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Statistical Bootstrap Model of Hadrons with Spin*

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We formulate the statistical bootstrap model of hadrons so that it includes angular momentum. Several types of solutions that the bootstrap admits are studied. One self-consistent solution, which has a strong physical motivation founded within the model, has a spin (J) distribution of resonances at mass m in which most resonances occur below a J proportional to $\sqrt{m}$, the constant of proportionality being bounded by the other parameters of the model. There is then a whole set of solutions, that have J proportional to $(\sqrt{m})^\gamma$, where γ lies between 0 and 1. This set of solutions cannot be excluded within the accuracy to which we can work. We also find a third class of solutions, very similar to the resonance spin spectrum of the Veneziano model, in which most resonances occur below a spin proportional to m. Finally, we have discovered a second solution to the bootstrap of the total level density.

I. INTRODUCTION

There is a remarkable coincidence between the predictions of the statistical bootstrap model of hadrons[1,2] and the dual-resonance model.[3] For, both of these apparently quite different models give the total level density, $n(m)$, of hadrons at mass m in the form

$$n(m)dm \sim \text{const}\, m^a e^{bm} \quad \text{as} \quad m \to \infty , \tag{1.1}$$

where, in the Veneziano model,[4,5]

$$b = 2\pi\left(\frac{D\alpha'}{6}\right)^{1/2} \quad \text{and} \quad a = -\frac{D+1}{2}$$

and $D = 4$ in the usual simple model, but may be a larger integer, to include different trajectory intercepts,[6] in which case

$$D = 5 \quad \text{giving} \quad a = -3$$

could be favored.[7,4] More generally, we have for

$$D = 4, 5, 6, \ldots ,$$

$$a = -\tfrac{5}{2}, -3, -\tfrac{7}{2}, \ldots ,$$

$$b^{-1} = 180, 174, 159, \ldots \text{ MeV} \quad (\text{for } \alpha' = 1 \text{ GeV}^{-2}).$$

The statistical model gives[8,9]

$$b^{-1} \approx 160 \text{ MeV}$$

and[2]

$$a < -\tfrac{5}{2} ,$$

but again a more exact treatment actually prefers[9,10]

$$a = -3 .$$

Further, for every extra internal quantum number (like charge) included, the value of a is effectively decreased in steps of $\frac{1}{2}$, if all exotic states are counted.[9]

This agreement between such conceptually disparate models is remarkable, but they have further results in common as well:

(i) Resonances couple and decay through two-particle channels;

(ii) The predominant (two-body) decay mode of a high-mass resonance is into one high-mass and one low-mass particle;

(iii) Both models rely upon the narrow-resonance approximation.

Result (i) is of course exact in the dual-resonance model, by construction: The scattering amplitude can always be decomposed completely into tree diagrams. In the statistical model, it is the predominant coupling,[2] accounting for ≈ 70% of decays. In fact, this model provides the first deductive argument[2] for the usual theoretical assumption of this result. The second result, (ii), is true in the statistical model, where it plays an important part in the self-consistency argument that leads to $a < -\frac{5}{2}$.[2] It has also been found to be true in the Veneziano model,[11] at least for the resonances on the leading few trajectories. The situation for low-lying resonances is, however, uncertain.

The general agreement of the two models on these problems may suggest that the general predictions of the Veneziano model follow from weaker assumptions than those embodied in the full model, and, in particular, from the limited number of assumptions that it does share with the statistical model. Indeed, a rapidly growing level density like Eq. (1.1) follows from local duality alone.[12] As we shall indicate below, there is an analogous assumption in the statistical model, involving resonance saturation.

To test this conjecture further, we compare the predictions of the two models for the distribution of spins over the resonances that occur at a high mass (m). We have previously calculated this for the Veneziano model,[13] and found that the fraction of resonances at mass m with spin J is given by

$$\bar{\rho}_m(J) = \frac{4c^2}{m^2}\,\frac{\sinh(c\hat{b})}{\cosh^3(c\hat{b})} \quad \text{as} \quad m \to \infty\,, \tag{1.2}$$

where $\hat{b}$ is the impact parameter ($\hat{b} \to 2J/m$) and[14]

$$c = \frac{\pi}{4}\left(\frac{D}{6\alpha'}\right)^{1/2} = \frac{b}{8\alpha}\,.$$

The distribution of z component of spin, J_z, from which Eq. (1.2) follows, is

$$\sigma_m(J_z) = \frac{c}{m}\,\text{sech}^2(c\hat{b})\,. \tag{1.3}$$

Our aim in this paper is to set up the formalism of the statistical bootstrap when spin is taken into account, and to explore the types of solutions it may admit. In Sec. II we outline the main assumptions of the model, and establish the spin formalism. For this, we need to undertake the necessary preliminary of a quantum-mechanical calculation of the orbital-angular-momentum z-component projection of two-body phase space, using an elegant method of Cerulus,[15] based upon group theory. In Sec. III, we deal in detail with one approximate trial solution, motivated by a physical argument,[16] and discuss some aspects of its uniqueness. This solution has most resonances occurring below spin $J \propto \sqrt{m}$, and is discussed in Sec. IV. In Sec. V, we give a mainly qualitative discussion of the effects of the many-body terms, arguing that they have very little influence on the achievement of self-consistency, or on the main features of the spin spectrum. Then in Sec. VI we investigate some other types of solutions, which, like the Veneziano model, have most resonances below $J \propto m$. It is found that this kind of solution is also allowed, at least to the accuracy to which one can work. We also show that, with this accuracy, solutions like that of Sec. III but with $J \propto (\sqrt{m})^{\gamma}$ $(0 < \gamma < 1)$ cannot be excluded. The situation is summarized in the discussion of Sec. VII.

In our treatment, we do not include charge (Q), or other internal quantum numbers, but these are easily allowed for[9]: If no restriction on exotic resonances is made, all level densities should be multiplied by

$$\left(\frac{d'}{\pi m}\right)^{1/2} e^{-d'Q^2/m}, \quad d' = \text{const} \approx 2m_\pi$$

for each such quantum number.[9] This effectively reduces a by $\frac{1}{2}$ for each quantum number, exactly like increasing D in the Veneziano model in order to include each additional different trajectory intercept.[6]

II. THE STATISTICAL BOOTSTRAP WITH SPIN

A. Assumptions of the Model

The model reflects the ideas of hadrons being composites of one another, and of resonance dominance of hadron interactions. Explicitly, one assumes that[1,2]

(i) resonances continue (indefinitely) to high mass;
(ii) hadrons are composites, built from each other self-consistently (bootstrap principle);
(iii) the mutual interactions of the hadron constituents can be completely represented by resonance formation;
(iv) the only other effect of interactions is to confine the constituents within a characteristic volume (V).

Here, (i) to (iii) are also assumed in the dual-resonance model, where the bootstrap condition is duality and Regge behavior. But the interaction radius [(iv)] has a different meaning.

It then follows that the constituent hadrons can be treated as free particles, provided that all resonance excitations are also included. The density of hadron energy levels is then determined by the free-particle phase space available. According to a theorem of Beth and Uhlenbeck,[17] this counting is exact for zero-width resonances: Attractive interaction gives a positive phase shift, $\delta_l(p)$; when this reaches π, one extra state can be incorporated into the phase space. Each narrow resonance would provide such a shift, so the effect of two-body interactions can be represented by ignoring the direct effects of the force, and then feeding in all the resonances needed to saturate it, and counting their free-particle phase space as well. The connection is that the level density (as a function of center-of-mass momentum p, at fixed spin l in an enclosure of radius R) is

$$\frac{\partial n(p,l)}{\partial p} = \frac{1}{\pi}\left(R + \frac{\partial \delta_l(p)}{\partial p}\right).$$

A decreasing phase shift cannot make this negative, since Wigner's theorem,[18] based upon causality, does not permit a phase shift to decrease faster than R.

Before turning to the detailed formulation of the model, we mention a physically interesting interpretation of it.[2] The bootstrap equation [Eq. (2.1) below] may be interpreted to say that there is a matching between the resonance level density, and the number of decay channels of that resonance into lighter ones. Roughly, there should be one resonance for each decay channel. So the supply of resonances available at any mass must increase with mass at just the right rate to fill up all of the open decay channels available. This relationship is also implied by duality.[12] This matching is exact if we count different momentum states of the same decay products as constituting different channels (which is not, of course, the normal practice). The effect of the phase space is to reduce a by $\frac{3}{2}$.

B. The Bootstrap Equation with Spin

The equation[1,2] that implements these ideas is that, as $m \to \infty$,

$$n(m) \sim \sum_{N=2}^{\infty} \frac{1}{N!}\left(\frac{V}{h^3}\right)^{N-1} \prod_{i=1}^{N} \int d^3p_i \int dm_i\, n(m_i)\delta^{(3)}\Big(\sum_i \vec{p}_i\Big)\delta\Big(\sum_i E_i - m\Big). \tag{2.1}$$

This is only required as $m \to \infty$, as the low-mass spectrum cannot be correctly represented by these rough concepts. The general term (N) comes from N-particle contributions to the level density, and the $1/N!$ factor gives some allowance for Bose statistics, and for double counting.[2]

If now $\sigma(m, J_z)$ is the number of resonances at mass m with spin component J_z, the bootstrap equation for resonances of definite J_z is

$$\sigma(m, J_z) \to \frac{1}{2h^3}\int dJ_{1z}\int dm_1\sigma(m_1, J_{1z})\int dJ_{2z}\int dm_2\sigma(m_2, J_{2z})\int dl_z\phi(m_1, m_2; m; l_z)\delta(J_z - J_{1z} - J_{2z} - l_z)$$
$$+ (\text{terms with } N \geq 3), \tag{2.2}$$

where $\phi\ (m_1, m_2; m; l_z)$ is that projection of the two-body phase-space integral

$$\Phi_{m_1, m_2}(m) = \int_V d^3r_1 \int_V d^3r_2 \int d^3p_1 \int d^3p_2\, \delta^{(3)}(\vec{R}^{\,\text{c.m.}})\delta^{(3)}(\vec{p}_1 + \vec{p}_2)\delta(E_1 + E_2 - m), \tag{2.3}$$

which ensures that the two particles (m_1 and m_2) have z component l_z of their orbital angular momentum in their center-of-mass frame.

For the present we shall drop the multiparticle ($N \geq 3$) terms in Eq. (2.2). We only treat the two-body term explicitly, but this cannot be expected to influence the main details of the solution, since it does not do so[2] in Eq. (2.1), which includes all J_z: If all J_z states couple strongest to two-body channels, this must also be true of almost all of the individual J_z states. We shall discuss the N-body terms in Sec. V.

