and written (Ref. 12) (reflecting a current trend) in terms of a hypergeometric function F_1 of two variables. This representation, however, is too cumbersome for our purpose and obscures the manner in which the various singularities arise.

[23] See the second part of Ref. 1.

[24] The method is not geared to obtaining information on any possible "C"-type Regge poles (Ref. 23), i.e., those that arise from zeros of $C(\xi)$ as $\nu\to 0$, in the region $\mathrm{Re}\,\xi<0$. We have already, in (27), obtained information on the leading pole which comes from a pole of $C(\xi)$ as $\nu\to 0$, in the region $\mathrm{Re}\,\xi>0$.

[25] Note that the general *requirement* (Ref. 26) that there be at least one Regge pole to the right of the line $\mathrm{Re}\,l=-\frac{1}{2}$ for s below threshold does not apply in our model; this requirement is derived from crossed-(t-) channel

unitarity, which is not satisfied by our set of diagrams. However we do have a Regge pole in the region $\mathrm{Re}\, l > -\frac{1}{2}$ when $s = 4m^2$, given by (27), that approaches $l = -\frac{1}{2}$ as the coupling goes to zero. This must be understood to be a result of the close similarity between our model and scattering off a superposition of Yukawa potentials. (Also, $s \sim 4m^2$ *is* the nonrelativistic region in the s channel). Regge (Ref. 27) has shown that for attractive potentials of the above class, there is at least one Regge pole in $\mathrm{Re}\, l > -\frac{1}{2}$.

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