Motion of Particles in Einstein's Relativistic Field Theory. II. Application of General Theory

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In this paper we apply the approximation procedure of paper I to the case of certain simple ideal particles located in an isolated region of space-time and find the equations of motion satisfied by the ideal particles to certain low orders of approximation. We also find the equations of change satisfied by those quantities which characterize the structure of the particles and the field equations satisfied by the external field which acts on each particle. We find that there are simple ideal particles which, in a finite region of space-time, interact with each other, to a good approximation, according to the laws of Maxwell-Lorentz electrodynamics. In a higher order of approximation we find an additional interaction among the particles which can be looked upon as a Lorentz-covariant generalization of the interaction given by Newton's gravitational theory.

I. SIMPLE IDEAL PARTICLES

A. Equations of Charge, Mass, Motion, and Spin

In this section we shall use the methods developed in paper I[1] to find the first-order equations of charge and the second-order equations of mass, motion, and spin satisfied by N simple ideal particles located in a perfectly isolated region of the continuum. In addition to finding these equations we shall also find the fourth-order equations of mass, motion, and spin satisfied by N interacting simple ideal particles in the special case where the interacting particles are, to second order, neutral and spinless.

We have as the solution to the homogeneous equations (I5.6) over a region of the continuum containing only simple ideal particles

$$j^H_\mu = \sum_{p=1}^{N} {}^{(p)}j^H_\mu , \tag{1.1}$$

$$\gamma^{EH}_\mu = \sum_{p=1}^{N} {}^{(p)}\gamma^{EH}_\mu , \quad \gamma^{MH}_\mu = \sum_{p=1}^{N} {}^{(p)}\gamma^{MH}_\mu , \tag{1.2}$$

$$\gamma^H_{(\mu\nu)} = \sum_{p=1}^{N} {}^{(p)}\gamma^H_{(\mu\nu)} , \tag{1.3}$$

where

$${}^{(p)}j^H_\mu = {}^{(p)}j^H_{\mu\,\mathrm{ret}} + {}^{(p)}j^H_{\mu\,\mathrm{adv}} , \tag{1.4}$$

$$\begin{aligned} {}^{(p)}\gamma^{EH}_\mu &= {}^{(p)}\gamma^{EH}_{\mu\,\mathrm{ret}} + {}^{(p)}\gamma^{EH}_{\mu\,\mathrm{adv}} , \\ {}^{(p)}\gamma^{MH}_\mu &= {}^{(p)}\gamma^{MH}_{\mu\,\mathrm{ret}} + {}^{(p)}\gamma^{MH}_{\mu\,\mathrm{adv}} , \end{aligned} \tag{1.5}$$

$${}^{(p)}\gamma^H_{(\mu\nu)} = {}^{(p)}\gamma^H_{(\mu\nu)\,\mathrm{ret}} + {}^{(p)}\gamma^H_{(\mu\nu)\,\mathrm{adv}} , \tag{1.6}$$

and

$${}^{(p)}j^H_{\mu\,\mathrm{ret(adv)}} = \frac{1}{c^2}\,{}^{(p)}a_{\mathrm{ret(adv)}}\,{}^{(p)}[(e^D/l^2)u_\mu(ru)^{-1}]_{\mathrm{ret(adv)}}, \tag{1.7}$$

$${}^{(p)}\gamma^{EH}_{\mu\,\mathrm{ret(adv)}} = \frac{1}{c^2}\,{}^{(p)}a_{\mathrm{ret(adv)}}\,{}^{(p)}[e^E u_\mu(ru)^{-1}]_{\mathrm{ret(adv)}}, \tag{1.8}$$

$$\begin{aligned} {}^{(p)}\gamma^{MH}_{\mu\,\mathrm{ret(adv)}} &= \frac{1}{c^3}\,{}^{(p)}a_{\mathrm{ret(adv)}}\,{}^{(p)}[e^M u_\mu(ru)^{-1}]_{\mathrm{ret(adv)}}, \\ {}^{(p)}\gamma^H_{(\mu\nu)} &= \frac{1}{c^2}\,{}^{(p)}a_{\mathrm{ret(adv)}}\,{}^{(p)}[(m^G u_\mu u_\nu + \tfrac{1}{2}\dot{S}_{\mu\rho}u^\rho u_\nu \\ &\qquad + \tfrac{1}{2}\dot{S}_{\nu\rho}u^\rho u_\mu)(ru)^{-1}]_{\mathrm{ret(adv)}} \\ &\quad + \frac{1}{c^2}\,{}^{(p)}a_{\mathrm{ret(adv)}}\,{}^{(p)}[(\tfrac{1}{2}S_{\mu\rho}u_\nu \\ &\qquad + \tfrac{1}{2}S_{\nu\rho}u_\mu)(ru)^{-1}]_{\mathrm{ret(adv)}}{}^{,\rho} . \end{aligned} \tag{1.9}$$

The quantities ${}^{(p)}e^D$, ${}^{(p)}e^E$, ${}^{(p)}e^M$, ${}^{(p)}m^G$, and ${}^{(p)}S_{\mu\nu}$ in (1.7)–(1.9) depend on the parameter κ introduced earlier and can be expanded in a power series in κ. We have

$$\begin{aligned} {}^{(p)}e^D &= \sum_{k=1}^{\infty}\kappa^k\,{}^{(p)}_{(k)}e^D , \\ {}^{(p)}e^E &= \sum_{k=1}^{\infty}\kappa^k\,{}^{(p)}_{(k)}e^E , \\ {}^{(p)}e^M &= \sum_{k=1}^{\infty}\kappa^k\,{}^{(p)}_{(k)}e^M , \\ {}^{(p)}m^G &= \sum_{k=1}^{\infty}\kappa^k\,{}^{(p)}_{(k)}m^G , \\ {}^{(p)}S_{\mu\nu} &= \sum_{k=1}^{\infty}\kappa^k\,{}^{(p)}_{(k)}S_{\mu\nu} . \end{aligned} \tag{1.10}$$

From the structure of the field equations it follows that any of the quantities ${}^{(p)}_{(1)}e^D$, ${}^{(p)}_{(1)}e^E$, ${}^{(p)}_{(1)}e^M$, ${}^{(p)}_{(1)}m^G$, and ${}^{(p)}_{(1)}S_{\mu\nu}$ can be taken to be zero. For convenience we shall assume that the region we are investigating contains only particles for which ${}^{(p)}_{(1)}m^G = 0$ and ${}^{(p)}_{(1)}S_{\mu\nu} = 0$. This assumption is a restriction on the structure of the simple ideal particles in the region we are investigating only in the sense that it determines the order in which various interaction terms appear in our approximation procedure. This assumption will mean that an electromagnetic interaction among the particles can appear in second order, while a gravitational interaction among the particles can only appear in fourth or higher order.

Making the assumption mentioned above, we find from (1.7)–(1.9) that

$$\overset{(p)}{\underset{[1]}{}} j^{H}_{\mu\,\mathrm{ret(adv)}} = \frac{1}{c^2}\,{}^{(p)}a_{\mathrm{ret(adv)}}{}^{(p)}[(e^D/l^2)u_\mu(ru)^{-1}]_{\mathrm{ret(adv)}}, \qquad (1.11)$$

$$\overset{(p)}{\underset{[1]}{}} \gamma^{EH}_{\mu\,\mathrm{ret(adv)}} = \frac{1}{c^2}\,{}^{(p)}a_{\mathrm{ret(adv)}}{}^{(p)}[e^E u_\mu(ru)^{-1}]_{\mathrm{ret(adv)}},$$

$$\overset{(p)}{\underset{[1]}{}} \gamma^{MH}_{\mu\,\mathrm{ret(adv)}} = \frac{1}{c^3}\,{}^{(p)}a_{\mathrm{ret(adv)}}{}^{(p)}[e^M u_\mu(ru)^{-1}]_{\mathrm{ret(adv)}}, \qquad (1.12)$$

$$\overset{(p)}{\underset{[1]}{}} \gamma^{H}_{(\mu\nu)} = 0, \qquad (1.13)$$

so that

$$\overset{(p)}{\underset{[1]}{}} j_{\mu\,\mathrm{ret(adv)}}{}^{,\mu} = {}^{(p)}a_{\mathrm{ret(adv)}}{}^{(p)}[(\dot{e}^D/c^2l^2)(ru)^{-1}]_{\mathrm{ret(adv)}}, \qquad (1.14)$$

$$\overset{(p)}{\underset{[1]}{}} \gamma^{EH}_{\mu\,\mathrm{ret(adv)}}{}^{,\mu} = {}^{(p)}a_{\mathrm{ret(adv)}}{}^{(p)}[(\dot{e}^E/c^2)(ru)^{-1}]_{\mathrm{ret(adv)}},$$

$$\overset{(p)}{\underset{[1]}{}} \gamma^{MH}_{\mu\,\mathrm{ret(adv)}}{}^{,\mu} = {}^{(p)}a_{\mathrm{ret(adv)}}{}^{(p)}[(\dot{e}^M/c^3)(ru)^{-1}]_{\mathrm{ret(adv)}}, \qquad (1.15)$$

$$\overset{(p)}{\underset{[1]}{}} \gamma^{H}_{(\mu\nu)}{}^{,\nu} = 0, \qquad (1.16)$$

and thus

$$\overset{(p)}{\underset{[1]}{}} C^{DH} = {}^{(p)}\dot{e}^D/c^2l^2, \qquad (1.17)$$

$$\overset{(p)}{\underset{[1]}{}} C^{EH} = {}^{(p)}\dot{e}^E/c^2, \quad \overset{(p)}{\underset{[1]}{}} C^{MH} = {}^{(p)}\dot{e}^M/c^3, \qquad (1.18)$$

$$\overset{(p)}{\underset{[1]}{}} C^{H}_{\mu} = 0, \quad \overset{(p)}{\underset{[1]}{}} C^{H}_{[\mu\nu]} = 0. \qquad (1.19)$$

We must next investigate the solutions to the inhomogeneous equations (I5.7).

Since

$$\overset{(p)}{\underset{[1]}{}} s_\mu = 0, \qquad (1.20)$$

we find that Eqs. (I5.7a) to first order take the form

$$\Box^2\,{}_{[1]}j^{I}_{\mu} = O(\kappa^2). \qquad (1.21)$$

The solutions to (1.21) are

$${}_{[1]}j^{I}_{\mu} = 0, \qquad (1.22)$$

so that

$$\overset{(p)}{\underset{[1]}{}} C^{DI} = 0. \qquad (1.23)$$

To first order, the diffuse electric charger acting on each particle is therefore zero, and the first-order equations of diffuse electric charge are

$${}^{(p)}\dot{e}^D = 0. \qquad (1.24)$$

Equations (I5.7c) to first order take the form

$$\Box^2\,{}_{[1]}\gamma^{I}_{\mu} = {}_{[1]}j_\mu + O(\kappa^2). \qquad (1.25)$$

From (1.11) and (1.22) we see that the solutions to Eqs. (1.25) can be put in the form

$${}_{[1]}\gamma^{I}_{\mu} = \sum_{p=1}^{N} \overset{(p)}{\underset{[1]}{}}\gamma^{I}_{\mu}, \qquad (1.26)$$

where

$$\Box^2\,\overset{(p)}{\underset{[1]}{}}\gamma^{I}_{\mu} = \overset{(p)}{\underset{[1]}{}} j^{H}_{\mu} + O(\kappa^2). \qquad (1.27)$$

Equations (1.27) are easily solved. We find as their solutions

$$\overset{(p)}{\underset{[1]}{}}\gamma^{I}_{\mu} = \overset{(p)}{\underset{[1]}{}}\gamma^{I}_{\mu\,\mathrm{ret}} + \overset{(p)}{\underset{[1]}{}}\gamma^{I}_{\mu\,\mathrm{adv}}, \qquad (1.28)$$

where

$$\overset{(p)}{\underset{[1]}{}}\gamma^{I}_{\mu\,\mathrm{ret(adv)}} = \frac{1}{c^2}\,{}^{(p)}a_{\mathrm{ret(adv)}}{}^{(p)}[-(e^D/2l^2)r_\mu]_{\mathrm{ret(adv)}}. \qquad (1.29)$$

Since this means

$${}_{[1]}\gamma^{I}_{\mu}{}^{,\mu\nu} = 0, \qquad (1.30)$$

we see that

$$\overset{(p)}{\underset{[1]}{}} C^{EI} = 0. \qquad (1.31)$$

To first order the localized electric charger acting on each particle is zero, and the first-order equations of localized electric and magnetic charge are

$${}^{(p)}\dot{e}^E = 0, \quad {}^{(p)}\dot{e}^M = 0. \qquad (1.32)$$

Since

$${}_{[1]}t_{\mu\nu} = 0, \qquad (1.33)$$

Eqs. (I5.7g) to first order take the form

$$\Box^2\,{}_{[1]}\gamma^{I}_{(\mu\nu)} = O(\kappa^2). \qquad (1.34)$$

The solutions to (1.34) are

$${}_{[1]}\gamma^{I}_{(\mu\nu)} = 0, \qquad (1.35)$$

so that

$$\overset{(p)}{\underset{[1]}{}} C^{I}_{\mu} = 0, \quad \overset{(p)}{\underset{[1]}{}} C^{I}_{[\mu\nu]} = 0. \qquad (1.36)$$

There are no first-order equations of mass, motion, and spin.