To treat Eq. (2.2) conveniently, we take its Fourier transform with respect to J_z, writing

$$\delta(J_z - J_{1z} - J_{2z} - l_z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha' \exp[i\alpha'(l_z + J_{1z} + J_{2z} - J_z)],$$

$$\tilde{\sigma}(m, \alpha) = \int_{-\infty}^{\infty} dJ_z e^{i\alpha J_z}\sigma(m, J_z), \tag{2.4}$$

$$\tilde{\phi}(m_1, m_2; m; \alpha) = \int_{-\infty}^{\infty} dl_z e^{i\alpha l_z}\phi(m_1, m_2; m; l_z), \tag{2.5}$$

giving our basic bootstrap equation: As $m \to \infty$,

$$\tilde{\sigma}(m, \alpha) \sim \frac{1}{2h^3}\int_{\mu}^{m-\mu} dm_2\tilde{\sigma}(m_2, \alpha)\int_{\mu}^{m-m_2-\mu} dm_1\tilde{\sigma}(m_1, \alpha) \times\tilde{\phi}(m_1, m_2; m; \alpha), \tag{2.6}$$

where μ is the lowest-mass state.

This simple form most clearly reflects the underlying physics: $\tilde{\phi}$ describes the free-particle orbital motion, and the m_1 and m_2 integrals describe the effects from resonance excitation. $\tilde{\phi}$ must be calculated from Eq. (2.3). At $\alpha = 0$, Eqs. (2.4) and (2.5) show that our bootstrap equation (2.6) becomes the original one of Frautschi, (2.1); in particular,

$$\tilde{\sigma}(m, 0) = n(m). \tag{2.7}$$

It is evident that this formulation for including spin leads to a bootstrap problem very similar to that of Frautschi,[2] except that we have to bootstrap both the m and α dependences, and that these are coupled by the phase-space factor. We shall first imagine imposing the bootstrap in the sense that the fractional difference of the two sides of Eq. (26) is only of order $1/m$ as $m \to \infty$. This will be relaxed later.

C. Angular Momentum Content of Two-Body Phase Space

We need to project the phase-space integral Φ, Eq. (2.3), so that it counts only those states with orbital angular momentum z component l_z. Without such a projection, the space integrals in Eq. (2.6) are uncoupled from the momentum integrals, and one has

$$\Phi_{m_1,m_2}(m) = V\int d^3p\,\delta(E_1+E_2-m) = 4\pi V\,(E_1E_2p/m)\,,$$

where

$$E_1(p)+E_2(p)\equiv m\,;\quad E_i(p)\equiv(p^2+m_i^2)^{1/2}\,. \tag{2.8}$$

Here, V is the integral $\int_V d^3r$ over the relative coordinate of the particles. It is determined by the kind of spatial cutoff assumed, but such details do not affect the problem at all when spin is not explicitly projected out. For particles in a box,[1,2] V is the volume of the box; if we have a less sharp cutoff, such as a Gaussian, then

$$\int_V d^3r \to \int d^3r\, e^{-\vec{r}^2/R^2} = (\sqrt{\pi}R)^3\,, \tag{2.9}$$

which may be defined to be V.

But when angular momentum $\vec{l}=\vec{r}\times\vec{p}$ is projected out, there is a coupling of the $\vec{r}$ and $\vec{p}$ dependences, and in this case the answer does depend upon the nature of the volume cutoff. But when this is sharp, only the very high-spin states (of which there are very few) show much sensitivity to the detailed cutoff, and, for purely technical reasons, we shall use the Gaussian, Eq. (2.9). A box, and a Yukawa, cutoff yield similar results.

The phase-space integral Φ is a count of momentum eigenstates inside the enclosure, weighted by their (uniform) spatial distribution upon performing the space integrals. Equation (2.3) gives

$$\Phi_{m_1,m_2}(m) = \int_V d^3r\int d^3p\,\delta(m-E_1(p)-E_2(p)) \times[e^{-i\vec{p}\cdot\vec{r}}e^{i\vec{p}\cdot\vec{r}}]\,. \tag{2.10}$$

When only states with a definite l_z are to be counted, their spatial distribution is no longer uniform, and is obtained, following Cerulus,[15] by projecting l_z eigenstates from the momentum eigenstate $e^{i\vec{p}\cdot\vec{r}}$. Then from Eq. (2.10)

$$\phi(m_1,m_2;m;l) = \int_V d^3r\int d^3p\,\delta(m-E_1(p)-E_2(p)) \times[e^{-i\vec{p}\cdot\vec{r}}\mathcal{P}_{l_z}e^{i\vec{p}\cdot\vec{r}}]\,, \tag{2.11}$$

where $\mathcal{P}_{l_z}$ is the projection operator onto a state of definite l_z. It is[15]

$$\mathcal{P}_{l_z} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, e^{-il_z\phi}e^{i\phi L_z}\,, \tag{2.12}$$

where $L_z=-i\partial/\partial\phi$ is the operator for the z component of orbital angular momentum. This procedure is essentially a partial-wave analysis. Equation (2.12) follows from the general Wigner method for constructing projection operators. It is based on the orthogonality properties of group representations, but in our simple case, its form is self-evident. In particular, $e^{i\phi L_z}$ is the transformation operator for a rotation of $-\phi$ about the z axis, and $(1/2\pi)\int_{-\pi}^{\pi} d\phi$ is a summation over this two-dimensional subgroup of O(3), normalized to the total "number" of transformations.

At this point, we must make a mathematical comment. In our formulation, we have used continuous l_z values, rather than the discrete (physical) l_z spectrum. Thus Eq. (2.2) has $\int dl_z$ rather than $\sum_{l_z}$, and a Dirac rather than a Kronecker δ. To account for this, and ensure that Eq. (2.12) projects out definite continuous l_z,[19] we should replace

$$\int_{-\pi}^{\pi}\frac{d\phi}{2\pi} \to \int_{-\infty}^{\infty}\frac{d\phi}{2\pi}$$

whenever the l_z spectrum is a continuum.

Now if $\vec{p}\,'$ is the vector $\vec{p}$ rotated by an angle $-\phi$ about the z axis (unit vector $\hat{z}$),

$$\vec{K}(\phi)\equiv\vec{p}\,'-\vec{p} \equiv(\vec{p}\times\hat{z})\sin\phi+(\vec{p}\times\hat{z})\times\hat{z}(1-\cos\phi) \tag{2.13}$$

and

$$e^{i\phi L_z}e^{i\vec{p}\cdot\vec{r}} = e^{i\vec{p}\,'\cdot\vec{r}}\,,$$

so

$$[e^{-i\vec{p}\cdot\vec{r}}e^{i\phi L_z}e^{i\vec{p}\cdot\vec{r}}] = e^{i\vec{K}(\phi)\cdot\vec{r}}\,.$$

So we have

$$\phi(m_1,m_2;m;l_z) = \int_V d^3r\int d^3p\,\delta(m-E_1(p)-E_2(p)) \times\int_{-\infty}^{\infty}\frac{d\phi}{2\pi}e^{-il_z\phi}e^{i\vec{K}(\phi)\cdot\vec{r}}\,,$$

which is already a Fourier integral. So the transform is

$$\tilde{\phi}(m_1,m_2;m;\alpha) = \int_V d^3r\int d^3p\,\delta(m-E_1(p)-E_2(p))\,e^{i\vec{K}(\alpha)\cdot\vec{r}}\,. \tag{2.14}$$

The space integral is understood in the sense of Eq. (2.9); it gives

$$\int_V d^3r\, e^{i\vec{K}(\alpha)\cdot\vec{r}} \to \int d^3r\, e^{-\vec{r}^{\,2}/R^2}e^{i\vec{K}(\alpha)\cdot\vec{r}} = (\pi R^2)^{3/2}e^{-R^2\vec{K}^2(\alpha)/4} \equiv Ve^{-R^2\vec{K}^2(\alpha)/4}\,,$$

and Eq. (2.13) gives

$$\vec{K}^2(\alpha) = 4\vec{p}_\perp^{\,2} C(\alpha),$$

where

$$C(\alpha) \equiv \tfrac{1}{2}(1-\cos\alpha) \qquad (2.15)$$

and $\vec{p}_\perp$ is the component of $\vec{p}$ normal to $\hat{z}$. We find

$$\begin{aligned}\tilde{\phi}(m_1, m_2; m; \alpha) &= V\int d^3p\,\delta(m - E_1(p) - E_2(p))\, e^{-R^2\vec{p}_\perp^{\,2}C(\alpha)} \\ &\equiv 4\pi V(E_1E_2p/m)\int_0^1 dz\,\exp[-R^2p^2C(\alpha)(1-z^2)]. \end{aligned} \qquad (2.16)$$

The one integral remaining in Eq. (2.16) is the $z \equiv \cos\theta$ integral from the momentum phase space. It leads to an error-function expression for $\tilde{\phi}$, but it is more convenient to leave the integral as it stands. As $\alpha \to 0$, $\tilde{\phi} \to \Phi$, as required.

It is interesting to compare this quantal calculation with the corresponding classical one. The distribution of l (not l_z) is classically a Gaussian,[20] and very nearly so in the quantum theory[15,21]:

$$\phi'(m_1, m_2; m; l) = 4\pi V\left(\frac{E_1E_2p}{m}\right)\frac{e^{-l^2/(2p^2R^2)}}{2p^2R^2} \qquad (2.17)$$

classically, but in the quantum theory,

$$\begin{aligned}\phi'(m_1, m_2; m; l) &= 4\pi V\left(\frac{E_1E_2p}{m}\right)\left(\frac{\pi}{2}\right)^{1/2}\frac{e^{-p^2R^2}}{pR}\, I_{l+1/2}(p^2R^2) \\ &\approx 4\pi V\left(\frac{E_1E_2p}{m}\right)\frac{\exp[-(l+\frac{1}{2})^2/(2p^2R^2)]}{2p^2R^2} \\ &\quad\times\left(1 - \frac{(l+\frac{1}{2})^2}{4p^2R^2}\right) \quad \text{if } l < \tfrac{1}{2}p^2R^2. \end{aligned} \qquad (2.18)$$

They differ only in the tail ($l \gtrsim pR$), where Eq. (2.18) decreases less fast than the Gaussian. The classical form (2.17) leads to an l_z distribution whose Fourier transform is identical to the quantal case, Eq. (2.16), except that classically

$$C(\alpha) = \tfrac{1}{4}\alpha^2,$$

rather than the quantal case,

$$\begin{aligned} C(\alpha) &= \tfrac{1}{2}(1-\cos\alpha) \\ &\approx \tfrac{1}{4}\alpha^2 \quad \text{for } \alpha \ll 1. \end{aligned} \qquad (2.15')$$

Quantum effects are therefore important only for large l ($l \gtrsim pR$), where the total number of phase-space states is very small, and where these calculations are unreliable in any case.