In summary, the first-order solutions to the field equations take the form

$${}_{[1]}\gamma_{[\mu\nu]} = {}_{[1]}\gamma^{E}_{[\mu\nu]} + {}_{[1]}\gamma^{M}_{[\mu\nu]}, \qquad (1.37)$$

$${}_{[1]}\gamma^{E}_{[\mu\nu]} = \sum_{p=1}^{N}\overset{(p)}{\underset{[1]}{}}\gamma^{E}_{[\mu\nu]},$$

$${}_{[1]}\gamma^{M}_{[\mu\nu]} = \sum_{p=1}^{N}\overset{(p)}{\underset{[1]}{}}\gamma^{M}_{[\mu\nu]}, \qquad (1.38)$$

$${}_{[1]}\gamma_{(\mu\nu)} = 0, \qquad (1.39)$$

where

$$\overset{(p)}{\underset{[1]}{}}\gamma^{E}_{[\mu\nu]} = \epsilon_{\mu\nu\rho\sigma}\,\overset{(p)}{\underset{[1]}{}}\gamma^{E\sigma,\rho},$$

$$\overset{(p)}{\underset{[1]}{}}\gamma^{M}_{[\mu\nu]} = \overset{(p)}{\underset{[1]}{}}\gamma^{M}_{\nu,\mu} - \overset{(p)}{\underset{[1]}{}}\gamma^{M}_{\mu,\nu}, \qquad (1.40)$$

$$
\begin{aligned}
{}^{(p)}_{[1]}\gamma^{E}_{\mu} &= {}^{(p)}_{[1]}\gamma^{E}_{\mu\,\mathrm{ret}} + {}^{(p)}_{[1]}\gamma^{E}_{\mu\,\mathrm{adv}}\,,\\
{}^{(p)}_{[1]}\gamma^{M}_{\mu} &= {}^{(p)}_{[1]}\gamma^{M}_{\mu\,\mathrm{ret}} + {}^{(p)}_{[1]}\gamma^{M}_{\mu\,\mathrm{adv}}\,,
\end{aligned}
\tag{1.41}
$$

and

$$
\begin{aligned}
{}^{(p)}_{[1]}\gamma^{E}_{\mu\,\mathrm{ret(adv)}} &= \frac{1}{c^2}\,{}^{(p)}a_{\mathrm{ret(adv)}}\,{}^{(p)}\big[e^{E}u_\mu(ru)^{-1} - (e^{D}/2l^2)r_\mu\big]_{\mathrm{ret(adv)}}\,,\\
{}^{(p)}_{[1]}\gamma^{M}_{\mu\,\mathrm{ret(adv)}} &= \frac{1}{c^3}\,{}^{(p)}a_{\mathrm{ret(adv)}}\,{}^{(p)}\big[e^{M}u_\mu(ru)^{-1}\big]_{\mathrm{ret(adv)}}\,.
\end{aligned}
\tag{1.42}
$$

To this order, the equations of charge associated with the pth particle are

$$ {}^{(p)}\dot{e}^{D} = 0\,, \tag{1.43} $$

$$ {}^{(p)}\dot{e}^{E} = 0\,, \quad {}^{(p)}\dot{e}^{M} = 0\,. \tag{1.44} $$

There are no first-order equations of mass, motion, and spin.

We now seek the equations of mass, motion, and spin to second order. To find these equations we will have to study the structure of ${}_{[2]}\gamma^{I}_{(\mu\nu)}$ in sufficient detail to determine ${}_{[2]}\gamma^{I}_{(\mu\nu)}{}^{,\nu}$ to second order. We will not need to know ${}_{[2]}\gamma^{I}_{[\mu\nu]}$, nor will we need to know any of the equations of charge beyond the first order.

From (I2.2), (I2.3), (I4.14), (I4.19), (I4.24), and (I5.7g) we find that ${}_{[2]}\gamma^{I}_{(\mu\nu)}$ satisfies the equations

$$ \Box^2\,{}_{[2]}\gamma^{I}_{(\mu\nu)} = {}_{[2]}t_{\mu\nu} + O(\kappa^3)\,, \tag{1.45} $$

where

$$
\begin{aligned}
{}_{[2]}t_{\mu\nu} = \eta^{\rho\sigma}\eta^{\tau\lambda}\big[&\tfrac{1}{2}\,{}_{[1]}\gamma_{[\rho\tau],\mu}\,{}_{[1]}\gamma_{[\sigma\lambda],\nu} + {}_{[1]}\gamma_{[\mu\tau],\rho}\,{}_{[1]}\gamma_{[\nu\sigma],\lambda} - {}_{[1]}\gamma_{[\mu\tau],\rho}\,{}_{[1]}\gamma_{[\nu\lambda],\sigma} - \tfrac{1}{4}\eta_{\mu\nu}\eta^{\alpha\beta}\,{}_{[1]}\gamma_{[\rho\tau],\alpha}\,{}_{[1]}\gamma_{[\sigma\lambda],\beta}\\
&- \tfrac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\,{}_{[1]}\gamma_{[\rho\tau],\alpha}\,{}_{[1]}\gamma_{[\sigma\beta],\lambda} + {}_{[1]}\gamma_{[\rho\tau]}\,{}_{[1]}\gamma_{[\mu\lambda],\nu\sigma} + {}_{[1]}\gamma_{[\rho\tau]}\,{}_{[1]}\gamma_{[\nu\lambda],\mu\sigma} - \tfrac{1}{2}\eta_{\mu\nu}\,{}_{[1]}\gamma_{[\rho\tau]}\Box^2\,{}_{[1]}\gamma_{[\sigma\lambda]}\big]\,.
\end{aligned}
\tag{1.46}
$$

Equations (1.45) have been studied in Appendix A and there we show that they have a second-order solution for which

$$ {}_{[2]}\gamma^{I}_{(\mu\nu)}{}^{,\nu} = \sum_{p=1}^{N}{}^{(p)}a_{\mathrm{ret}}\,{}^{(p)}\big[{}_{[2]}C^{I}_{\mu}(ru)^{-1}\big]_{\mathrm{ret}} + \sum_{p=1}^{N}{}^{(p)}a_{\mathrm{adv}}\,{}^{(p)}\big[{}_{[2]}C^{I}_{\mu}(ru)^{-1}\big]_{\mathrm{adv}} + O(\kappa^3)\,, \tag{1.47} $$

where

$$ {}^{(p)}_{[2]}C^{I}_{\mu} = -(1/c^2)\,{}^{(p)}\big[e^{E}\Box^2\,{}_{[1]}\gamma^{*\mathrm{ext}}_{[\mu\nu]}u^{\nu} + (e^{D}/l^2)\,{}_{[1]}\gamma^{*\mathrm{ext}}_{[\mu\nu]}u^{\nu} + \tfrac{4}{3}a(e^{E}e^{D}/c^2l^2)(\ddot{u}_\mu + \dot{u}_\nu\dot{u}^{\nu}u_\mu) - (e^{M}/c)\Box^2\,{}_{[1]}\gamma^{\mathrm{ext}}_{[\mu\nu]}u^{\nu}\big] \tag{1.48} $$

and

$$ {}^{(p)}_{[1]}\gamma^{*\mathrm{ext}}_{[\mu\nu]} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\,{}^{(p)}_{[1]}\gamma^{\mathrm{ext}[\rho\sigma]}\,, \quad {}^{(p)}_{[1]}\gamma^{\mathrm{ext}}_{[\mu\nu]} = \sum_{p'\neq p}{}^{(p')}_{[1]}\gamma_{[\mu\nu]}\,, \tag{1.49} $$

$$ {}^{(p)}a = {}^{(p)}a_{\mathrm{adv}} - {}^{(p)}a_{\mathrm{ret}}\,. \tag{1.50} $$

The fields ${}^{(p)}_{[1]}\gamma^{*\mathrm{ext}}_{[\mu\nu]}$ and ${}^{(p)}_{[1]}\gamma^{\mathrm{ext}}_{[\mu\nu]}$ in (1.48) are understood as evaluated at the position of the pth particle.

From (I6.3) and (1.47)–(1.50) we find that the force to second order acting on the pth particle is

$$
\begin{aligned}
{}^{(p)}f_\mu = {}^{(p)}\big[&e^{E}\Box^2\gamma^{*\mathrm{ext}}_{[\mu\nu]}u^{\nu} + (e^{D}/l^2)\gamma^{*\mathrm{ext}}_{[\mu\nu]}u^{\nu}\\
&+ \tfrac{4}{3}a(e^{E}e^{D}/c^2l^2)(\ddot{u}_\mu + \dot{u}_\rho\dot{u}^{\rho}u_\mu)\\
&- (e^{M}/c)\Box^2\gamma^{\mathrm{ext}}_{[\mu\nu]}u^{\nu}\big]\,,
\end{aligned}
\tag{1.51}
$$

and the spin torque is zero. The second-order equations of mass, motion, and spin take the form

$$ {}^{(p)}\dot{P}_\mu = {}^{(p)}f_\mu\,, \tag{1.52} $$

$$ {}^{(p)}(\dot{S}_{\mu\nu} - \dot{S}_{\mu\rho}u^{\rho}u_\nu + \dot{S}_{\nu\rho}u^{\rho}u_\mu) = 0\,, \tag{1.53} $$

where ${}^{(p)}f_\mu$ is given in (1.51) and

$$ {}^{(p)}\dot{e}^{D} = 0\,, \tag{1.54} $$

$$ {}^{(p)}\dot{e}^{E} = 0\,, \quad {}^{(p)}\dot{e}^{M} = 0\,, \tag{1.55} $$

$$ {}^{(p)}S_{\mu\rho}\,{}^{(p)}u^{\rho} = 0\,. \tag{1.56} $$

We are using the notation

$$ {}^{(p)}P_\mu = {}^{(p)}(m^{G}u_\mu + \dot{S}_{\mu\rho}u^{\rho}) \tag{1.57} $$

in (1.52). From (1.52)–(1.57) we find that

$$ {}^{(p)}\dot{m}^{G} = 0\,. \tag{1.58} $$

We now seek the fourth-order equations of motion in the special case where the interacting simple ideal particles we are investigating are neutral and spinless to second order. In this case ${}_{[2]}\gamma_{(\mu\nu)}$ is given by

$$ {}_{[2]}\gamma_{(\mu\nu)} = \sum_{p=1}^{N}{}^{(p)}_{[2]}\gamma_{(\mu\nu)}\,, \tag{1.59} $$

where

$$ {}^{(p)}_{[2]}\gamma_{(\mu\nu)} = {}^{(p)}_{[2]}\gamma_{(\mu\nu)\,\mathrm{ret}} + {}^{(p)}_{[2]}\gamma_{(\mu\nu)\,\mathrm{adv}}\,, \tag{1.60} $$

and

$$ {}^{(p)}_{[2]}\gamma_{(\mu\nu)\,\mathrm{ret(adv)}} = \frac{1}{c^2}\,{}^{(p)}a_{\mathrm{ret(adv)}}\,{}^{(p)}\big[m^{G}u_\mu u_\nu(ru)^{-1}\big]_{\mathrm{ret(adv)}}\,. \tag{1.61} $$

The second-order equations of mass and motion satisfied by the particles are

$$ {}^{(p)}(m^{G}\dot{u}_\mu) = 0\,, \quad {}^{(p)}\dot{m}^{G} = 0\,. \tag{1.62} $$

There are no equations of spin to this order.

From (I2.2), (I2.3), (I4.14), (I4.19), (I4.24), and (I5.7g), we find that ${}_{[4]}\gamma^{I}_{(\mu\nu)}$ must satisfy the equations

$$\Box^2\,{}_{[4]}\gamma^{I}_{(\mu\nu)} = {}_{[4]}t_{\mu\nu} + O(\kappa^5), \tag{1.63}$$

where

$$\begin{aligned}{}_{[4]}t_{\mu\nu} = \eta^{\rho\sigma}\eta^{\tau\lambda}\big[&\tfrac{1}{2}\,{}_{[2]}\gamma_{(\rho\tau),\mu}\,{}_{[2]}\gamma_{(\sigma\lambda),\nu} - \tfrac{1}{4}\,{}_{[2]}\gamma_{(\rho\sigma),\mu}\,{}_{[2]}\gamma_{(\tau\lambda),\nu} + {}_{[2]}\gamma_{(\mu\rho),\tau}\,{}_{[2]}\gamma_{(\tau\sigma),\lambda} + {}_{[2]}\gamma_{(\mu\rho),\tau}\,{}_{[2]}\gamma_{(\nu\lambda),\sigma}\\ &- {}_{[2]}\gamma_{(\mu\rho),\tau}\,{}_{[2]}\gamma_{(\sigma\lambda),\nu} - {}_{[2]}\gamma_{(\nu\rho),\tau}\,{}_{[2]}\gamma_{(\sigma\lambda),\mu} - \tfrac{1}{4}\eta_{\mu\nu}\eta^{\alpha\beta}\,{}_{[2]}\gamma_{(\rho\tau),\alpha}\,{}_{[2]}\gamma_{(\sigma\lambda),\beta}\\ &+ \tfrac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\,{}_{[2]}\gamma_{(\alpha\rho),\tau}\,{}_{[2]}\gamma_{(\beta\lambda),\sigma} + \tfrac{1}{8}\eta_{\mu\nu}\eta^{\alpha\beta}\,{}_{[2]}\gamma_{(\rho\sigma),\alpha}\,{}_{[2]}\gamma_{(\tau\lambda),\beta} - {}_{[2]}\gamma_{(\rho\tau)}\,{}_{[2]}\gamma_{(\mu\nu),\sigma\lambda}\big].\end{aligned} \tag{1.64}$$

Equations (1.63) have been studied in Appendix B and there it is shown that they have a fourth-order solution for which

$${}_{[4]}\gamma_{(\mu\nu)}{}^{,\nu} = \sum_{p=1}^{N}{}^{(p)}a_{\text{ret}}\,{}^{(p)}\big[{}_{[4]}C^{I}_{\mu}(ru)^{-1}\big]_{\text{ret}} + \sum_{p=1}^{N}{}^{(p)}a_{\text{adv}}\,{}^{(p)}\big[{}_{[4]}C^{I}_{\mu}(ru)^{-1}\big]_{\text{adv}} + O(\kappa^5), \tag{1.65}$$

where

$${}^{(p)}_{[4]}C_{\mu} = -(1/c^2)^{(p)}\big[m^{G}\big(\tfrac{1}{4}\eta^{\rho\sigma}\,{}_{[2]}\gamma^{\text{ext}}_{(\rho\sigma),\mu} - \tfrac{1}{2}\,{}_{[2]}\gamma^{\text{ext}}_{(\rho\sigma),\mu}u^{\rho}u^{\sigma} + {}_{[2]}\gamma^{\text{ext}}_{(\mu\rho),\sigma}u^{\rho}u^{\sigma} - \tfrac{1}{4}\eta^{\rho\sigma}\,{}_{[2]}\gamma^{\text{ext}}_{(\rho\sigma),\tau}u^{\tau}u_{\mu} - \tfrac{1}{2}\,{}_{[2]}\gamma^{\text{ext}}_{(\rho\sigma),\tau}u^{\rho}u^{\sigma}u^{\tau}u_{\mu}\big)\big] \tag{1.66}$$

and

$${}^{(p)}_{[2]}\gamma^{\text{ext}}_{(\mu\nu)} = \sum_{p'\neq p}{}^{(p')}_{[2]}\gamma_{(\mu\nu)}\,. \tag{1.67}$$

The fields ${}^{(p)}_{[2]}\gamma^{\text{ext}}_{(\mu\nu)}$ in (1.66) are understood to be evaluated at the position of the pth particle.

From (I6.3) and (1.65)–(1.67) we find that the force to fourth order acting on the pth particle is

$${}^{(p)}f_{\mu} = {}^{(p)}\big[m^{G}\big(\tfrac{1}{4}\eta^{\rho\sigma}\gamma^{\text{ext}}_{(\rho\sigma),\mu} - \tfrac{1}{2}\gamma^{\text{ext}}_{(\rho\sigma),\mu}u^{\rho}u^{\sigma} + \gamma^{\text{ext}}_{(\mu\rho),\sigma}u^{\rho}u^{\sigma} - \tfrac{1}{4}\eta^{\rho\sigma}\gamma_{(\rho\sigma),\tau}u^{\tau}u_{\mu} - \tfrac{1}{2}\gamma_{(\rho\sigma),\tau}u^{\rho}u^{\sigma}u^{\tau}u_{\mu}\big)\big], \tag{1.68}$$

and the spin torque is zero. The fourth-order equations of mass, motion, and spin satisfied by simple ideal particles which are neutral and spinless to second order are[2]

$${}^{(p)}\dot{P}_{\mu} = {}^{(p)}f_{\mu}\,, \tag{1.69}$$

$${}^{(p)}(\dot{S}_{\mu\nu} - \dot{S}_{\mu\rho}u^{\rho}u_{\nu} + \dot{S}_{\nu\rho}u^{\rho}u_{\mu}) = 0\,, \tag{1.70}$$

where ${}^{(p)}f_{\mu}$ is given in (1.68) and

$${}^{(p)}S_{\mu\nu}\,{}^{(p)}u^{\nu} = 0\,. \tag{1.71}$$

We are using the notation

$${}^{(p)}P_{\mu} = {}^{(p)}(m^{G}u_{\mu} + S_{\mu\nu}u^{\nu}) \tag{1.72}$$

in (1.69). From (1.69)–(1.72) we find that

$${}^{(p)}\dot{m}^{G} = 0\,. \tag{1.73}$$

Generalized Maxwellian particles. In this subsection we shall consider only simple ideal particles which possess no magnetic charge, that is, particles for which

$${}^{(p)}e^{M} = 0\,. \tag{1.74}$$

The condition (1.74) must be satisfied if a simple ideal particle is to approximate, at distances sufficiently far from its center, a nonsingular solution to Einsteins's field equations.

From (1.51)–(1.58) we see that there are two conditions which must be satisfied if simple ideal particles possessing no magnetic charge are to appear spinless and are to interact among themselves in the lowest order of approximation through forces which are proportional to the localized electric charge on each particle and repulsive, at least over a certain range of distances, when acting between like particles. First, the spin associated with each particle must vanish to second order, that is,

$${}^{(p)}_{(1)}S_{\mu\nu} = {}^{(p)}_{(2)}S_{\mu\nu} = 0\,. \tag{1.75}$$

Second, in the lowest order of approximation the coefficient of diffuse electric charge characterizing each particle must be proportional to, and of opposite sign to, the localized electric charge associated with the same particle. If the proportionality constant is absorbed into l, this second requirement is equivalent to

$${}^{(p)}_{(1)}e^{D} = -{}^{(p)}_{(1)}e^{E}\,. \tag{1.76}$$

A set of simple ideal particles which satisfies (1.74) and the two conditions mentioned above will be known as a set of generalized Maxwellian particles.

In investigating the interaction among generalized Maxwellain particles in a perfectly isolated region of the continuum we shall find it convenient to introduce practical units. We shall define the electromagnetic field $F_{[\mu\nu]}$ in practical units through the equations

$$F_{[\mu\nu]}=(c^2/8\pi\epsilon_0 l^2 G)^{1/2}\gamma^*_{[\mu\nu]}\,, \tag{1.77}$$

where ϵ_0 is the permittivity of free space and G is the gravitational constant. We shall define the mass ${}^{(p)}m$ and the electric charge ${}^{(p)}e$ in practical units through the equations

$$\begin{aligned}{}^{(p)}m&=(1/4G)\,{}^{(p)}m^G\,,\\ {}^{(p)}e&=(2\pi\epsilon_0/Gl^2)^{1/2}\,{}^{(p)}e^E\,.\end{aligned} \tag{1.78}$$

In addition to the introduction of practical units we shall find it convenient to assume that (1.76) applies to the generalized Maxwellian particles in the region we are investigating. This involves no loss in generality, but does mean that the magnitude of l in our equations must from now on be considered a property of the generalized Maxwellian particles we are investigating.

In the practical units defined above, and using (1.76), the second-order equations of motion satisfied by interacting generalized Maxwellian particles in a perfectly isolated region of the continuum take the form

$${}^{(p)}m\,{}^{(p)}\dot{u}_\mu={}^{(p)}F_\mu\,, \tag{1.79}$$

$$\begin{aligned}{}^{(p)}F_\mu={}^{(p)}[&(e/2c)l^2\Box^2F^{\text{ext}}_{[\mu\nu]}u^\nu-(e/2c)F^{\text{ext}}_{[\mu\nu]}u^\nu\\ &-\tfrac{2}{3}a(e^2/4\pi\epsilon_0c^2)(\ddot{u}_\mu+\dot{u}_\rho\dot{u}^\rho u_\mu)]\,,\end{aligned} \tag{1.80}$$

where

$${}^{(p)}\dot{m}=0\,,\quad {}^{(p)}\dot{e}=0\,, \tag{1.81}$$

$${}^{(p)}a={}^{(p)}a_{\text{adv}}-{}^{(p)}a_{\text{ret}}\,. \tag{1.82}$$

In these equations the external electromagnetic field ${}^{(p)}F^{\text{ext}}_{[\mu\nu]}$ is understood to be evaluated at the position of the pth particle. This field to first order can be written in the form

$${}^{(p)}F^{\text{ext}}_{[\mu\nu]}=\sum_{p'\neq p}{}^{(p')}F_{[\mu\nu]}\,, \tag{1.83}$$

where ${}^{(p)}F_{[\mu\nu]}$ is the electromagnetic field produced by the pth particle. The electromagnetic field produced by the pth particle is given by

$${}^{(p)}F_{[\mu\nu]}={}^{(p)}A_{\mu,\nu}-{}^{(p)}A_{\nu,\mu}\,, \tag{1.84}$$

where

$$\begin{aligned}{}^{(p)}A_\mu&={}^{(p)}A_{\mu\,\text{ret}}+{}^{(p)}A_{\mu\,\text{adv}}\,,\\ {}^{(p)}A_{\mu\,\text{ret(adv)}}&={}^{(p)}a_{\text{ret(adv)}}\,{}^{(p)}\{(e/4\pi\epsilon_0c)[u_\mu(ru)^{-1}\\ &\qquad+(1/2l^2)r_\mu]\}_{\text{ret(adv)}}\,,\end{aligned} \tag{1.85}$$

$${}^{(p)}a_{\text{ret}}+{}^{(p)}a_{\text{adv}}=1\,. \tag{1.86}$$

This field satisfies the electromagnetic field equations

$${}^{(p)}F_{[\mu\nu]}{}^{,\nu}=\mu_0\,{}^{(p)}J_\mu\,,\quad {}^{(p)}F_{[\mu\nu,\rho]}=0\,, \tag{1.87}$$

where

$$\mu_0=1/\epsilon_0c^2\,,\qquad {}^{(p)}J_\mu={}^{(p)}J^L_\mu+{}^{(p)}J^D_\mu\,, \tag{1.88}$$

and

$${}^{(p)}J^L_\mu=ec\int\delta(x^\mu-{}^{(p)}\xi^\mu)\,{}^{(p)}u^\mu\,d({}^{(p)}\tau)\,, \tag{1.89}$$

$$\begin{aligned}{}^{(p)}J^D_\mu&={}^{(p)}J^D_{\mu\,\text{ret}}+{}^{(p)}J^D_{\mu\,\text{adv}}\,,\\ {}^{(p)}J^D_{\mu\,\text{ret(adv)}}&={}^{(p)}a_{\text{ret(adv)}}\,{}^{(p)}[-(ec/4\pi l^2)\,u_\mu(ru)^{-1}]_{\text{ret(adv)}}\,.\end{aligned} \tag{1.90}$$

We are using the symbol $\delta(x^\sigma-{}^{(p)}\xi^\sigma)$ to denote the four-dimensional Dirac δ function centered at the point ${}^{(p)}\xi^\sigma$. Note that electromagnetic disturbances (light) propagate in empty space with the velocity c.