With the phase-space projection $\tilde{\phi}$ [Eq. (2.16)], we can now attempt to solve for $\tilde{\sigma}$ in the nonlinear bootstrap equation (2.6).

III. A SOLUTION OF THE BOOTSTRAP EQUATION

We shall discuss in detail one solution of the bootstrap equation (2.6). The form of this solution was suggested by mathematical arguments, but it is more plausible to motivate it by an intuitive argument due to Frautschi.[16] This argument reflects the main physical effects that are operating; although it is very approximate, our more detailed calculations show that the corrections needed are insignificant.

First we outline this physical reasoning which is essentially the same as that used for the distribution of charge,[9] and then write down the general form of our trial solution. Simple generalizations of this would also appear to be equally satisfactory, so we then present arguments that systematically eliminate all of these generalizations. Finally we give our detailed calculations that confirm these assertions.

A. Intuitive Argument of the Trial Solution

In the bootstrap not specifying spin[2] [Eq. (2.1), where we keep only the $N=2$ term[22]], the dominant contribution comes from one heavy and one light particle:

$$\begin{aligned} m_1 &\approx m - \mu, \\ m_2 &\approx \mu, \end{aligned} \qquad (3.1)$$

and vice versa. The light particle has limited kinetic energy (of order μ). These results follow from $a < -\frac{5}{2}$ and $b \approx 1/\mu$ in Eq. (1.1).

That $\alpha < -\frac{5}{2}$ rather than < -1 comes from the phase-space factor (E_1E_2p/m), Eq. (2.8), and it is this condition that enables the bootstrap to be satisfied to leading order of m.

Because of the relatively low kinetic energy of the two constituents in the most important mass configuration, the orbital angular momentum is limited, so there are mainly S and P waves, say. This picture is confirmed by numerical studies,[9] where it is found that, on average (with radius $R \approx 1/\mu$),

$m_2 \approx \mu$,

kinetic energy $\approx 1.5\mu$,

(so $m - m_1 \approx 2.5\mu$).

So a resonance of mass m is mainly built by successive additions of masses μ with kinetic energy $\propto \mu$, and orbital angular momentum component $l_z \sim \pm 1$. To achieve mass m, we need ν additions, where

$$\nu \approx \frac{m}{1.5\mu}$$

and the spin component (J_z) of the resonances is built by a random-walk addition of the orbital angular momentum. Then J_z has a distribution

$$\sigma(m, J_z) \approx n(m)\frac{\exp[-J_z^2/2(m/1.5\mu)x]}{[2\pi(m/1.5\mu)x]^{1/2}}. \tag{3.2}$$

Here, $x=1$ if there is only $l_z=\pm 1$, but as $l_z=0$ also (S and P waves), we expect that $0<x<1$, some steps ($l_z=0$) being ineffective in building up resonance spin J_z. Of course, there will be $D, F, \ldots$ waves as well (possibly suppressed), but their effect on Eq. (3.2) and x may be small.

Equation (3.2) suggests that we take the Fourier transform of $\sigma(m, J_z)$ to have the form

$$\tilde{\sigma}(m,\alpha)=n(m)e^{-\alpha^2 m d}, \tag{3.3}$$

but we prefer to use [c.f. Eq. (2.15)]

$$\tilde{\sigma}(m,\alpha)=n(m)e^{-m d C(\alpha)} \tag{3.4}$$

although the difference is not actually significant. Here d is a constant, and from Eq. (3.2), one expects that

$$0<d \leqslant O(1/\mu). \tag{3.5}$$

Our later calculations show that Eq. (3.4) does reproduce itself through the bootstrap equation, but only to within an over-all function of α, so our trial solution is

$$\tilde{\sigma}(m,\alpha)=n(m)f(\alpha)e^{-mC(\alpha)d} \tag{3.6}$$

and where $f(\alpha)$ is a function determined by the bootstrap. It later emerges that there is no significant difference between the cases $f(\alpha)=1$ and the $f(\alpha)$ produced by Eq. (2.6), so the above intuitive argument is confirmed.

B. Elimination of Some Generalizations

The form (3.6) suggests that we should also consider more general functions of the type

$$\tilde{\sigma}(m,\alpha)=n(m)f(\alpha)\exp[-m^{q+1}C(\alpha)^{r+1}d]. \tag{3.7}$$

We now present arguments that eliminate each of the following cases:

$r \neq$ integer. Here, $\tilde{\sigma}(m,\alpha)$ is not even in α, so that the Fourier transform, $\sigma(m, J_z)$, is complex.[23] This is not allowed physically.

$r=1,2,3,\ldots$. Numerical calculation shows that $\sigma(m, J_z)$ oscillates with J_z, and has negative regions. This is shown in Fig. 1. Although we have no general argument for this phenomenon, it can be understood qualitatively by considering the dominant region of integration of the Fourier transform, and it evidently eliminates these cases.

$r=-1$. The whole α dependence now disappears, apart from $f(\alpha)$, which is now linear in $C(\alpha)$ [shown by our later calculation with $d=0$; see Eq.

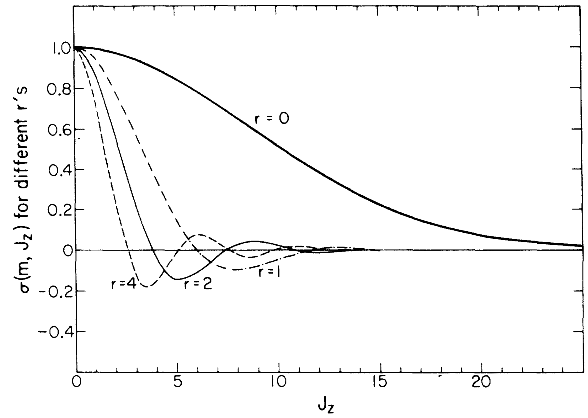

FIG. 1. The spectrum $\sigma(m, J_z)$ as a function of spin component J_z, at constant mass, corresponding to Fourier transforms of the form $\tilde{\sigma}(m,\alpha)=e^{[-dmC(\alpha)^{r+1}]}$. Shown are $r=0$, 1, 2, 4, with $dm=150$ ($m\approx 21$ GeV if $d=1/m_\pi$). The curves are normalized to unity at $J_z=0$. The exact function and Gaussian approximation fall on top of each other in the range plotted.

(3.13)]. Then $\sigma(m, J_z)$ is a sum of $\delta_{J_z,0}$ and $\delta_{J_z,1}$ which is unphysical. For example, it violates strict conservation of angular momentum. So it is also eliminated.

$r=-2,-3,-4,\ldots$. As $\alpha \to 0$, these cases give

$$\tilde{\sigma}(m,0)\to 0 \text{ for } d>0$$
$$\to\infty \text{ for } d<0,$$

so contradicting the normalization requirement, Eq. (2.7).

These cases exhaust all except for $r=0$, Eq. (3.6).

$q>0$. The peaking effect, Eq. (3.1), enables us to write[2]

$$m_1^{q+1} \approx m^{q+1}-(q+1)\mu m,$$
$$m_2^{q+1} \approx \mu^{q+1},$$

and the two sides of the bootstrap equation are now unbalanced by a factor of

$$\exp\{[dC(\alpha)(q+1)\mu]m^q\} \tag{3.8}$$

as $m\to\infty$. For $q>0$, this is manifestly impermissible. Further, it would destroy the peaking towards low kinetic energy, that is necessary for self-consistency, when $d>0$, as it must be physically (Sec. IV).

$-1<q<0$. The lack of balance is the factor, Eq. (3.8), which as $m\to\infty$ is now

$$\approx 1+[d\mu C(\alpha)(q+1)]m^q \to 1+O(m^q).$$

The fractional difference between the sides of Eq. (2.6) now dies slower than $1/m$, which is not permitted if we work to $1/m$ accuracy. However, if the bootstrap cannot be satisfied to such accuracy, this cannot be excluded.

$q=-1$. The exponential in Eq. (3.7) is now independent of m, so this can be factored out of the mass integrals. Equation (2.6) is unbalanced by $e^{-dC(\alpha)}$ so the bootstrap cannot be achieved.

$q<-1$. As $m\to\infty$, Eq. (3.7) makes $\tilde{\sigma}(m,\alpha)$ independent of α, apart from $f(\alpha)$. This is just like the case $r=-1$, and is thus rejected.

These arguments leave only the original form, Eq. (3.6) ($r=q=0$), as a possible solution, except that the case $-1<q<0$ may also be allowed (less accurately).

C. Detailed Calculations

The trial solution

$$\tilde{\sigma}(m,\alpha)=n(m)f(\alpha)e^{-dmC(\alpha)} \tag{3.6}$$

satisfies the bootstrap as a function of m and α almost exactly when $f(\alpha)$ is set equal to one. More precisely, the two sides of Eq. (2.6) differ by an over-all function of α that is independent of m. This function determines the $f(\alpha)$ needed for self-consistency. So we insert Eq. (3.6) into Eq. (2.6), show the consistency in m, and perform the remaining integrals to find $f(\alpha)$.