Equation (1.80) for the force acting on a particle can be put into a more easily analyzable form if we make use of the fact that to first order,

$$l^2\Box^2\,{}^{(p)}F_{[\mu\nu]}=-{}^{(p)}F^L_{[\mu\nu]}\,. \tag{1.91}$$

The field ${}^{(p)}F^L_{[\mu\nu]}$ in (1.91) is the electromagnetic field produced by the localized electric charge associated with the pth particle. The field produced by the diffuse electric charge associated with the pth particle will be denoted by ${}^{(p)}F^D_{[\mu\nu]}$. Thus

$${}^{(p)}F_{[\mu\nu]}={}^{(p)}F^L_{[\mu\nu]}+{}^{(p)}F^D_{[\mu\nu]}\,, \tag{1.92}$$

where

$${}^{(p)}F^L_{[\mu\nu]}{}^{,\nu}=\mu_0\,{}^{(p)}J^L_\mu\,,\quad {}^{(p)}F^L_{[\mu\nu,\rho]}=0\,, \tag{1.93}$$

$${}^{(p)}F^D_{[\mu\nu]}{}^{,\nu}=\mu_0\,{}^{(p)}J^D_\mu\,,\quad {}^{(p)}F^D_{[\mu\nu,\rho]}=0\,. \tag{1.94}$$

Making use of (1.91) and (1.92)–(1.94), we find that the equations of motion (1.79)–(1.80) can be written in the form

$${}^{(p)}m\,{}^{(p)}\dot{u}_\mu={}^{(p)}F_\mu\,, \tag{1.95}$$

$$\begin{aligned}{}^{(p)}F_\mu={}^{(p)}[&(e/c)u^\lambda F^{L\,\text{ext}}_{[\lambda\mu]}+(e/2c)u^\lambda F^{D\,\text{ext}}_{[\lambda\mu]}\\ &-\tfrac{2}{3}a(e^2/4\pi\epsilon_0c^2)(\ddot{u}_\mu+\dot{u}_\rho\dot{u}^\rho u_\mu)]\,,\end{aligned} \tag{1.96}$$

where

$$\begin{aligned}{}^{(p)}F^{L\,\text{ext}}_{[\mu\nu]}&=\sum_{p'\neq p}{}^{(p')}F^L_{[\mu\nu]}\,,\\ {}^{(p)}F^{D\,\text{ext}}_{[\mu\nu]}&=\sum_{p'\neq p}{}^{(p')}F^D_{[\mu\nu]}\,,\end{aligned} \tag{1.97}$$

$${}^{(p)}a={}^{(p)}a_{\text{adv}}-{}^{(p)}a_{\text{ret}}\,. \tag{1.98}$$

The fields ${}^{(p)}F^{L\,\text{ext}}_{[\mu\nu]}$ and ${}^{(p)}F^{D\,\text{ext}}_{[\mu\nu]}$ in (1.96) are understood to be evaluated at the position of the pth particle.

From the work above we see that a generalized Maxwellian particle can be regarded as a localized electric charge surrounded by a diffuse electric charge of opposite sign. The charge structure follows from (1.87)–(1.90). The current associated with the localized electric charge of the pth particle is given by ${}^{(p)}J^L_\mu$ and that associated with the diffuse electric charge is given by ${}^{(p)}J^D_\mu$. For an isolated particle at rest, the density of the diffuse electric charge associated with a particle is inversely proportional to the distance from the center of the particle. The length l in (1.90) can be regarded as the distance from the center of the particle within which one finds an amount of diffuse electric charge equal in magnitude to one half the localized electric charge. From (1.96) it follows that the field produced by the diffuse electric charge associated with a particle is only half as effective in producing an acceleration on another particle as is the field produced by the localized electric charge.

If we assume that all interparticle distances within the perfectly isolated region we are investigating are negligible[3] in comparison to l, then the second term of ${}^{(p)}A_{\mu\text{ ret}}$ or ${}^{(p)}A_{\mu\text{ adv}}$ in (1.85) can be neglected in comparison with the first, the current ${}^{(p)}J^D_\mu$ can be neglected in comparison with ${}^{(p)}J^L_\mu$, and ${}^{(p)}F^D_{[\mu\nu]}$ can be neglected in comparison with ${}^{(p)}F^L_{[\mu\nu]}$. In such a region, with a negligible loss in accuracy, the second-order equations of motion for interacting generalized Maxwellian particles take the form

$$ {}^{(p)}m\, {}^{(p)}\dot{u}_\mu = {}^{(p)}F_\mu\,, \tag{1.99} $$

$$ {}^{(p)}F_\mu = {}^{(p)}[(e/c)\,u^\lambda F^{\text{ext}}_{[\lambda\mu]} - \tfrac{2}{3}a(e^2/4\pi\epsilon_0 c^2)(\ddot{u}_\mu + \dot{u}_\rho \dot{u}^\rho u_\mu)]\,, \tag{1.100} $$

where the electromagnetic field produced by each particle is given by

$$ {}^{(p)}F_{[\mu\nu]} = {}^{(p)}A_{\mu,\nu} - {}^{(p)}A_{\nu,\mu}\,, \tag{1.101} $$

$$ \begin{aligned} &{}^{(p)}A_\mu = {}^{(p)}A_{\mu\text{ ret}} + {}^{(p)}A_{\mu\text{ adv}}\,, \\ &{}^{(p)}A_{\mu\text{ ret(adv)}} = {}^{(p)}a_{\text{ret(adv)}}\, {}^{(p)}[(e/4\pi\epsilon_0 c)\,u_\mu (ru)^{-1}]_{\text{ret(adv)}}\,, \end{aligned} \tag{1.102} $$

and satisfies the electromagnetic field equations

$$ {}^{(p)}F_{[\mu\nu]}{}^{,\nu} = \mu_0\, {}^{(p)}J_\mu\,, \quad {}^{(p)}F_{[\mu\nu,\rho]} = 0\,, \tag{1.103} $$

where

$$ {}^{(p)}J_\mu = {}^{(p)}ec \int \delta(x^\sigma - {}^{(p)}\xi^\sigma)\, {}^{(p)}u^\mu\, d({}^{(p)}\tau)\,. \tag{1.104} $$

Equations (1.99)–(1.104) are the equations of motion and the electromagnetic field equations of Maxwell-Lorentz electrodynamics. Particles whose interaction is described through such equations will be known as Maxwellian particles. We see that Einstein's relativistic field theory admits simple ideal particles which, in a finite and perfectly isolated region of the space-time continuum, interact among themselves, to a high degree of accuracy, as Maxwellian particles.[4]

Generalized Newtonian particles. Generalized Maxwellian particles for which

$$ \begin{aligned} &{}^{(p)}_{(1)}e^D = {}^{(p)}_{(2)}e^D = 0\,, \\ &{}^{(p)}_{(1)}e^E = {}^{(p)}_{(2)}e^E = 0\,, \end{aligned} \tag{1.105} $$

$$ {}^{(p)}_{(3)}S_{\mu\nu} = {}^{(p)}_{(4)}S_{\mu\nu} = 0\,, \tag{1.106} $$

will be known as generalized Newtonian particles. Generalized Newtonian particles are thus generalized Maxwellian particles which are neutral to second order and spinless to fourth order. Introducing the gravitational field $F_{(\mu\nu)}$ in practical units through the equations

$$ F_{(\mu\nu)} = \tfrac{1}{4}c^2\gamma_{(\mu\nu)}\,, \tag{1.107} $$

we find from (1.68)–(1.73) that the fourth-order equations of motion satisfied by interacting generalized Newtonian particles in a perfectly isolated region of the continuum take the form[5]

$$ {}^{(p)}m\, {}^{(p)}\dot{u}_\mu = {}^{(p)}F_\mu\,, \tag{1.108} $$

$$ \begin{aligned} {}^{(p)}F_\mu = {}^{(p)}[(m/c^2)(&\eta^{\rho\sigma}F^{\text{ext}}_{(\rho\sigma),\mu} - 2F^{\text{ext}}_{(\rho\sigma),\mu}u^\rho u^\sigma \\ &+ 4F^{\text{ext}}_{(\mu\rho),\sigma}u^\rho u^\sigma - \eta^{\rho\sigma}F^{\text{ext}}_{(\rho\sigma),\tau}u^\tau u_\mu \\ &- 2F^{\text{ext}}_{(\rho\sigma),\tau}u^\rho u^\sigma u^\tau u_\mu)]\,, \end{aligned} \tag{1.109} $$

where

$$ {}^{(p)}\dot{m} = 0\,. \tag{1.110} $$

In (1.109) the external gravitational field ${}^{(p)}F^{\text{ext}}_{(\mu\nu)}$ can be written

$$ {}^{(p)}F^{\text{ext}}_{(\mu\nu)} = \sum_{p'\neq p} {}^{(p')}F_{(\mu\nu)}\,, \tag{1.111} $$

where ${}^{(p)}F_{(\mu\nu)}$ is the gravitational field produced by the pth particle. This field is given by

$$ \begin{aligned} &{}^{(p)}F_{(\mu\nu)} = {}^{(p)}F_{(\mu\nu)\text{ ret}} + {}^{(p)}F_{(\mu\nu)\text{ adv}}\,, \\ &{}^{(p)}F_{(\mu\nu)\text{ ret(adv)}} = {}^{(p)}a_{\text{ret(adv)}}\, {}^{(p)}[Gm_\mu u_\nu (ru)^{-1}]_{\text{ret(adv)}}\,, \end{aligned} \tag{1.112} $$

$$ {}^{(p)}a_{\text{ret}} + {}^{(p)}a_{\text{adv}} = 1\,, \tag{1.113} $$

and satisfies the gravitational field equations

$$ \Box^2\, {}^{(p)}F_{(\mu\nu)} = (4\pi G/c^2)\, {}^{(p)}T_{\mu\nu}\,, \tag{1.114} $$

where

$$ {}^{(p)}T_{\mu\nu} = {}^{(p)}mc^2 \int {}^{(p)}u_\mu\, {}^{(p)}u_\nu\, \delta(x^\sigma - {}^{(p)}\xi^\sigma)\, d({}^{(p)}\tau)\,. \tag{1.115} $$

If the velocities of each of the interacting particles are very much less than the velocity of light, we can, if the quantities in the equations are expressed as functions of t instead of ${}^{(p)}\tau$ or x^4, expand both sides of (1.108)–(1.113) in power series in c^{-1} and keep, with a negligible loss in accuracy, only the lowest-order terms in c^{-1}. Doing this and making use of the notation

$$ {}^{(p)}F_{(44)} = {}^{(p)}\varphi\,, \quad {}^{(p)}T_{44} = {}^{(p)}\rho c^2\,, \tag{1.116}$$

we find as the equations of motion for interacting generalized Newtonian particles in a perfectly isolated region of the continuum,

$$ {}^{(p)}m\, d^2({}^{(p)}\xi_s)/dt^2 = {}^{(p)}F_s'\,, \tag{1.117}$$

$$ {}^{(p)}F_s' = {}^{(p)}m\, {}^{(p)}\varphi^{\text{ext}}{}_{,s}\,, \tag{1.118}$$

where

$$ d({}^{(p)}m)/dt = 0\,. \tag{1.119}$$

In this approximation the external gravitational potential ${}^{(p)}\varphi^{\text{ext}}$ takes the form

$$ {}^{(p)}\varphi^{\text{ext}} = \sum_{p'\neq p} {}^{(p')}\varphi\,, \tag{1.120}$$

where ${}^{(p)}\varphi$ is the gravitational potential produced by the pth particle. This potential is given by

$$ {}^{(p)}\varphi = G\, {}^{(p)}m\, {}^{(p)}|\vec{r}\,|^{-1} \tag{1.121}$$

and satisfies the gravitational field equations

$$ \nabla^2\, {}^{(p)}\varphi = -4\pi G\, {}^{(p)}\rho\,, \tag{1.122}$$

where

$$ {}^{(p)}\rho = {}^{(p)}m\,\delta(x^s - {}^{(p)}\xi^s)\,. \tag{1.123}$$

We are using the symbol $\delta(x^s - {}^{(p)}\xi^s)$ to denote the three-dimensional Dirac δ function centered at the point ${}^{(p)}\xi^s$.

Equations (1.117)–(1.123) are the equations of motion and the field equations of Newtonian gravitational theory.[6] Particles whose interaction is described through such equations will be known as Newtonian particles. We see that Einstein's relativistic field theory admits simple ideal particles which, in a finite and perfectly isolated region of the space-time continuum, interact among themselves, in the lowest order of approximation with respect to an expansion in the parameter c^{-1}, as Newtonian particles.

An isolated spinless simple ideal particle. In 1960 Bandyopadhyay,[7] making use of the previous work of Papapetrou[8] and Wyman,[9] found the general static spherically symmetric solution to the field equations of Einstein's relativistic field theory which represents a particle with finite mass. The solution he found is singular and can be written in the form

$$ g_{\mu\nu} = \begin{bmatrix} -\alpha & 0 & 0 & w \\ 0 & -\beta & f\sin\theta & 0 \\ 0 & -f\sin\theta & -\beta\sin^2\theta & 0 \\ -w & 0 & 0 & \gamma \end{bmatrix}\,, \tag{1.124}$$

where

$$ f + i\beta = \frac{m_B{}^2(1+ih_B)\operatorname{sech}^2[\tfrac{1}{2}(1+ih_B)^{1/2}\ln\delta + a_B]}{\delta(c_B + i)}\,, \tag{1.125}$$

$$ \alpha = \frac{(f^2+\beta^2)\delta'^2}{4m_B{}^2\delta}\,, \quad w = \frac{\alpha\delta}{f^2+\beta^2}\, l_B{}^2\,, \quad \gamma = \delta + \frac{w^2}{\alpha}\,, \tag{1.126}$$

with

$$ \delta = e^x\,. \tag{1.127}$$

The variable x in (1.127) is to be regarded as an arbitrary function of $|\vec{r}\,|$ where

$$ |\vec{r}\,| = (x^s x^s)^{1/2} \tag{1.128}$$

and

$$ x^1 = |\vec{r}\,|\sin\theta\cos\varphi\,, \quad x^2 = |\vec{r}\,|\sin\theta\sin\varphi\,, \quad x^3 = |\vec{r}\,|\cos\theta\,. \tag{1.129}$$

The quantity δ' in (1.126) is the derivative of δ with respect to $|\vec{r}\,|$. The quantity a_B is a complex constant. The quantities m_B, h_B, c_B, and l_B are real constants. If the boundary conditions

$$ \alpha \to 1, \quad \beta \to \infty, \quad \gamma \to 1, \quad \text{as } |\vec{r}\,| \to \infty \tag{1.130}$$

are to be satisfied, the quantity a_B must satisfy the equation

$$ \sinh^2 a_B = -1\,. \tag{1.131}$$

If the coordinates x^1, x^2, and x^3 are to be part of a set of harmonic coordinates, then α, β, γ, f, and w must satisfy the equation

$$ \frac{(f^2+\beta^2)^{1/2} d}{\beta(\alpha\gamma - w^2)^{1/2} d|\vec{r}\,|}\left[\frac{\gamma(f^2+\beta^2)^{1/2}}{(\alpha\gamma - w^2)^{1/2}}\right] - 2|\vec{r}\,| = 0\,. \tag{1.132}$$

If both (1.131) and (1.132) are satisfied, Bandyopadhyay's solution to Einstein's field equations can be shown to represent a spinless simple ideal particle isolated in space-time. Introducing the notation

$$ e^D = 2c^2 l_B{}^2 c_B\,, \quad e^E = -\tfrac{1}{3}c^2 m_B{}^2 h_B\,, \quad e^M = -c^3 l_B\,, \tag{1.133}$$

$$ m^G = 4c^2 m_B\,, \tag{1.134}$$

and expanding α, β, γ, f, and w in a power series in e^D, e^E, e^M, and m^G, we find, using (1.131) and (1.132), that

$$ \begin{aligned} &\alpha = 1 + (m^G/2c^2)|\vec{r}\,|^{-1} + \alpha^N\,, \\ &\beta = |\vec{r}\,|^2[1 + (m^G/2c^2)|\vec{r}\,|^{-1}] + \beta^N\,, \qquad (1.135) \\ &\gamma = 1 - (m^G/2c^2)|\vec{r}\,|^{-1} + \gamma^N\,, \\ &f = -(e^E/c^2) - (e^D/2c^2 l^2)|\vec{r}\,|^2 + f^N\,, \\ &w = -(e^M/c^3)|\vec{r}\,|^{-2} + w^N\,. \qquad (1.136) \end{aligned}$$

The quantities α^N, β^N, γ^N, f^N, and w^N in (1.135) and (1.136) contain only terms nonlinear in e^D, e^E, e^M, and m^G. In harmonic coordinates we find, keeping only terms linear in e^D, e^E, e^M, and m^G,

$$g_{[st]}=\epsilon_{stk}\gamma^E{}_{,k}, \quad g_{[4s]}=-\gamma^M{}_{,s}, \tag{1.137}$$

$$g_{(st)}=-1-\tfrac{1}{2}\delta_{st}\gamma^G, \quad g_{(4s)}=0, \quad g_{(44)}=1-\tfrac{1}{2}\gamma^G, \tag{1.138}$$

where

$$\gamma^E=(e^E/c^2)|\vec{r}|^{-1}-(e^D/2c^2l^2)|\vec{r}|, \quad \gamma^M=(e^M/c^3)|\vec{r}|^{-1}, \tag{1.139}$$

$$\gamma^G=(m^G/c^2)|\vec{r}|^{-1}. \tag{1.140}$$

This means, in a harmonic coordinate system, keeping only terms linear in e^D, e^E, e^M, and m^G, that

$$\gamma_{[\mu\nu]}=\gamma^E_{[\mu\nu]}+\gamma^M_{[\mu\nu]}, \tag{1.141}$$

where

$$\gamma^E_{[\mu\nu]}=\epsilon_{\mu\nu\rho\sigma}\gamma^{E\sigma,\rho}, \quad \gamma^M_{[\mu\nu]}=\gamma^M_{\nu,\mu}-\gamma^M_{\mu,\nu}, \tag{1.142}$$

with

$$\gamma^E_s=0, \quad \gamma^E_4=(e^E/c^2)|\vec{r}|^{-1}-(e^D/2c^2l^2)|\vec{r}|, \tag{1.143}$$

$$\gamma^M_s=0, \quad \gamma^M_4=(e^M/c^3)|\vec{r}|^{-1}, \tag{1.144}$$

and that

$$\gamma_{(4s)}=\gamma_{(s4)}=0, \quad \gamma_{(st)}=0, \tag{1.145}$$

$$\gamma_{(44)}=(m^G/c^2)|\vec{r}|^{-1}. \tag{1.146}$$

We see that Bandyopadhyay's solution to Einstein's field equations does indeed, if the conditions (1.131) and (1.132) are satisfied, represent a spinless simple ideal particle isolated in space-time.

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APPENDIX A: ON SOLUTIONS TO THE FIELD EQUATIONS $\Box^2\,{}_{[2]}\gamma^I_{(\mu\nu)}={}_{[2]}t_{\mu\nu}+O(\kappa^3)$

We shall study the solutions to Eqs. (1.45) for which ${}_{[2]}\gamma^I_{(\mu\nu)}{}^{,\nu}$ takes form (I5.59). If all quantities in the equations are expressed as functions of t instead of as functions of ${}^{(p)}\tau$ or x^4, we can expand both sides of Eqs. (1.45) in a power series in c^{-1} and solve the field equations step by step with respect to powers of c^{-1}. Proceeding in this manner we find, in the neighborhood of the pth particle,

$$\begin{aligned}
{}_{[2]}\gamma^I_{(00)}=&({}^{(p)}e^E/c^2)\epsilon_{stk}{}^{(p)}[\tfrac{1}{4}(krl)^0\,{}_{[1]}\bar\gamma_{[st],rl}-\tfrac{1}{2}(krl)^0\xi''^l\,{}_{[1]}\bar\gamma_{[st],r}\\
&-\tfrac{1}{2}(r)^0\xi''^r\,{}_{[1]}\bar\gamma_{[st],k}+\tfrac{1}{4}(k)^0\,{}_{[1]}\bar\gamma_{[st],44}-(k)^0\,{}_{[1]}\bar\gamma_{[s4],t4}\\
&+\tfrac{3}{4}(krl)^0\xi''^r\xi''^l\,{}_{[1]}\bar\gamma_{[st]}-\tfrac{3}{4}(k)^0\xi''^r\xi''^r\,{}_{[1]}\bar\gamma_{[st]}+\tfrac{3}{2}(r)^0\xi''^r\xi''^k\,{}_{[1]}\bar\gamma_{[st]}\\
&-2(k)^0\xi''^t\,{}_{[1]}\bar\gamma_{[s4]}-(k)^0\xi''^t\,{}_{[1]}\bar\gamma_{[s4],4}+\tfrac{1}{2}(kr)^{-1}\,{}_{[1]}\bar\gamma_{[st],r}\\
&+\tfrac{1}{2}|\vec{r}|^{-1}\,{}_{[1]}\bar\gamma_{[st],k}-\tfrac{1}{2}(kr)^{-1}\xi''^r\,{}_{[1]}\bar\gamma_{[st]}+(kr)^{-1}\xi'^t\,{}_{[1]}\bar\gamma_{[s4],r}\\
&+(kr)^{-1}\xi'^r\,{}_{[1]}\bar\gamma_{[s4],t}-\tfrac{1}{2}|\vec{r}|^{-1}\xi''^k\,{}_{[1]}\bar\gamma_{[st]}-4(kr)^{-1}\xi'^r\xi''^t\,{}_{[1]}\bar\gamma_{[s4]}-2(kr)^{-1}\xi''^r\xi'^t\,{}_{[1]}\bar\gamma_{[s4]}\\
&-\tfrac{3}{2}(krl)^{-2}\xi'^r\xi'^l\,{}_{[1]}\bar\gamma_{[st]}+\tfrac{1}{2}(k)^{-2}\xi'^r\xi'^r\,{}_{[1]}\bar\gamma_{[st]}+2(k)^{-2}\xi'^t\,{}_{[1]}\bar\gamma_{[s4]}]\\
&+({}^{(p)}e^D/c^2l^2)\epsilon_{stk}{}^{(p)}[\tfrac{1}{4}(k)^0\,{}_{[1]}\bar\gamma_{[st]}]+({}^{(p)}e^M/c^3)^{(p)}[-\tfrac{1}{2}(str)^0\,{}_{[1]}\bar\gamma_{[s4],tr}\\
&+(rst)^0\xi''^r\,{}_{[1]}\bar\gamma_{[s4],t}-(s)^0\xi''^t\,{}_{[1]}\bar\gamma_{[s4],t}+(t)^0\xi''^s\,{}_{[s4],t}\\
&+\tfrac{1}{2}(s)^0\,{}_{[1]}\bar\gamma_{[s4],44}-\tfrac{3}{2}(srl)^0\xi''^r\xi''^l\,{}_{[1]}\bar\gamma_{[s4]}+\tfrac{3}{2}(s)^0\xi''^r\xi''^r\,{}_{[1]}\bar\gamma_{[s4]}\\
&-3(r)^0\xi''^r\xi''^s\,{}_{[1]}\bar\gamma_{[s4]}+2(s)^0\xi'''^t\,{}_{[1]}\bar\gamma_{[st]}+2(s)^0\xi''^t\,{}_{[1]}\bar\gamma_{[st],4}-(st)^{-1}\,{}_{[1]}\bar\gamma_{[s4],t}\\
&-(sr)^{-1}\xi'^t\,{}_{[1]}\bar\gamma_{[st],r}+(sr)^{-1}\xi''^r\,{}_{[1]}\bar\gamma_{[s4]}-(sr)^{-1}\xi'^r\,{}_{[1]}\bar\gamma_{[s4],4}\\
&+|\vec{r}|^{-1}\xi''^s\,{}_{[1]}\bar\gamma_{[s4]}+4(sr)^{-1}\xi'^r\xi''^t\,{}_{[1]}\bar\gamma_{[st]}+2(sr)^{-1}\xi''^r\xi'^t\,{}_{[1]}\bar\gamma_{[st]}\\
&+3(srl)^{-2}\xi'^r\xi'^l\,{}_{[1]}\bar\gamma_{[s4]}-(s)^{-2}\xi'^r\xi'^r\,{}_{[1]}\bar\gamma_{[s4]}-2(s)^{-2}\xi'^t\,{}_{[1]}\bar\gamma_{[st]}]+{}^{(p)}_{[2]}\overset{+}{\gamma}_{(00)}+O(\kappa^3),
\end{aligned} \tag{A1a}$$