We have

$$\tilde{\sigma}(m,\alpha)=\frac{1}{2h^3}\int_\mu^{m-\mu}dm_1\tilde{\sigma}(m_1,\alpha)\times\int_\mu^{m-m_1}dm_2\tilde{\sigma}(m_2,\alpha)\tilde{\phi}(p^2;\alpha), \tag{2.6$'$}$$

where

$$\tilde{\phi}(p^2;\alpha)=4\pi V\left(\frac{E_1E_2p}{m}\right)\int_0^1 dz\,\exp[-R^2p^2C(\alpha)(1-z^2)]\,, \tag{2.16$'$}$$

$$\left.\begin{aligned}&p^2\approx 2m_2(\bar{m}-m_2)\,,\\ &(E_1E_2p/m)\approx\bar{m}(\bar{m}-m_2)^{1/2}(2m_2)^{1/2}\\ &\qquad\approx(\bar{m}-m_2)^{1/2}m_2(2m_2)^{1/2}\end{aligned}\right\}\quad \begin{aligned}\bar{m}&\equiv m-m_1\\ &\approx m_2\,.\end{aligned} \tag{3.9}$$

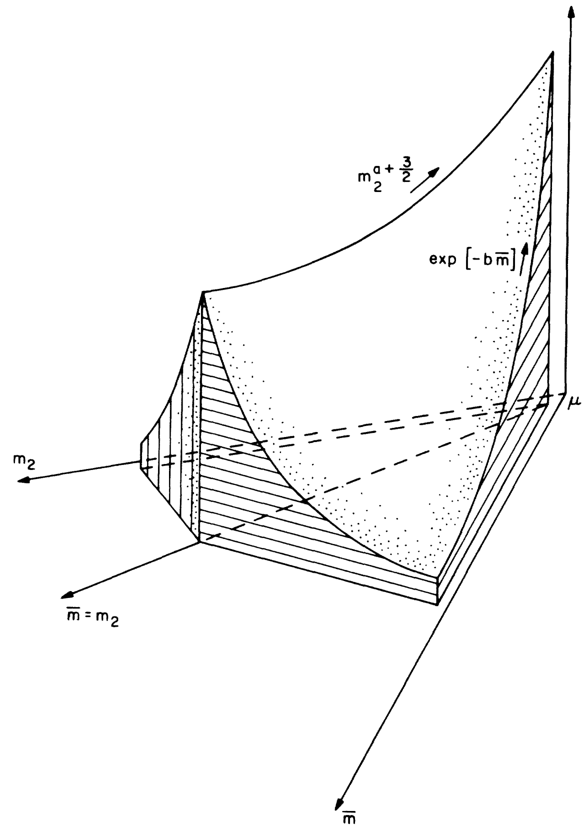

FIG. 2. Picture illustrating the strong peaking of the integrand (of the two-body term) towards $\bar{m}$ and m_2 equal to μ.

Equations (3.8) and (3.9) are the approximations for the kinematic quantities[24] when $m_1\approx m-\mu$, $m_2\approx\mu$, which is one of the dominant regions of integration. The other is $m_2\approx m-\mu$, $m_1\approx\mu$, but by the symmetry of Eq. (2.6), we may simply cut off the m_1 integral at $\frac{1}{2}(m-\mu)$ and double the answer.[25] We use this peaking effect [Eq. (3.1)] in any approximations that follow. The peaking of the integrand (at $\alpha=0$) is shown in Fig. 2.

So Eq. (2.6) is, with $m_1{}^a\approx m^a$ and K the constant of Eq. (1.1),

$$KF(\alpha)m^ae^{bm}e^{-dmC(\alpha)}=(4\pi V)\left(\frac{K^2\sqrt{2}}{h^3}\right)f^2(\alpha)m^ae^{bm}e^{-dmC(\alpha)}\int_\mu^{(m-\mu)/2}dm_2m_2{}^{a+3/2}\int_0^1dz\int_{m_2}^{m-\mu}d\bar{m}(\bar{m}-m_2)^{1/2}\times\exp\{-(\bar{m}-m_2)[b-dC(\alpha)+2m_2R^2(1-z^2)C(\alpha)]\}\,. \tag{3.10}$$

The exponential m dependence, and main $C(\alpha)$ dependence, cancel, so

$$1=4\pi V\left(\frac{K\sqrt{2}}{h^3}\right)f(\alpha)\int_\mu^{(m-\mu)/2}dm_2\,m_2{}^{a+3/2}\int_0^1dz\int_{m_2}^{m-\mu}d\bar{m}(\bar{m}-m_2)^{1/2}\times\exp\{-(\bar{m}-m_2)[b-dC(\alpha)+2m_2R^2(1-z^2)C(\alpha)]\}\,. \tag{3.11}$$

The integrals are performed explicitly in the order of Eq. (3.11).

$\bar{m}$ integral. Put

$$(\bar{m}-m_2)[b-dC(\alpha)+2m_2R^2(1-z^2)C(\alpha)]=u\,.$$

The integral equals

$$\frac{1}{[b-dC(\alpha)+2m_2R^2(1-z^2)C(\alpha)]^{3/2}}\int_0^{\bar{m}=m-\mu} du\,u^{1/2}e^{-u}\to\frac{\frac{1}{2}\sqrt{\pi}}{[b-dC(\alpha)+2m_2R^2(1-z^2)C(\alpha)]^{3/2}}\quad\text{as}\quad m\to\infty.$$

z integral. This is

$$\frac{\sqrt{\pi}}{2}\int_0^1\frac{dz}{[b-dC(\alpha)+2m_2R^2C(\alpha)(1-z^2)]^{3/2}}=\frac{\sqrt{\pi}}{2}\int\frac{dz}{(A-Bz^2)^{3/2}}\,,$$

where

$$A=b-dC(\alpha)+B\,,$$

$$B=2m_2R^2C(\alpha)\,.$$

Put $z=\sin\phi$; the integral is then

$$\frac{\frac{1}{2}\sqrt{\pi}}{A(A-B)^{1/2}}=\frac{\frac{1}{2}\sqrt{\pi}}{[b-dC(\alpha)+2m_2R^2C(\alpha)][b-dC(\alpha)]^{1/2}}\,.$$

m_2 integral. The final integral is

$$\frac{\frac{1}{2}\sqrt{\pi}}{[b-dC(\alpha)]^{1/2}}\int_\mu^{(m-\mu)/2}\frac{dm_2\,m_2^{a+3/2}}{b-dC(\alpha)+m_2[2R^2C(\alpha)]}$$

$$\approx\frac{\frac{1}{2}\sqrt{\pi}}{[b-dC(\alpha)]^{1/2}[b-dC(\alpha)+2\mu R^2C(\alpha)]}\int_\mu^{(m-\mu)/2}dm_2\,m_2^{a+3/2}$$

$$=\frac{1}{[1-(d/b)C(\alpha)]^{1/2}[1-(d/b-2\mu R^2/b)C(\alpha)]}\,\frac{\sqrt{\pi}\,\mu^{a+5/2}}{2b^{3/2}[-(a+\frac{5}{2})]}\left[1+O\left(\left(\frac{m}{\mu}\right)^{a+5/2}\right)\right]. \tag{3.12}$$

From Eq. (3.11) we obtain

$$f(\alpha)=[1-(d/b)C(\alpha)]^{1/2}[1-C(\alpha)(d/b-2\mu R^2/b)] \tag{3.13}$$

and

$$K=\frac{1}{V}\left[\left(\frac{b}{2\pi}\right)^{1/2}h\right]^3\left[\frac{-(a+\frac{5}{2})}{\mu^{a+5/2}}\right],$$

although the value of K is unreliable as it is determined by the spectrum at low mass which is not correctly represented by the asymptotic form, Eq. (1.1).

We summarize with some remarks about the above calculation:

(i) It relies upon the particles having low kinetic energy ($p^2\ll m_1^{\,2}$ and $m_2^{\,2}$), for all α. This damps out large momenta, which is necessary for self-consistency.[2] So the bootstrap requires d to be bounded [Eq. (2.15)]:

$$d<b\,.$$

(ii) We have actually assumed that this momentum damping makes the motion nonrelativistic. This is true if

$$b\gg 1/\mu \quad\text{and}\quad b-d\gg 1/\mu\,.$$

Since[8,9] $b\approx 1/\mu$, this is not actually satisfied, but the effect[2] is just to change the value of K. Our expression for p^2, Eq. (3.8), comes from the nonrelativistic approximation

$$m\equiv E_1(p)+E_2(p)$$

$$\approx m_1+m_2+p^2(1/2m_1+1/2m_2)$$

so

$$\bar{m}\approx m_2+p^2/2m_2\quad\text{as}\quad m_1\approx m-\mu\to\infty\,,$$

but we have estimated the resulting error to be insignificant.

(iii) To ensure the peaking effect in the masses, Eq. (3.1),

$$a<-\tfrac{5}{2}\,.$$

This is necessary to make only one particle have mass of order m; otherwise, self-consistency is again impossible[2] [see Eq. (3.12)].

(iv) The calculation is correct to a fractional error of order $1/m$, except for the very last step, Eq. (3.12). This is also correct to this order, if

$$a \leq -\tfrac{7}{2}\,. \tag{3.14}$$

(v) Even though the value of K is unreliable, the form of $f(\alpha)$ is reliable, provided that the mass-peaking effect is operating. This is because the form of $f(\alpha)$ in the integrand comes from the asymptotic regions.

(vi) We show below that positivity of the level density implies

$$d > 0\,.$$

(vii) The treatment of the mass integrals closely parallels that of Frautschi,[2] whose case is $\alpha = 0$. The (unreliable) value of K is the same, so we expect that

$$f(\alpha)e^{-dmC(\alpha)}$$

should be normalized to unity at $\alpha = 0$. Equation (3.13) shows that it is.

The solution of the bootstrap that we find is thus

$$\frac{\tilde{\sigma}(m,\alpha)}{\tilde{\sigma}(m,0)} = \left[1 - \frac{d}{b}C(\alpha)\right]^{1/2}\left[1 - C(\alpha)\left(\frac{d}{b} - \frac{2\mu R^2}{b}\right)\right]e^{-dmC(\alpha)}\,, \tag{3.15}$$

where $0 < d < b$ and $\tilde{\sigma}(m,0) \equiv n(m)$, Eq. (1.1).

From this, we now calculate the spin distribution, and show it to differ insignificantly from the intuitive form, Eq. (3.2).