$$ {}_{[2]}\gamma^{I}_{(0m)} = ({}^{(p)}e^{E}/c^2)\epsilon_{stk}{}^{(p)}[\tfrac18(mkr)^0{}_{[1]}\bar\gamma_{[st],r4}+\tfrac38(m)^0{}_{[1]}\bar\gamma_{[st],k4}-\tfrac18(k)^0{}_{[1]}\bar\gamma_{[st],m4}-\tfrac18\delta_{km}(r)^0{}_{[1]}\bar\gamma_{[st],r4} $$

$$ -\tfrac14(mkr)^0{}_{[1]}\bar\gamma_{[s4],tr}-\tfrac34(k)^0{}_{[1]}\bar\gamma_{[s4],tm}+\tfrac14\delta_{km}(r)^0{}_{[1]}\bar\gamma_{[s4],tr}-\tfrac14(mkr)^0\xi'''^r{}_{[1]}\bar\gamma_{[st]}-\tfrac14(m)^0\xi'''^k{}_{[1]}\bar\gamma_{[st]} $$

$$ -\tfrac38(mkr)^0\xi''^r{}_{[1]}\gamma_{[st],4}-\tfrac18(m)^0\xi''^k{}_{[1]}\bar\gamma_{[st],4}+\tfrac18(k)^0\xi''^m{}_{[1]}\bar\gamma_{[st],4}-\tfrac38\delta_{km}(r)^0\xi''^r{}_{[1]}\bar\gamma_{[st],4} $$

$$ +\tfrac14(mkr)^0\xi''^r{}_{[1]}\bar\gamma_{[s4],t}+\tfrac14(m)^0\xi''^k{}_{[1]}\bar\gamma_{[s4],t}-\tfrac34(k)^0\xi''^m{}_{[1]}\bar\gamma_{[s4],t}+\tfrac14\delta_{km}(r)^0\xi''^r{}_{[1]}\bar\gamma_{[s4],t} $$

$$ +\tfrac12(mkr)^0\xi''^t{}_{[1]}\bar\gamma_{[s4],r}-\tfrac34\delta_{km}(t)^0\xi''^r{}_{[1]}\bar\gamma_{[s4],r}-\tfrac14\delta_{km}(r)^0\xi''^t{}_{[1]}\bar\gamma_{[s4],r} $$

$$ +\tfrac14(k)^0\xi''^t{}_{[1]}\bar\gamma_{[s4],m}-\tfrac12\delta_{km}(t)^0{}_{[1]}\bar\gamma_{[s4],44}-\tfrac32(mkr)^0\xi''^r\xi''^t{}_{[1]}\bar\gamma_{[s4]} $$

$$ +\tfrac38\delta_{km}(trl)^0\xi''^r\xi''^l{}_{[1]}\bar\gamma_{[s4]}+\tfrac32(k)^0\xi''^t\xi''^m{}_{[1]}\bar\gamma_{[s4]}-\tfrac38\delta_{km}(t)^0\xi''^r\xi''^r{}_{[1]}\bar\gamma_{[s4]} $$

$$ +\tfrac34\delta_{km}(r)^0\xi''^r\xi''^t{}_{[1]}\bar\gamma_{[s4]}+\tfrac34(mkrl)^{-1}\xi'^l{}_{[1]}\bar\gamma_{[st],r}+\tfrac14(mk)^{-1}\xi'^r{}_{[1]}\bar\gamma_{[st],r} $$

$$ -\tfrac14(mr)^{-1}\xi'^k{}_{[1]}\bar\gamma_{[st],r}-\tfrac14(kr)^{-1}\xi'^m{}_{[1]}\bar\gamma_{[st],r}+\tfrac14(kr)^{-1}\xi'^r{}_{[1]}\bar\gamma_{[st],m} $$

$$ -\tfrac14\delta_{km}(rl)^{-1}\xi'^l{}_{[1]}\bar\gamma_{[st],r}+\tfrac14(mk)^{-1}{}_{[1]}\bar\gamma_{[st],4}-\tfrac12(mk)^{-1}{}_{[1]}\bar\gamma_{[s4],t} $$

$$ -\tfrac12|\vec r|^{-1}\xi'^m{}_{[1]}\bar\gamma_{[st],k}-\tfrac14|\vec r|^{-1}\xi'^k{}_{[1]}\bar\gamma_{[st],m}+\tfrac34\delta_{km}|\vec r|^{-1}\xi'^r{}_{[1]}\bar\gamma_{[st],r} $$

$$ +\tfrac14\delta_{km}|\vec r|^{-1}{}_{[1]}\bar\gamma_{[st],4}-\tfrac12\delta_{km}|\vec r|^{-1}{}_{[1]}\bar\gamma_{[s4],t}-\tfrac94(mkrl)^{-1}\xi'^r\xi''^l{}_{[1]}\bar\gamma_{[st]} $$

$$ +\tfrac34(mk)^{-1}\xi'^r\xi''^r{}_{[1]}\bar\gamma_{[st]}-\tfrac14(mr)^{-1}\xi'^r\xi''^k{}_{[1]}\bar\gamma_{[st]}+\tfrac14(mr)^{-1}\xi''^r\xi'^k{}_{[1]}\bar\gamma_{[st]} $$

$$ +\tfrac14(kr)^{-1}\xi'^r\xi''^m{}_{[1]}\bar\gamma_{[st]}+\tfrac12(kr)^{-1}\xi''^r\xi'^m{}_{[1]}\bar\gamma_{[st]}+(mk)^{-1}\xi''^t{}_{[1]}\bar\gamma_{[s4]} $$

$$ -\tfrac12\delta_{km}(tr)^{-1}\xi''^r{}_{[1]}\bar\gamma_{[s4]}-\delta_{km}(tr)^{-1}\xi'^r{}_{[1]}\bar\gamma_{[s4],4}+\tfrac14|\vec r|^{-1}\xi'^k\xi''^m{}_{[1]}\bar\gamma_{[st]} $$

$$ +\tfrac12|\vec r|^{-1}\xi''^k\xi'^m{}_{[1]}\bar\gamma_{[st]}-\tfrac12\delta_{km}|\vec r|^{-1}\xi''^t{}_{[1]}\bar\gamma_{[s4]}-\delta_{km}|\vec r|^{-1}\xi'^t{}_{[1]}\bar\gamma_{[s4],4} $$

$$ +\tfrac32(mkr)^{-2}\xi'^r{}_{[1]}\bar\gamma_{[st]}-\tfrac12(m)^{-2}\xi'^k{}_{[1]}\bar\gamma_{[st]}+\delta_{km}(t)^{-2}{}_{[1]}\bar\gamma_{[s4]} $$

$$ +3(mkr)^{-2}\xi'^r\xi'^t{}_{[1]}\bar\gamma_{[s4]}-\tfrac32\delta_{km}(trl)^{-2}\xi'^r\xi'^l{}_{[1]}\bar\gamma_{[s4]}-(k)^{-2}\xi'^t\xi'^m{}_{[1]}\bar\gamma_{[s4]} $$

$$ +\tfrac12\delta_{km}(t)^{-2}\xi'^r\xi'^r{}_{[1]}\bar\gamma_{[s4]}]+({}^{(p)}e^{M}/c^3){}^{(p)}[\tfrac12(s)^0{}_{[1]}\bar\gamma_{[m4],s4} $$

$$ +\tfrac12(s)^0{}_{[1]}\bar\gamma_{[s4],m4}+\tfrac12(msr)^0\xi'''^r{}_{[1]}\bar\gamma_{[s4]}+\tfrac12(m)^0\xi'''^s{}_{[1]}\bar\gamma_{[s4]} $$

$$ +\tfrac12(msr)^0\xi''^r{}_{[1]}\bar\gamma_{[s4],4}+\tfrac12(s)^0\xi''^m{}_{[1]}\bar\gamma_{[s4],4}+\tfrac12(s)^0\xi''^s{}_{[1]}\bar\gamma_{[m4],4} $$

$$ -\tfrac12(msr)^0\xi''^t{}_{[1]}\bar\gamma_{[st],r}-(s)^0\xi''^r{}_{[1]}\bar\gamma_{[sm],r}+\tfrac32(msr)^0\xi''^r\xi''^t{}_{[1]}\bar\gamma_{[st]} $$

$$ +\tfrac38(srl)^0\xi''^r\xi''^l{}_{[1]}\bar\gamma_{[sm]}-\tfrac32(s)^0\xi''^t\xi''^m{}_{[1]}\bar\gamma_{[st]}-\tfrac38(s)^0\xi''^r\xi''^r{}_{[1]}\bar\gamma_{[sm]} $$

$$ +\tfrac34(r)^0\xi''^r\xi''^s{}_{[1]}\bar\gamma_{[sm]}-\tfrac32(msrl)^{-1}\xi'^r{}_{[1]}\bar\gamma_{[s4],l}+\tfrac12(sr)^{-1}\xi'^m{}_{[1]}\gamma_{[s4],r} $$

$$ -\tfrac12|\vec r|^{-1}\xi'^r{}_{[1]}\bar\gamma_{[m4],r}-\tfrac12|\vec r|^{-1}\xi'^s{}_{[1]}\bar\gamma_{[s4],m}+\tfrac92(msrl)^{-1}\xi'^r\xi''^l{}_{[1]}\bar\gamma_{[s4]} $$

$$ -\tfrac32(ms)^{-1}\xi'^r\xi''^r{}_{[1]}\bar\gamma_{[s4]}+\tfrac12(mr)^{-1}\xi'^r\xi''^s{}_{[1]}\bar\gamma_{[s4]}-\tfrac12(mr)^{-1}\xi''^r\xi'^s{}_{[1]}\bar\gamma_{[s4]} $$

$$ -(sr)^{-1}\xi''^r\xi'^m{}_{[1]}\bar\gamma_{[s4]}-\tfrac12(sr)^{-1}\xi'^r\xi''^m{}_{[1]}\bar\gamma_{[s4]}-(ms)^{-1}\xi''^t{}_{[1]}\bar\gamma_{[st]} $$

$$ -\tfrac12(ms)^{-1}\xi'^t{}_{[1]}\bar\gamma_{[st],4}-\tfrac12(sr)^{-1}\xi''^r{}_{[1]}\bar\gamma_{[sm]}-\tfrac32(sr)^{-1}\xi'^r{}_{[1]}\bar\gamma_{[sm],4} $$

$$ -\tfrac12|\vec r|^{-1}\xi'^s\xi''^m{}_{[1]}\bar\gamma_{[s4]}-|\vec r|^{-1}\xi''^s\xi'^m{}_{[1]}\bar\gamma_{[s4]}-\tfrac12|\vec r|^{-1}\xi''^s{}_{[1]}\bar\gamma_{[sm]} $$

$$- 3(msr)^{-2}\xi'^{r}\,{}_{[1]}\overline{\gamma}_{[s4]} + (m)^{-2}\xi'^{s}\,{}_{[1]}\overline{\gamma}_{[s4]} + (s)^{-2}\,{}_{[1]}\overline{\gamma}_{[sm]}$$

$$- 3(msr)^{-2}\xi'^{r}\,\xi'^{t}\,{}_{[1]}\overline{\gamma}_{[st]} - \tfrac{3}{2}(srl)^{-2}\xi'^{r}\,\xi'^{l}\,{}_{[1]}\overline{\gamma}_{[sm]}$$

$$+(s)^{-2}\xi'^{t}\,\xi'^{m}\,{}_{[1]}\overline{\gamma}_{[st]} + \tfrac{1}{2}(s)^{-2}\xi'^{r}\,\xi'^{r}\,{}_{[1]}\overline{\gamma}_{[sm]}\Big] + {}^{(p)}_{[2]}\gamma^{+}_{(0m)} + O(\kappa^3), \qquad \text{(A1b)}$$

$${}_{[2]}\gamma^{I}_{(mn)} = ({}^{(p)}e^{E}/c^2)\epsilon_{stk}\,{}^{(p)}\Big[-\tfrac{1}{4}(mnkrl)^0\,{}_{[1]}\overline{\gamma}_{[st],rl}$$

$$- \tfrac{1}{24}(mnk)^0\nabla^2\,{}_{[1]}\overline{\gamma}_{[st]} + \tfrac{1}{6}(mnr)^0\,{}_{[1]}\overline{\gamma}_{[st],kr} - \tfrac{1}{12}(mkr)^0\,{}_{[1]}\overline{\gamma}_{[st],rn}$$

$$- \tfrac{1}{12}(nkr)^0\,{}_{[1]}\overline{\gamma}_{[st],rm} + \tfrac{1}{12}\delta_{km}(nrl)^0\,{}_{[1]}\overline{\gamma}_{[st],rl} + \tfrac{1}{12}\delta_{kn}(mrl)^0\,{}_{[1]}\overline{\gamma}_{[st],rl}$$

$$- \tfrac{1}{6}\delta_{mn}(krl)^0\,{}_{[1]}\overline{\gamma}_{[st],rl} + \tfrac{1}{3}(m)^0\,{}_{[1]}\overline{\gamma}_{[st],kn} + \tfrac{1}{3}(n)^0\,{}_{[1]}\overline{\gamma}_{[st],km}$$

$$- \tfrac{2}{3}(k)^0\,{}_{[1]}\overline{\gamma}_{[st],mn} + \tfrac{1}{24}\delta_{km}(n)^0\nabla^2\,{}_{[1]}\overline{\gamma}_{[st]} + \tfrac{1}{24}\delta_{kn}(m)^0\nabla^2\,{}_{[1]}\overline{\gamma}_{[st]}$$

$$+ \tfrac{1}{24}\delta_{mn}(k)^0\nabla^2\,{}_{[1]}\gamma_{[st]} + \tfrac{1}{12}\delta_{km}(r)^0\,{}_{[1]}\gamma_{[st],rn} + \tfrac{1}{12}\delta_{kn}(r)^0\,{}_{[1]}\gamma_{[st],rm} - \tfrac{1}{6}\delta_{mn}(r)^0\,{}_{[1]}\gamma_{[st],kr}$$

$$+\tfrac{3}{8}(mnkrl)^0\xi''^{l}\,{}_{[1]}\overline{\gamma}_{[st],r} + \tfrac{1}{8}(mnk)^0\xi''^{r}\,{}_{[1]}\overline{\gamma}_{[st],r}$$

$$+\tfrac{1}{8}(mnr)^0\xi''^{k}\,{}_{[1]}\overline{\gamma}_{[st],r} - \tfrac{1}{8}(mnr)^0\xi''^{r}\,{}_{[1]}\overline{\gamma}_{[st],k} - \tfrac{1}{8}(mkr)^0\xi''^{n}\,{}_{[1]}\overline{\gamma}_{[st],r}$$

$$- \tfrac{1}{8}(nkr)^0\xi''^{m}\,{}_{[1]}\overline{\gamma}_{[st],r} + \tfrac{1}{8}(mkr)^0\xi''^{r}\,{}_{[1]}\overline{\gamma}_{[st],n} + \tfrac{1}{8}(nkr)^0\xi''^{r}\,{}_{[1]}\overline{\gamma}_{[st],m}$$

$$- \tfrac{1}{8}\delta_{km}(nrl)^0\xi''^{l}\,{}_{[1]}\overline{\gamma}_{[st],r} - \tfrac{1}{8}\delta_{kn}(mrl)^0\xi''^{l}\,{}_{[1]}\overline{\gamma}_{[st],r}$$

$$+\tfrac{1}{8}\delta_{mn}(krl)^0\xi''^{l}\,{}_{[1]}\overline{\gamma}_{[st],r} - \tfrac{1}{8}(m)^0\xi''^{n}\,{}_{[1]}\overline{\gamma}_{[st],k} - \tfrac{1}{8}(n)^0\xi''^{m}\,{}_{[1]}\overline{\gamma}_{[st],k}$$

$$+ \tfrac{1}{8}(m)^0\xi''^{k}\,{}_{[1]}\overline{\gamma}_{[st],n} + \tfrac{1}{8}(n)^0\xi''^{k}\,{}_{[1]}\overline{\gamma}_{[st],m} - \tfrac{1}{8}(k)^0\xi''^{m}\,{}_{[1]}\overline{\gamma}_{[st],n}$$

$$- \tfrac{1}{8}(k)^0\xi''^{n}\,{}_{[1]}\overline{\gamma}_{[st],m} - \tfrac{1}{8}\delta_{km}(n)^0\xi''^{r}\,{}_{[1]}\overline{\gamma}_{[st],r} - \tfrac{1}{8}\delta_{kn}(m)^0\xi''^{r}\,{}_{[1]}\overline{\gamma}_{[st],r}$$

$$+ \tfrac{1}{8}\delta_{mn}(k)^0\xi''^{r}\,{}_{[1]}\overline{\gamma}_{[st],r} + \tfrac{1}{8}\delta_{km}(r)^0\xi''^{n}\,{}_{[1]}\overline{\gamma}_{[st],r} + \tfrac{1}{8}\delta_{kn}(r)^0\xi''^{m}\,{}_{[1]}\overline{\gamma}_{[st],r} + \tfrac{1}{8}\delta_{mn}(r)^0\xi''^{k}\,{}_{[1]}\overline{\gamma}_{[st],r}$$

$$+ \tfrac{1}{8}\delta_{km}(r)^0\xi''^{r}\,{}_{[1]}\overline{\gamma}_{[st],n} + \tfrac{1}{8}\delta_{kn}(r)^0\xi''^{r}\,{}_{[1]}\overline{\gamma}_{[st],m}$$

$$- \tfrac{1}{8}\delta_{mn}(r)^0\xi''^{r}\,{}_{[1]}\overline{\gamma}_{[st],k} + \tfrac{1}{8}(mnk)^0\,{}_{[1]}\overline{\gamma}_{[st],44} - \tfrac{1}{8}\delta_{km}(n)^0\,{}_{[1]}\overline{\gamma}_{[st],44}$$

$$- \tfrac{1}{8}\delta_{kn}(m)^0\,{}_{[1]}\overline{\gamma}_{[st],44} + \tfrac{1}{8}\delta_{mn}(k)^0\,{}_{[1]}\overline{\gamma}_{[st],44} - \tfrac{9}{16}(mnkrl)^0\xi''^{r}\,\xi''^{l}\,{}_{[1]}\overline{\gamma}_{[st]}$$

$$+ \tfrac{3}{16}(mnk)^0\xi''^{r}\,\xi''^{r}\,{}_{[1]}\overline{\gamma}_{[st]} - \tfrac{3}{8}(mnr)^0\xi''^{r}\,\xi''^{k}\,{}_{[1]}\overline{\gamma}_{[st]} + \tfrac{3}{8}(mkr)^0\xi''^{r}\,\xi''^{n}\,{}_{[1]}\overline{\gamma}_{[st]} + \tfrac{3}{8}(nkr)^0\xi''^{r}\,\xi''^{m}\,{}_{[1]}\overline{\gamma}_{[st]}$$

$$+ \tfrac{3}{16}\delta_{km}(nrl)^0\xi''^{r}\,\xi''^{l}\,{}_{[1]}\overline{\gamma}_{[st]} + \tfrac{3}{16}\delta_{kn}(mrl)^0\xi''^{r}\,\xi''^{l}\,{}_{[1]}\overline{\gamma}_{[st]}$$

$$- \tfrac{3}{16}\delta_{mn}(krl)^0\xi''^{r}\,\xi''^{l}\,{}_{[1]}\overline{\gamma}_{[st]} + \tfrac{3}{8}(m)^0\xi''^{k}\,\xi''^{n}\,{}_{[1]}\overline{\gamma}_{[st]}$$

$$+ \tfrac{3}{8}(n)^0\xi''^{k}\,\xi''^{m}\,{}_{[1]}\overline{\gamma}_{[st]} - \tfrac{3}{8}(k)^0\xi''^{m}\xi''^{n}\,{}_{[1]}\overline{\gamma}_{[st]} - \tfrac{3}{16}\delta_{km}(n)^0\xi''^{r}\,\xi''^{r}\,{}_{[1]}\overline{\gamma}_{[st]}$$

$$- \tfrac{3}{16}\delta_{kn}(m)^0\xi''^{r}\,\xi''^{r}\,{}_{[1]}\overline{\gamma}_{[st]} + \tfrac{3}{16}\delta_{mn}(k)^0\xi''^{r}\,\xi''^{r}\,{}_{[1]}\overline{\gamma}_{[st]} + \tfrac{3}{8}\delta_{km}(r)^0\xi''^{r}\,\xi''^{n}\,{}_{[1]}\overline{\gamma}_{[st]} + \tfrac{3}{8}\delta_{kn}(r)^0\xi''^{r}\,\xi''^{m}\,{}_{[1]}\overline{\gamma}_{[st]}$$

$$- \tfrac{3}{8}\delta_{mn}(r)^0\xi''^{r}\,\xi''^{k}\,{}_{[1]}\overline{\gamma}_{[st]} + \tfrac{1}{2}(mnk)^0\xi'''^{t}\,{}_{[1]}\overline{\gamma}_{[s4]} - \tfrac{1}{2}\delta_{km}(n)^0\xi'''^{t}\,{}_{[1]}\overline{\gamma}_{[s4]}$$

$$- \tfrac{1}{2}\delta_{kn}(m)^0\xi'''^{t}\,{}_{[1]}\overline{\gamma}_{[s4]} + \tfrac{1}{2}\delta_{mn}(k)^0\xi'''^{t}\,{}_{[1]}\overline{\gamma}_{[s4]}$$

$$+ \tfrac{1}{2}(mnk)^0\xi''^{t}\,{}_{[1]}\overline{\gamma}_{[s4],4} - \tfrac{1}{2}\delta_{km}(n)^0\xi''^{t}\,{}_{[1]}\overline{\gamma}_{[s4],4} - \tfrac{1}{2}\delta_{kn}(m)^0\xi''^{t}\,{}_{[1]}\overline{\gamma}_{[s4],4}$$

$$+ \tfrac{1}{2}\delta_{mn}(k)^0\xi''^{t}\,{}_{[1]}\overline{\gamma}_{[s4],4} + \tfrac{1}{2}\delta_{km}(t)^0\xi'''^{n}\,{}_{[1]}\overline{\gamma}_{[s4]} + \tfrac{1}{2}\delta_{kn}(t)^0\xi'''^{m}\,{}_{[1]}\overline{\gamma}_{[s4]} + \delta_{km}(t)^0\xi''^{n}\,{}_{[1]}\overline{\gamma}_{[s4],4}$$