IV. THE SPIN DISTRIBUTION

To find $\sigma(m, J_z)$, we perform the Fourier transform of $\tilde{\sigma}(m,\alpha)$, Eq. (3.15). Since this is periodic in α, the J_z spectrum is discrete, and so [recalling the remarks before Eq. (2.13)]

$$\sigma(m, J_z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha\, e^{-i\alpha J_z}\tilde{\sigma}(m,\alpha)\,.$$

This is evaluated by expanding the algebraic factor $f(\alpha)$ multiplying the exponential in Eq. (3.15) in powers of $C(\alpha)$. This is valid, since

$$0 < (d/b) < 1 \quad \text{and} \quad 0 < C(\alpha) < 1\,.$$

Formally, we have [with only the first term (1) eventually significant]

$$\tilde{\sigma}(m,\alpha) = \tilde{\sigma}(m,0)\left(1 + \frac{1}{b}\frac{\partial}{\partial m}\right)^{1/2} \times\left[1 + \left(\frac{1}{b} - \frac{2\mu R^2}{bd}\right)\frac{\partial}{\partial m}\right]e^{-dmC(\alpha)}$$

$$\equiv \tilde{\sigma}(m,0)\left[1 + \sum_{k=1}^{\infty} a_k \frac{\partial^k}{\partial m^k}\right]e^{-dmC(\alpha)}$$

giving, with Eq. (2.15),

$$\sigma(m,J_z) = \tilde{\sigma}(m,0)\left(1 + \frac{1}{b}\frac{\partial}{\partial m}\right)^{1/2} \times\left[1 + \left(\frac{1}{b} - \frac{2\mu R^2}{bd}\right)\frac{\partial}{\partial m}\right]I_{J_z}(\tfrac{1}{2}dm)e^{-(d/2)m} \tag{4.1}$$

upon using[26]

$$e^{x\cos\alpha} = I_0(x) + 2\sum_{l=1}^{\infty}\cos(l\alpha)I_l(x)\,. \tag{4.2}$$

Here, $I_l(x)$ is the modified Bessel function:

$$I_l(x) \equiv i^{-l}J_l(ix)\,.$$

Now for $J_z < \frac{1}{4}dm$,

$$e^{-(d/2)m}I_{J_z}(\tfrac{1}{2}dm) \approx \frac{1}{(\pi dm)^{1/2}}e^{-J_z^2/dm}\left(1 - \frac{J_z^2}{d^2m^2}\cdots\right) \tag{4.3}$$

showing that all of the derivative terms in Eq. (4.1) are suppressed by some power of m, in the region $J_z \ll \frac{1}{4}dm$ as $m \to \infty$. In the accuracy to which we can work, these terms are not significant, and so

$$\sigma(m,J_z) \approx n(m)\frac{1}{(dm\pi)^{1/2}}e^{-J_z^2/dm}\left(1 - \frac{J_z^2}{d^2m^2}\cdots\right) \quad \text{for } J_z < \tfrac{1}{4}dm\,. \tag{4.4}$$

This is our main result, and confirms the intuition, Eq. (3.2). It is essentially a Gaussian, of width (standard deviation)

$$\Delta J = (\tfrac{1}{2}d)^{1/2}\sqrt{m}\,,$$

where $0 < d < b$. That $d > 0$ follows from Eq. (4.1), and the property [apparent from Eq. (4.2)] that[27]

$$I_l(-x) = e^{i\pi l}I_l(x)\,,$$

which makes $\sigma(m,J_z)$ negative at alternate physical J_z values unless $d > 0$.

We note that the approximation, Eq. (4.4), holds well into the Gaussian tail (well beyond $J_z \propto \sqrt{m}$) and so up to a point where $\sigma(m,J_z)$ is very small. Beyond this point, our function $\sigma(m,J_z)$ decreases slower than a Gaussian[28]:

$$e^{-(d/2)m}I_{J_z}(\tfrac{1}{2}dm) \sim \frac{1}{(2\pi J_z)^{1/2}}e^{-(d/2)m}\left(\frac{edm}{4J_z}\right)^{J_z} \quad \text{for } J_z^2 \gg (\tfrac{1}{2}dm)^2\,.$$

In this region ($J_z \gg mR$), the result is sensitive to the volume cutoff used. Apart from the fact that

$\tilde{\sigma}(m, J_z)$ is very small and continues to decrease with increasing J_z, our calculation has no reliability here.

The physical interpretation of our result has already been discussed in Sec. III A, in terms of the full spin being built up as a random walk. In particular, we have the characteristics

$$\Delta J_z = (\tfrac{1}{2}d)^{1/2}\sqrt{m}$$

and

$$d < b \approx 1/\mu .$$

It appears that the configuration of one massive and one light particle, with a small, limited, kinetic energy, is so dominant that it provides the overwhelming physical mechanism for building composite states.

The number of resonances, $\bar{\rho}(m, J)$, with spin J at mass m is easily found from the J_z distribution. For $\sigma(m, J_z)$ has contributions from all resonances whose spin $J \geq J_z$, so that, with

$$\sigma(m, J_z) = \sigma(m, -J_z),$$

$$\sigma(m, J_z) \approx \int_{|J_z|}^{\infty} dJ' \bar{\rho}(m, J'),$$

or

$$\bar{\rho}(m, J) \approx -\left.\frac{\partial\sigma(m, J_z)}{\partial J_z}\right|_{J_z = J} .$$

Equation (4.4) yields

$$\bar{\rho}(m, J) \approx n(m)\frac{2}{[\pi(dm)^3]^{1/2}} J e^{-J^2/dm} \quad \text{for } J \ll \tfrac{1}{4}dm \tag{4.5}$$

which has a linear suppression for low spin $[J \ll (\tfrac{1}{2}dm)^{1/2}]$ and a Gaussian cutoff for high spins, $J \gg (\tfrac{1}{2}dm)^{1/2}$.

The average spin of the resonances at mass m is

$$\langle J \rangle = \int_0^{\infty} dJ(2J+1)\bar{\rho}(m, J) \approx 2(dm/\pi)^{1/2} \tag{4.6}$$

using the approximation, Eq. (4.5). Similarly,

$$\langle J^2 \rangle \approx \tfrac{3}{2}dm$$

so that the spin distribution has standard deviation

$$\Delta J \approx 0.48\,(dm)^{1/2} . \tag{4.7}$$

So most resonances occur in a band of width $\sim \sqrt{m}$ centered about the mean value also $\sim \sqrt{m}$.

In Fig. 1 we show the form of $\sigma(m, J_z)$, and in Fig. 3 the distribution of number of resonances $\bar{\rho}(m, J)$, and also the total number of states,

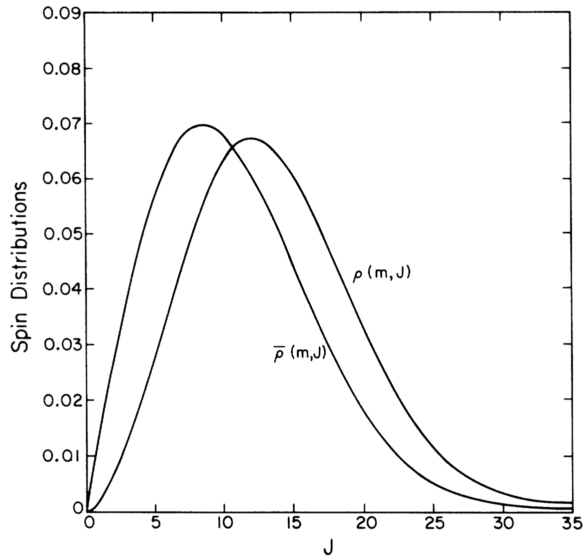

FIG. 3. The density of resonances $\bar{\rho}(m, J)$ as a function of spin J at fixed mass m, and the total density of spin-J states $\rho(m, J) = (2J+1)\,\bar{\rho}(m, J)$, both normalized to unit total number. Here $dm = 150$ ($m \approx 21$ GeV if $d = 1/m_\pi$). The exact curves and Gaussian approximations are indistinguishable in the range shown.

$$\rho(m, J) = (2J+1)\bar{\rho}(m, J) .$$

Both of these latter curves are so normalized to make the integrated areas equal to unity.

It must be remarked that this solution is totally different from the Veneziano-model spin spectrum[13] [Eqs. (1.2) and (1.3)], where J_z (or J) scales like

$$J_z \propto m$$

so making the spectrum (apart from its normalization) a function of impact parameter alone. Also, for $J \gg m$, the Veneziano spectrum falls exponentially, and not as a Gaussian. The only common feature is the linear suppression of very low-spin resonances. But this is a largely kinematic effect, following from $\sigma(m, J_z)$ having zero slope and curvature at $J_z = 0$ [see Eqs. (1.3) and (4.3)]. As $\sigma(m, J_z)$ must be symmetric in J_z by rotational symmetry, such a flattening at $J_z = 0$ must be present if $\sigma(m, J_z)$ is to be smooth here.

V. MANY-PARTICLE CONSTITUENTS

In the bootstrap equation (2.6), we have actually dropped all of those terms that arise from three or more constituents. In the case when J_z is integrated out [Eq. (2.1)], these terms do not affect the form of the solution, although they do assist in eliminating some generalizations. Their treatment proceeds as follows.[2] The dominant configuration is again one massive particle, and $N-1$

light ones (μ), with low total kinetic energy. These latter are grouped together, and the level density reproduces itself due to the contribution of the one massive particle. The remaining $N-1$ particles give a contribution that is independent of m as $m \to \infty$, and so determine the constant K.[2] One obtains the summable series[29]

$$1=\sum_{N=2}^{\infty} N \frac{x^{N-1}}{N!}.$$

Therefore,

$$1=e^x-1,$$

where

$$x \approx \frac{(2\pi)^{3/2}}{h^3} \frac{KV}{b^{3/2}} \frac{\mu^{a+5/2}}{-(a+\frac{5}{2})}, \tag{5.1}$$

giving one constraint on the parameters (a, b, K, V, μ). This implies that multiparticle constituents are strongly suppressed: They account for only $\approx 30\%$ of the hadron content, and the mean number of constituents (≈ 2.4) is hardly more than two.[2] So, multiparticle couplings are expected to be very small.

With orbital angular momentum included, this suppression of multiparticle constituents should still hold at most spins, since it does so for the sum over all spins. So we again have $N-1$ light particles, and one massive one, all with rather low energy and angular momentum relative to their center of mass. We expect that the multi-body terms do not affect the main features of the spin distribution; the $N-1$ light particles collectively behave like a single object with rather low mass and spin, as $m\to\infty$.

To the extent that all orbital motion is small, it is possible to see that the N-body terms all reproduce the same exponentials that dominate $\tilde{\sigma}(m,\alpha)$; only the unimportant function $f(\alpha)$ and constant K are changed. The missing terms in Eq. (2.6) are like

$$\frac{1}{N!}\frac{1}{h^{3(N-1)}}\int dm_1 \cdots dm_N \tilde{\sigma}(m_1,\alpha)\tilde{\sigma}(m_2,\alpha)\cdots \times\tilde{\sigma}(m_N,\alpha)\tilde{\phi}(m_1,\ldots,m_N;m;\alpha). \tag{5.2}$$

To find $\tilde{\sigma}_N(m_1,\ldots,m_N;m;\alpha)$, we need the spin projection of the N-body phase space, which is only known for equal masses. So we group the $N-1$ light particles together: all of

$$m_2, m_3, \ldots, m_N \approx \mu, \tag{5.3}$$

and treat these spinless particles as a single object of mass

$$E \equiv \sum_{i=2}^{N} E_i \approx (N-1)\mu,$$

with spin distribution given by the $N-1$ equal-particle phase-space projection. Only the asymptotic limit ($N \gg 1$) is known. The derivation is nonrelativistic and classical, with the Gaussian cutoff of Eq. (2.9).[30] It only has a convenient form for small l[31]:

$$l^2 \ll NE\mu R^2 \approx N(E/\mu) \quad \text{if} \quad R=1/\mu, \tag{5.4}$$

in which case the l_z projection (not l) is a Gaussian, which can most easily be understood in terms of the classical random coupling of the $\vec{l}$ vectors[32]:

$$\phi_{N'}(\mu;E;l_z)=e^{-l_z^2/2gE}\frac{1}{(2\pi gE)^{1/2}}\Phi_{N'}(\mu;E), \tag{5.5}$$

where

$$g \equiv \tfrac{2}{3}\mu R^2 = \text{const}$$

and

$$\begin{aligned}\Phi_{N'}(\mu;E) &= \frac{(3\pi^2 gE)^{3(N'-1)/2}}{EN'^2\Gamma(\frac{3}{2}(N'-1))} \\ &\equiv \frac{V^{(N'-1)}(2\pi\mu E)^{3(N'-1)/2}}{EN'^2\Gamma(\frac{3}{2}(N'-1))} \\ &\underset{N'\to\infty}{\sim} \left(\frac{2\pi^2 egE}{N'}\right)^{3(N'-1)/2}\frac{1}{EN'^2}\left(\frac{\frac{3}{2}N'}{2\pi}\right)^{1/2}\end{aligned} \tag{5.6}$$

is the total phase-space integral for N equal-mass (μ) particles at total energy E. In deriving Eq. (5.5), we have used the l (not l_z) projection,[30,31] and the fact that

$$\phi'_{N'}(\mu;E;l)=\frac{-\partial}{\partial l_z}\phi_{N'}(\mu;E;l_z)\Big|_{l_z=l}$$

and our normalization

$$\int_{-\infty}^{\infty} dl_z\,\phi_{N'}(\mu;E;l_z)=\int_0^{\infty} dl(2l+1)\,\phi'_{N'}(\mu;E;l) = \Phi_{N'}(\mu;E).$$

A relativistic calculation, including the Lorentz contraction of the volume,[33] gives essentially the same result. Since, for the $N-1$ masses μ,

$$E/N \approx \mu,$$

we see from Eq. (5.6) the factor $(V\mu^{3/2})^{(N-1)}$ that is expected from the momentum phase space of the $N-1$ particles that have small mass.

These $N-1$ particles together then act like a single mass-E object with spin distribution given by Eq. (5.5). Its Fourier transform is

$$\tilde{\phi}_{N-1}(\mu; E; \alpha) = e^{-\alpha^2 g E/2}\,\Phi_{N-1}(\mu, E). \tag{5.7}$$

Now add in the massive particle m_1:

$$m_1 \approx m - N\mu,$$
$$E_1 \approx m_1.$$

It has a spin distribution, using Eq. (3.6),

$$\tilde{\sigma}(m_1, \alpha) = Kf(\alpha) m_1{}^a e^{bm_1} e^{-dm_1 C(\alpha)}$$

$$\approx [Km^a e^{bm} e^{-dmC(\alpha)} f(\alpha)]\,[e^{-b\bar{m}} e^{d\bar{m}\alpha^2/4}]$$
$$= [\tilde{\sigma}(m, \alpha)]\,[e^{-b\bar{m}} e^{d\bar{m}\alpha^2/4}], \tag{5.8}$$

where

$$\bar{m} \equiv m - m_1$$
$$\approx \mu(N-1).$$

Its motion relative to the $N-1$ masses μ gives a spin distribution described by the two-body phase space [Eq. (2.16)]:

$$\tilde{\sigma}(m_1, E, m; \alpha) = 4\pi V\left(\frac{E_1 E_{(N-1)} p}{m}\right)\int_0^1 dz\, \exp[-R^2 p^2 \tfrac{1}{4}\alpha^2(1-z^2)]$$
$$\approx 4\pi V(\bar{m}-E)^{1/2}\bar{m}(2E)^{1/2}\int_0^1 dz\, \exp[-\tfrac{1}{4}R^2\alpha^2(\bar{m}-E)2E(1-z^2)]$$

as only small α (the dominant region) is reliable.

In Eq. (5.8), the first bracket is the required function, so at least this main part does reproduce itself term by term in the N-body series. The remaining integrals serve to determine $f(\alpha)$ and K. Canceling off the $\tilde{\sigma}(m, \alpha)$ factor that does bootstrap, we are left with the remaining integral from Eq. (5.2): Very roughly, it is[29]

$$\frac{N}{N!}\left(\frac{\mu^{a+5/2} K f(\alpha)}{-(a+\frac{5}{2})h^3 b^{3/2}}\right)^{(N-1)}\left(\frac{\Phi_{N-1}(\mu, E)}{\mu^{3(N-1)/2}}\right)\int_E^\infty d\bar{m}\,(\bar{m}-E)^{1/2}\,\bar{m}\,e^{-b\bar{m}} e^{d\bar{m}\alpha^2/4} e^{-gE\alpha^2/2}$$
$$\times\int_0^1 dz\,\exp[-\tfrac{1}{4}R^2\alpha^2(\bar{m}-E)2E(1-z^2)].$$

This can be evaluated by the general method of Sec. III C. It leads to a series like Eq. (5.1), that determines K and $f(\alpha)$. Again, we can expand the answer

$$f(\alpha) = 1 + \sum_{k=1}^{\infty}\left(\frac{\alpha^2}{4}\right)^k a_k$$

and the detailed nature of $f(\alpha)$ does not affect the spin spectrum significantly for $J_z \ll dm$, which is the region where there is little sensitivity to the tail of the volume cutoff. But the important point is that the spectrum is reproduced by the N-body terms. Although the argument is complex, the mechanism is simple: The phase-space angular momentum, being small, does not contribute to the spectrum in a substantial way. This is also true in the two-body case, Sec. III C. The situation is thus really very close to the charge-distribution case[9] (where the phase-space factor is not present). It follows that the spectrum is not dominated by the radius of the enclosure, except for very high-spin states which can only be built by relying upon high orbital angular momentum ($l \gg mR$). It is worth pointing out that the N-particle terms are in any case strongly suppressed by the N dependence apparent in Eq. (5.6), in accordance with the results of Frautschi.[2]

VI. OTHER SOLUTIONS

So far we have stressed the solution, Eq. (4.1), which is most strongly suggested on physical grounds. But outside the arguments of Sec. III B (to which we shall shortly return), there is no argument for the uniqueness of this solution. Of particular interest is whether one can find a spectrum with

$$\Delta J \propto m$$

as in the Veneziano model.[13] It appears that bootstrap solutions with this characteristic are indeed possible. Although the precise Veneziano solution fails, there are very similar ones that succeed in bootstrapping themselves, to the leading order in m.

A. The Veneziano Solution

In this case,[13,14]

$$\frac{\tilde{\sigma}(m, \alpha)}{\tilde{\sigma}(m, 0)} = \frac{d'\alpha m}{\sinh(d'\alpha m)},$$

with

$$d' = (6\alpha'/D)^{1/2}$$
$$= \text{const}. \tag{6.1}$$

This function [or Eq. (6.1) times $f(\alpha)$, where $f(0) = 1$] cannot reproduce itself in the bootstrap equation due to the α factor in the numerator. This is easily seen by using the method of Sec. III C.

B. Veneziano - Type Solutions

We may try

$$\frac{\tilde{\sigma}(m,\alpha)}{\tilde{\sigma}(m,0)}=\frac{f_1(\alpha)}{\cosh(d'\alpha m)}\,,\quad f_1(0)=1 \tag{6.2}$$

or

$$=\frac{(d'm)f_2(\alpha)}{\sinh(d'\alpha m)}\,,\quad f_2(\alpha)\sim\alpha \text{ as } \alpha\to 0 \tag{6.3}$$

where d' is a constant.

As $\alpha m\to\infty$, Eqs. (6.2) and (6.3) behave like

$$\sim e^{-d'\alpha m} \tag{6.4}$$

and so show the asymptotic m dependence which, from Sec. III B, is sufficient to achieve the bootstrap. Their significant feature is that they differ from Eq. (3.6) in the region

$$|\alpha|\lesssim O(1/d'm)\,, \tag{6.5}$$

where there is no argument for requiring a particular form. This region of α dominates the Fourier transform, which yields a J_z (or J) distribution in which J_z (say) scales like

$$J_z\sim m$$

[as in the Veneziano case, Eq. (6.1), as is seen in Eqs. (1.2) and (1.3)].

In view of Eq. (6.4), one expects to be able to achieve a bootstrap at least in the region $|d'\alpha m| >> 1$, so we first concentrate on this limit, and then examine the more significant region, Eq. (6.5). We shall refer explicitly to the first case, Eq. (6.2); the second, Eq. (6.3), is closely analogous.

To proceed, we turn back to Sec. III C, set $d=0$, and insert into the integrand of Eq. (3.10) an additional factor of

$$\frac{1}{\cosh(d'\alpha m_1)\cosh(d'\alpha m_2)}\,.$$

The peaking effect, Eq. (3.1), still holds, so this factor becomes

$$\approx\frac{1}{\cosh[d'\alpha(m-\bar{m})]\cosh(d'\alpha m_2)}\,,$$

where $\bar{m}\to\mu$ eventually.

So we gain an extra factor of

$$\frac{1}{\cosh(d'\alpha m)}\,\frac{1}{\cosh^2(d'\alpha\mu)[1-\tanh(d'\alpha m)\tanh(d'\alpha\,\mu)]}$$

$$\to\frac{1}{\cosh(d'\alpha m)}\,\frac{1}{\cosh^2(d'\alpha\mu)[1-|\tanh(d'\alpha\mu)|]}$$

for $|d'\alpha m|>>1$. The first term shows that $\tilde{\sigma}(m,\alpha)$ reproduces itself within $f_1(\alpha)$, which is determined by the second term. With Eq. (3.13), one finds

$$f_1(\alpha)=\cosh^2(d'\alpha\mu)[1-|\tanh(d'\alpha\mu)|]\left[1+\frac{2\mu R^2}{b}C(\alpha)\right]. \tag{6.6}$$

When $|\alpha|\lesssim O(1/d'm)$, Eq. (6.6) shows that the two sides of the bootstrap equation (2.6) fail to match by the factor

$$\frac{1-|\tanh(d'\alpha\mu)|}{1-\tanh(d'\alpha m)\tanh(d'\alpha\mu)}\,.$$

But in this region, one notices that this differs from unity by terms that are only of order $1/m$, so that self-consistency is achieved here as well.

Since the Fourier transform is dominated by the region $|d'\alpha m|\lesssim 1$, we may rather roughly replace $f_1(\alpha)$ by $f_1(0)=1$, and obtain a J_z spectrum by contour integration:

$$\sigma(m,J_z)\approx n(m)\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\alpha\, e^{i\alpha J_z}}{\cosh(d'\alpha m)}$$

$$=\frac{n(m)}{2d'm}\,\frac{1}{\cosh(\pi J_z/2d'm)}\quad\text{for } J_z\lesssim d'm\,.$$

As expected, $\Delta J_z\propto m$, and the resonance spin spectrum $\bar{\rho}(m,J)$ is easily obtained as before.

The same considerations apply to Eq. (6.3). One finds

$$f_2(\alpha)=\sinh(d'\alpha\mu)\cosh(d'\alpha\mu)$$

$$\times[1-|\tanh(d'\alpha\mu)|][1+(2\mu R^2/b)C(\alpha)]$$

and

$$\sigma(m,J_z)\approx\frac{n(m)}{(4/\pi)d'm}\,\frac{1}{\cosh^2(\pi J_z/2d'm)}\,.$$

Figures 4 and 5 show this $\sigma(m,J_z)$, and the corresponding $\bar{\rho}(m,J)$ and $\rho(m,J)$ distributions [c.f. Veneziano case,[13] Eqs. (1.2) and (1.3)].

The Fourier transforms can be performed more accurately by the following device. For example, in Eq. (6.6), make the replacement

$$1-|\tanh(d'\alpha\mu)|\to 1-\tanh(d'\alpha\mu)\tanh(d'\alpha m)\,,$$

which never makes the error larger than of order $1/m$. The second term contributes to $\tilde{\sigma}(m,\alpha)$ an extra term

$$\approx(d'\alpha\mu)\frac{\sinh(d'\alpha m)}{\cosh^2(d'\alpha m)}$$

$$=-\mu\,\frac{\partial}{\partial m}\,\frac{1}{\cosh(d'\alpha m)}\,,$$

and so to $\sigma(m,J_z)$ a term

$$-n(m)\,\frac{\pi J_z}{4d'^2m^3}\,\frac{\sinh(\pi J_z/2d'm)}{\cosh^2(\pi J_z/2d'm)}\,,$$

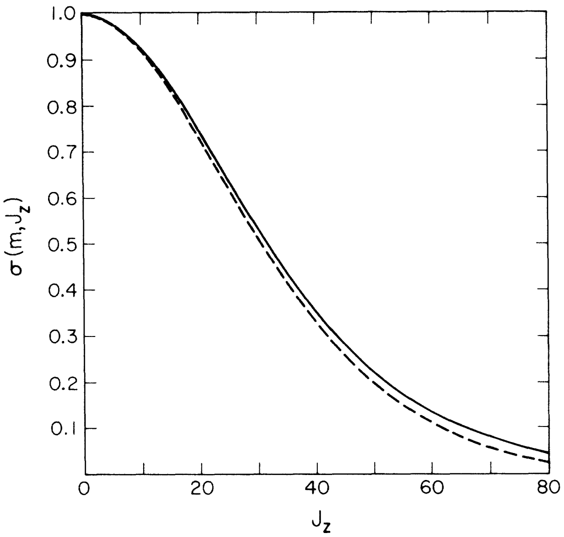

FIG. 4. Distribution $\sigma(m, J_z)$ for the Veneziano-type solution of Sec. VI B, normalized to unity at $J_z = 0$. The dashed curve is the exact Veneziano-model spectrum. Here $d'm = 57$ ($m \approx 8$ GeV if $d' \approx 1/m_\pi$).

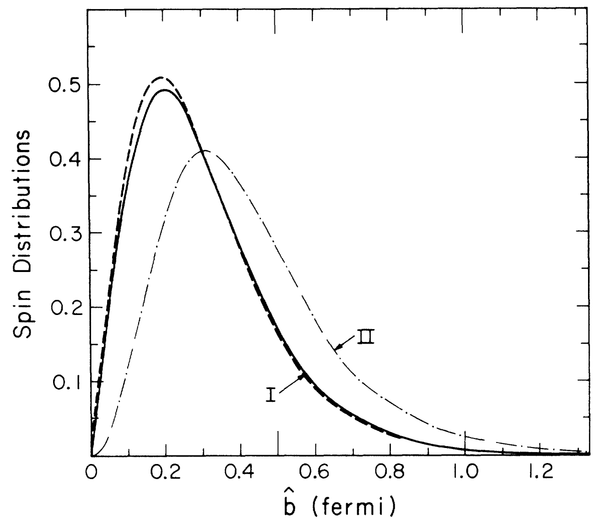

FIG. 5. I is the number of resonances, $\bar{\rho}(m, J)$. The full curve is the Veneziano-type solution, Eq. (6.2); the dashed curve is the exact Veneziano-model spectrum. II is the total number of spin states, $\rho(m, J)$, for the Veneziano-type solution. All are normalized to unit area and are plotted as a function of impact parameter $\hat{b} = 2J/m$ at fixed mass m: $d'm = 57$ ($m \approx 8$ GeV if $d' \approx 1/m_\pi$).

which is suppressed by an extra power of m when $J_z << d'm$.

Again for these solutions, there is a constraint $|d'| < b$.

C. Solutions with $\Delta J \propto m^{\gamma/2}$, $0 < \gamma < 1$

In Sec. III B, we could only eliminate the case

$$\tilde{\sigma}(m, \alpha) = f(\alpha) e^{-dm^\gamma C(\alpha)}, \quad 0 < \gamma < 1 \tag{6.7}$$

if the fractional error permitted in satisfying the bootstrap is of order $1/m$ or less. Equation (6.7) gives an error of order $1/m^{1-\gamma}$, whereas the calculation of Sec. III C has error $1/m^\delta$, where $\delta = \min\{1, -(a+\frac{5}{2})\}$. So one would require $a \leq -\frac{7}{2}$. But the preferred value of a is[9, 10] -3, so it appears that such accuracy cannot be achieved. Then Eq. (6.7) cannot strictly be ruled out, even though it is less accurate a solution than Eq. (3.6)

It gives a spin distribution just like Eq. (4.3), except that

$$(dm) \to (dm^\gamma)$$

and so [Eqs. (4.6) and (4.7)]

$$\langle J \rangle \sim m^{\gamma/2} < O(m^{1/2}),$$

$$\Delta J \sim m^{\gamma/2} < O(m^{1/2}).$$

This solution has no constraint on the size of d (except for $d > 0$).

The distribution is narrower. In terms of the physical picture of Sec. III A, which is well confirmed numerically,[9] the orbital angular momentum of the light constituent decreases as m increases as

$$\frac{1}{(\sqrt{m})^{(1-\gamma)}},$$

a circumstance that is physically most unlikely. This, and the fact that this solution actually reproduces itself less convincingly than Eq. (3.6), leads us to believe that a more detailed analysis of Eq. (2.2), analogous to the numerical treatment of Eq. (2.1),[9] would fail to find Eq. (6.6) as the bootstrap solution.

This ambiguity is similar to one that occurs in the bootstrap of the total level density, Eq. (2.1).[2,3] It is possible to bootstrap a form like

$$n(m) = \text{const}\, m^a e^{bm} e^{-b'm^\gamma}, \quad 0 < \gamma < 1, \; b' > 0 \tag{6.8}$$

with a fractional error of order

$$\frac{1}{m^{1-\gamma}} \to 0 \text{ as } m \to \infty.$$

The condition $b' < 0$ is necessary to give the peaking effect, Eq. (3.1), that is so essential for self-consistency. Indeed it is a much stronger effect than from m^a (with $a < -\frac{5}{2}$), so that a is now unconstrained. Unless one tries to work to higher accuracy ($1/m$), this case cannot be excluded. But by going beyond the asymptotic bootstrap condition,[2] and requiring agreement at finite masses as well, it can be excluded. This can be seen[16] by using the method of Nahm,[10] and the fits of Hamer and Frautschi[9] also argue strongly against this form: They find that a definite value of a ($a=-3$) is strongly favored on performing fits to numerical calculations over a very wide range of masses.

Similarly, a spin spectrum like Eq. (6.6) could only be ruled out by working more accurately. Although we believe that a solution like Eqs. (3.6) and (4.1), with $\Delta J_z \propto \sqrt{m}$, is preferred, we cannot at this stage exclude Eq. (6.6), with $\Delta J_z \propto m^{\gamma/2}$, $0<\gamma<1$, or the completely different type of solutions of Sec. VI B, which have $\Delta J_z \propto m$. One can also look at the N-particle terms with these other solutions. Because of the overwhelming effect of the dominance by $N-1$ light particles, these solutions do reproduce themselves, although Eq. (6.7) not as accurately as does Eq. (3.6). Much of this ambiguity is associated with the fact that the constraints are asymptotic. Thus, a function exactly like Eq. (6.4) for all α and m is impermissible: It would not be even in α.[23] But we can have functions that are even in α and yet behave asymptotically like Eq. (6.4).

VII. DISCUSSION

We have found that the statistical bootstrap model permits a variety of spin distributions. These solutions are like

$$A.\quad \sigma(m,J_z)\approx n(m)e^{-J_z^2/dm}\frac{1}{(\pi d m)^{1/2}}\,,\quad 0<d<b$$

$$B.\quad \sigma(m,J_z)\approx n(m)e^{-J_z^2/dm^\gamma}\frac{1}{(\pi d m^\gamma)^{1/2}}\,,\quad 0<\gamma<1, 0<d$$

$$C.\quad \sigma(m,J_z)\approx \frac{n(m)}{2d'm}\,\frac{1}{\cosh(\pi J_z/2d'm)}$$

$$\text{and}\quad \approx \frac{n(m)}{(4/\pi)d'm}\,\frac{1}{\cosh^2(\pi J_z/2d'm)}\,,\quad |d'|<b\,.$$

Of these, A has the strongest motivation in terms of the physical picture of hadron structure that is strongly suggested by the model.[1,2,9] We shall discuss this case first. Solutions B are implausible when viewed from this standpoint, but we cannot rule them out. Solutions of type C are very similar to the Veneziano model, and again we are unable to eliminate them. Only more accurate working (on the lines of Refs. 9 and 10) would enable one to see if all of these are really allowed.

Solution A corresponds to the physical picture described in Sec. III A. This picture is confirmed by numerical studies[9] of the bootstrap for $n(m)$. It leads to hadrons being composed predominantly of only two constituents, whose masses are

$$m_1 \to m,\qquad m_2\to \mu\,, \tag{7.1}$$

and vice versa, with average momentum[9] (when $R\approx 1/\mu$) of

$$\langle p\rangle \approx \tfrac{3}{2}\mu$$

and average spacing between resonances of[9]

$$\Delta m\approx \tfrac{3}{2}\mu\,.$$

So the orbital angular momentum l of m_1 and m_2 is small:

$$\langle l_z\rangle \lesssim \langle p\rangle R\approx \tfrac{3}{2}\mu R$$

and as described in Sec. III A, this makes the resonance spin spectrum a Gaussian, following from the random-walk addition of the ν orbital angular momenta $\langle l_z\rangle$, where

$$\nu=\frac{m}{\Delta m}\approx\frac{m}{\frac{3}{2}\mu}\,.$$

This Gaussian will be reliable, if J_z is much less than the maximum physically possible:

$$J_z \ll \langle l_z\rangle\nu\approx \tfrac{3}{2}\mu R\,\frac{m}{\frac{3}{2}\mu} \tag{7.2}$$

or

$$J_z\ll mR\,.$$

Our calculations give a Gaussian like A for [Eq. (4.3)]

$$J_z\ll m\tfrac{1}{4}d \tag{7.3}$$

and since $d<b$, and also since b scales linearly with R for μ small[9] [and numerically, $b\approx 1/\mu$ when $R\approx 1/\mu$ (Ref. 9)], so that

$$b\approx R\,,$$

we see that Eqs. (7.2) and (7.3) are consistent, and leads us to believe that solution A is reliable up to

$$J_z\ (\text{or } J)\lesssim mR\,. \tag{7.4}$$

One can get as far as $J_z\approx mR$ by just relying upon small orbital angular momenta feeding the resonance spin. This is the dominant configuration. To get much further, one has to rely more and more on higher orbital angular momenta, which are kinematically suppressed. So, much higher-spin resonances will be much more scarce. But the precise high-spin distribution of resonances will be determined by the precise volume cutoff used, and as this is rather arbitrary, it is not meaningful to go much beyond $J\approx mR$.

In this paper, we have used a Gaussian cutoff, e^{-r^2/R^2}. We have also looked at a box cutoff, and a

Yukawa form,

$$\frac{e^{-r/R'}}{r/R'}.$$

In the latter case, we have calculated the first correction to the distribution, by finding the coefficient of $C(\alpha)$ in $f(\alpha)$, and comparing with the Gaussian case, Eq. (3.13). When R' is fixed by requiring the volumes to be equal, in the sense that

$$\int d^3r\,\frac{e^{-r/R'}}{r/R'}=\int d^3r\,e^{-r^2/R^2},$$

the coefficient of $C(\alpha)$ is the same in both cases, within 1%. Thus, a cutoff with a larger tail does not affect the spectrum for $J<mR$.

The above physical picture suggests that resonances are, on average, evenly spaced in m, and not in m^2. This is in fact found numerically,[9] where

$$\Delta m \approx \tfrac{3}{2}\mu .$$

This can be seen analytically. For two fixed masses, the total level density is [Eqs. (2.1) and (2.8)]

$$\frac{dN(m)}{dm}\equiv n(m)=\frac{4\pi V}{2h^3}\,\frac{E_1E_2p}{m}, \tag{7.5}$$

where $N(m)$ is the total number of resonances up to mass m. In the dominant configuration, Eq. (7.1),

$$E_1E_2p/m \approx \tfrac{3}{2}\mu^2=\text{const},$$

so to increase $N(m)$ by one, m must increase by

$$\begin{aligned}\Delta m&=\text{const}\\ &\approx h^3/3\pi\mu^2V\\ &\propto 1/R\end{aligned}$$

which is expected for two particles confined to a volume V.

Since this dominant effect has [Eq. (4.6)]

$$\langle J\rangle \propto \sqrt{m}$$

and

$$\Delta m=\text{const},$$

one may be concerned that the solution A does not produce a leading linear Regge trajectory of resonances – a sequence of resonances with spin up to

$$J\sim \alpha' m^2;\quad \alpha'=\text{const}.$$

This is far outside the reliable range, Eq. (7.4), and indeed our spin distribution, Eq. (4.1), for this dominant configuration decreases rapidly with m increasing on the lines $J/m^2=\text{const}$. There is no sign of a linear leading trajectory, whose existence must be expected to depend very sensitively upon the precise volume cutoff (which may need to be mass- and spin-dependent). We note that a fixed volume is not relativistically invariant, whereas linear trajectories are believed to reflect relativistic phenomena.

We may use the above argument to inquire about the average resonance spacing and spin distribution of resonances coming from nondominant configurations [which our calculation of Sec. III C misses altogether, on account of the approximations that it relies upon]. We consider two cases:

(i) Both constituents relativistic:

$$m_1 \text{ and } m_2 \ll m,$$

$$p\approx \tfrac{1}{2}m .$$

Most spin comes from the orbital motion, and not from the spins of the constituents.

$$(E_1E_2p/m)\sim \tfrac{1}{8}m^2,$$

so $\Delta N(m)=1$ requires

$$\Delta m \propto 1/m^2 \quad\text{or}\quad \Delta(m^2)\propto 1/m,$$

and by Eq. (2.18)

$$\begin{aligned}\bar\rho_{m_1,m_2}(m,J)&\approx \bar\rho_{m_1,m_2}(m,l)\\ &\approx \frac{\pi^{3/2}V}{\sqrt{2}\,h^3R}\,m e^{-m^2R^2/4}\,I_{l+1/2}(m^2\tfrac{1}{4}R^2)\\ &\approx \frac{\pi V}{h^3R^2}\exp[-(l+\tfrac{1}{2})^2/(m^2R^2/2)]\end{aligned}$$

for $l<\frac{1}{8}m^2R^2$.

(ii) Both constituents very massive and nonrelativistic:

$$m_1 \text{ and } m_2\approx \tfrac{1}{2}m,$$

$$p\approx \text{const} \sim\mu .$$

Most spin comes from the constituents, as the orbital motion is so small.

$$(E_1E_2p/m)\propto m,$$

so $\Delta(m^2)=\text{const}$, with [Eq. (4.3)]

$$\bar\rho_{m_1,m_2}(m,J)\approx \text{const}\, m^{1/2}Je^{-4J^2/dm}$$

for $J<\frac{1}{8}dm$.

In neither case is there any sign of linear trajectories, although of course neither is reliable on such lines. Case (ii) does, however, have the average resonance spaced evenly in m^2.

Again, we notice that in solution A, the resonance spectrum has a mean impact parameter

$$\begin{aligned}\langle\hat b\rangle&=2\,\frac{\langle J\rangle}{m}\\ &\approx 4(d/\pi)^{1/2}\,1/\sqrt{m}\end{aligned}$$

that decreases with m. If it were assumed that each spin couples with equal strength, it would give a near forward scattering amplitude whose width (in t) expands in proportion to $m \equiv \sqrt{s}$, and which has fixed power (and not Regge) behavior. So at least this assumption is wrong.

Solutions B are very similar to A, except that the width of the Gaussian spectrum is

$$\Delta J_z \propto (\sqrt{m})^\gamma$$

and so decreases with m. In terms of the physical picture suggested by A and by numerical studies,[9] the average orbital angular momentum of the heavy and light constituent decreases with mass:

$$\langle l_z \rangle \propto \frac{1}{(\sqrt{m})^{(1-\gamma)}},$$

which is physically most unexpected. This leads us to believe that more accurate calculations would rule out case B.

For our third solutions, C, the behavior of the spectrum is identical to the Veneziano model, Eq. (1.3):

$$\Delta J_z \propto m$$

and is naturally parametrized by an impact parameter $\hat{b}$: For class C,

$$\sigma(m, J_z) = \text{const}\, \frac{n(m)}{m} f(c\hat{b}),$$

where

$$c' = \pi/4d'$$

so that the "radius" is

$$\langle \hat{b} \rangle \propto d' \quad \text{and} \quad |d'| < b \approx R. \tag{7.6}$$

In the Veneziano model, Eq. (1.3),

$$\langle \hat{b} \rangle \propto \alpha'/b. \tag{7.7}$$

Equations (7.6) and (7.7) suggest that even in case C, the two models may not be very close. As mentioned in Sec. II A, this may be because "radius" has a different meaning. The non-Gaussian form of solutions C shows that the physical picture of Sec. III A does not apply to them. In the Veneziano case, the non-Gaussian form is a consequence of the restrictions of Bose statistics[13] for the "constituent" oscillators,[34] each carrying spin 0 and 1. This may be the physical origin of this type of solution of the statistical bootstrap model: Ultimately, each hadron is dominated by ν mass-μ constituents, all with low energy and small, bounded orbital angular momentum. In the statistical model,

$$\nu \approx (m/\mu), \tag{7.8}$$

whereas in the Veneziano model,[35]

$$\nu \approx D(m/m_0) \ln(m/m_0), \tag{7.9}$$

where

$$m_0 = \pi(D/6\alpha')^{1/2}$$
$$\equiv (2/\alpha')b$$

so that in terms of "fundamental" constituents, both models are similar.

The ambiguity between solutions A and B is of the same nature as that which occurs in the case of the total level density. There, we have found that the original solution of the asymptotic bootstrap, Eq. (1.1),

$$n(m) = K m^a e^{bm},$$

where $a < -\frac{5}{2}$, can always be multiplied by

$$e^{-b' m^\gamma},$$

where $0 < \gamma < 1$ and $b' < 0$ [as in Eq. (6.8)]. In this case, a is not constrained. Such ambiguities can only be removed by requiring the bootstrap to hold at large finite masses, as well as in the asymptotic limit, $m \to \infty$.

The reason why we can find such a variety of bootstrap solutions is probably that our calculations concentrate on the dominant configuration, which has low orbital angular momentum, rather independent of the geometrical "size" R of the hadron. Only by looking more accurately, particularly at high resonance spins, would it be possible to separate the various possibilities. But one suspects that the model would be particularly unreliable at such high spins.

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[23]One recalls that $\sigma(m,J_z)$ is a number of states, and so is semi-positive definite, and real. Reality requires $\tilde{\sigma}(m,\alpha)$ to be even in α.

[24]The full expressions for the kinematic quantities are

$$p^2 = \frac{1}{4m^2}(m-m_1-m_2)(m-m_1+m_2)(m+m_1-m_2)(m+m_1+m_2),$$

$$E_1 = \frac{1}{2m}(m^2+m_1^2-m_2^2),$$

$$E_2 = \frac{1}{2m}(m^2+m_2^2-m_1^2).$$

[25]The precise cutoff is irrelevant, as the upper region of the integrand contributes nothing as $m\to\infty$.

[26]*Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1968), Sec. 9.6.34.

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