$$+\delta_{kn}(t)^0\xi''^m{}_{[1]}\bar{\gamma}_{[s4],4}-\tfrac{3}{4}(mnkr)^{-1}{}_{[1]}\bar{\gamma}_{[st],r}+\tfrac{1}{4}(mn)^{-1}{}_{[1]}\bar{\gamma}_{[st],k}-\tfrac{1}{4}(mk)^{-1}{}_{[1]}\bar{\gamma}_{[st],n}-\tfrac{1}{4}(nk)^{-1}{}_{[1]}\bar{\gamma}_{[st],m}$$

$$+\tfrac{1}{4}\delta_{km}(nr)^{-1}{}_{[1]}\bar{\gamma}_{[st],r}+\tfrac{1}{4}\delta_{kn}(mr)^{-1}{}_{[1]}\bar{\gamma}_{[st],r}-\tfrac{1}{4}\delta_{mn}(kr)^{-1}{}_{[1]}\bar{\gamma}_{[st],r}-\tfrac{1}{4}\delta_{km}|\vec{r}|^{-1}{}_{[1]}\bar{\gamma}_{[st],n}$$

$$-\tfrac{1}{4}\delta_{kn}|\vec{r}|^{-1}{}_{[1]}\bar{\gamma}_{[st],m}+\tfrac{1}{4}\delta_{mn}|\vec{r}|^{-1}{}_{[1]}\bar{\gamma}_{[st],k}+\tfrac{3}{4}(mnkr)^{-1}\xi''^r{}_{[1]}\bar{\gamma}_{[st]}+\tfrac{1}{4}(mn)^{-1}\xi''^k{}_{[1]}\bar{\gamma}_{[st]}$$

$$-\tfrac{1}{4}(mk)^{-1}\xi''^n{}_{[1]}\bar{\gamma}_{[st]}-\tfrac{1}{4}(nk)^{-1}\xi''^m{}_{[1]}\bar{\gamma}_{[st]}-\tfrac{1}{4}\delta_{km}(nr)^{-1}\xi''^r{}_{[1]}\bar{\gamma}_{[st]}$$

$$-\tfrac{1}{4}\delta_{kn}(mr)^{-1}\xi''^r{}_{[1]}\bar{\gamma}_{[st]}+\tfrac{1}{4}\delta_{mn}(kr)^{-1}\xi''^r{}_{[1]}\bar{\gamma}_{[st]}+\tfrac{3}{4}(mnkr)^{-1}\xi'^r{}_{[1]}\bar{\gamma}_{[st],4}+\tfrac{1}{4}(mn)^{-1}\xi'^k{}_{[1]}\bar{\gamma}_{[st],4}$$

$$-\tfrac{1}{2}(mk)^{-1}\xi'^n{}_{[1]}\bar{\gamma}_{[st],4}-\tfrac{1}{2}(nk)^{-1}\xi'^m{}_{[1]}\bar{\gamma}_{[st],4}-\tfrac{1}{4}\delta_{km}(nr)^{-1}\xi'^r{}_{[1]}\bar{\gamma}_{[st],4}-\tfrac{1}{4}\delta_{kn}(mr)^{-1}\xi'^r{}_{[1]}\bar{\gamma}_{[st],4}$$

$$+\tfrac{1}{4}\delta_{mn}(kr)^{-1}\xi'^r{}_{[1]}\bar{\gamma}_{[st],4}-\tfrac{3}{2}(mnkr)^{-1}\xi'^t{}_{[1]}\bar{\gamma}_{[s4],r}+\tfrac{1}{2}(mn)^{-1}\xi'^k{}_{[1]}\bar{\gamma}_{[s4],t}+\delta_{km}(nr)^{-1}\xi'^t{}_{[1]}\bar{\gamma}_{[s4],r}$$

$$+\delta_{kn}(mr)^{-1}\xi'^t{}_{[1]}\bar{\gamma}_{[s4],r}-\tfrac{1}{2}\delta_{km}(nr)^{-1}\xi'^r{}_{[1]}\bar{\gamma}_{[s4],t}-\tfrac{1}{2}\delta_{kn}(mr)^{-1}\xi'^r{}_{[1]}\bar{\gamma}_{[s4],t}-\tfrac{1}{2}\delta_{km}(tr)^{-1}\xi'^n{}_{[1]}\bar{\gamma}_{[s4],r}$$

$$-\tfrac{1}{2}\delta_{kn}(tr)^{-1}\xi'^m{}_{[1]}\bar{\gamma}_{[s4],r}+\tfrac{1}{2}\delta_{km}(tr)^{-1}\xi'^r{}_{[1]}\bar{\gamma}_{[s4],n}+\tfrac{1}{2}\delta_{kn}(tr)^{-1}\xi'^r{}_{[1]}\bar{\gamma}_{[s4],m}-\tfrac{3}{2}\delta_{mn}(kr)^{-1}\xi'^t{}_{[1]}\bar{\gamma}_{[s4],r}$$

$$+\delta_{mn}(kr)^{-1}\xi'^r{}_{[1]}\bar{\gamma}_{[s4],t}-\tfrac{1}{4}\delta_{km}|\vec{r}|^{-1}\xi''^n{}_{[1]}\bar{\gamma}_{[st]}-\tfrac{1}{4}\delta_{kn}|\vec{r}|^{-1}\xi''^m{}_{[1]}\bar{\gamma}_{[st]}+\tfrac{1}{4}\delta_{mn}|\vec{r}|^{-1}\xi''^k{}_{[1]}\bar{\gamma}_{[st]}$$

$$-\tfrac{1}{2}\delta_{km}|\vec{r}|^{-1}\xi'^n{}_{[1]}\bar{\gamma}_{[st],4}-\tfrac{1}{2}\delta_{kn}|\vec{r}|^{-1}\xi'^m{}_{[1]}\bar{\gamma}_{[st],4}+\tfrac{1}{4}\delta_{mn}|\vec{r}|^{-1}\xi'^k{}_{[1]}\bar{\gamma}_{[st],4}+\delta_{km}|\vec{r}|^{-1}\xi'^n{}_{[1]}\bar{\gamma}_{[s4],t}$$

$$+\delta_{kn}|\vec{r}|^{-1}\xi'^m{}_{[1]}\bar{\gamma}_{[s4],t}-\tfrac{3}{2}\delta_{mn}|\vec{r}|^{-1}\xi'^k{}_{[1]}\bar{\gamma}_{[s4],t}-\delta_{km}|\vec{r}|^{-1}\xi'^t{}_{[1]}\bar{\gamma}_{[s4],n}-\delta_{kn}|\vec{r}|^{-1}\xi'^t{}_{[1]}\bar{\gamma}_{[s4],m}$$

$$+3(mnkr)^{-1}\xi'^r\,\xi''^t{}_{[1]}\bar{\gamma}_{[s4]}+\tfrac{3}{2}(mnkr)^{-1}\xi''^r\,\xi'^t{}_{[1]}\bar{\gamma}_{[s4]}+\tfrac{1}{2}(mn)^{-1}\xi'^t\,\xi''^k{}_{[1]}\bar{\gamma}_{[s4]}-(mk)^{-1}\xi''^t\,\xi'^n{}_{[1]}\bar{\gamma}_{[s4]}$$

$$-(nk)^{-1}\xi''^t\,\xi'^m{}_{[1]}\bar{\gamma}_{[s4]}-\tfrac{1}{2}(mk)^{-1}\xi'^t\,\xi''^n{}_{[1]}\bar{\gamma}_{[s4]}-\tfrac{1}{2}(nk)^{-1}\xi'^t\,\xi''^m{}_{[1]}\bar{\gamma}_{[s4]}$$

$$-\delta_{km}(nr)^{-1}\xi'^r\,\xi''^t{}_{[1]}\bar{\gamma}_{[s4]}-\delta_{kn}(mr)^{-1}\xi'^r\,\xi''^t{}_{[1]}\bar{\gamma}_{[s4]}+\delta_{mn}(kr)^{-1}\xi'^r\,\xi''^t{}_{[1]}\bar{\gamma}_{[s4]}-\tfrac{1}{2}\delta_{km}(nr)^{-1}\xi''^r\,\xi'^t{}_{[1]}\bar{\gamma}_{[s4]}$$

$$-\tfrac{1}{2}\delta_{kn}(mr)^{-1}\xi''^r\,\xi'^t{}_{[1]}\bar{\gamma}_{[s4]}+\tfrac{1}{2}\delta_{mn}(kr)^{-1}\xi''^r\,\xi'^t{}_{[1]}\bar{\gamma}_{[s4]}+\delta_{km}(tr)^{-1}\xi'^r\,\xi''^n{}_{[1]}\bar{\gamma}_{[s4]}+\delta_{kn}(tr)^{-1}\xi'^r\,\xi''^m{}_{[1]}\bar{\gamma}_{[s4]}$$

$$+\tfrac{1}{2}\delta_{km}(tr)^{-1}\xi''^r\,\xi'^n{}_{[1]}\bar{\gamma}_{[s4]}+\tfrac{1}{2}\delta_{kn}(tr)^{-1}\xi''^r\,\xi'^m{}_{[1]}\bar{\gamma}_{[s4]}-\tfrac{1}{2}\delta_{km}|\vec{r}|^{-1}\xi'^t\,\xi''^n{}_{[1]}\bar{\gamma}_{[s4]}-\tfrac{1}{2}\delta_{kn}|\vec{r}|^{-1}\xi'^t\,\xi''^m{}_{[1]}\bar{\gamma}_{[s4]}$$

$$+\tfrac{1}{2}\delta_{km}|\vec{r}|^{-1}\xi''^t\,\xi'^n{}_{[1]}\bar{\gamma}_{[s4]}+\tfrac{1}{2}\delta_{kn}|\vec{r}|^{-1}\xi''^t\,\xi'^m{}_{[1]}\bar{\gamma}_{[s4]}+\tfrac{1}{2}\delta_{mn}|\vec{r}|^{-1}\xi'^t\,\xi''^k{}_{[1]}\bar{\gamma}_{[s4]}-\tfrac{3}{2}(mnk)^{-2}{}_{[1]}\bar{\gamma}_{[st]}$$

$$+\tfrac{1}{2}\delta_{km}(n)^{-2}{}_{[1]}\bar{\gamma}_{[st]}+\tfrac{1}{2}\delta_{kn}(m)^{-2}{}_{[1]}\bar{\gamma}_{[st]}-\tfrac{1}{2}\delta_{mn}(k)^{-2}{}_{[1]}\bar{\gamma}_{[st]}$$

$$+\tfrac{15}{4}(mnkrl)^{-2}\xi'^r\,\xi'^l{}_{[1]}\bar{\gamma}_{[st]}-\tfrac{3}{4}(mnk)^{-2}\xi'^r\,\xi'^r{}_{[1]}\bar{\gamma}_{[st]}-\tfrac{3}{2}(mkr)^{-2}\xi'^r\,\xi'^n{}_{[1]}\bar{\gamma}_{[st]}-\tfrac{3}{2}(nkr)^{-2}\xi'^r\,\xi'^m{}_{[1]}\bar{\gamma}_{[st]}$$

$$-\tfrac{3}{4}\delta_{kn}(nrl)^{-2}\xi'^r\,\xi'^l{}_{[1]}\bar{\gamma}_{[st]}-\tfrac{3}{4}\delta_{kn}(mrl)^{-2}\xi'^r\,\xi'^l{}_{[1]}\bar{\gamma}_{[st]}+\tfrac{3}{4}\delta_{mn}(krl)^{-2}\xi'^r\,\xi'^l{}_{[1]}\bar{\gamma}_{[st]}+\tfrac{1}{2}(k)^{-2}\xi'^m\xi'^n{}_{[1]}\bar{\gamma}_{[st]}$$

$$+\tfrac{1}{4}\delta_{km}(n)^{-2}\xi'^r\,\xi'^r{}_{[1]}\bar{\gamma}_{[st]}+\tfrac{1}{4}\delta_{kn}(m)^{-2}\xi'^r\,\xi'^r{}_{[1]}\bar{\gamma}_{[st]}-\tfrac{1}{4}\delta_{mn}(k)^{-2}\xi'^r\,\xi'^r{}_{[1]}\bar{\gamma}_{[st]}-3(mnk)^{-2}\xi'^t{}_{[1]}\bar{\gamma}_{[s4]}$$

$$+\delta_{km}(n)^{-2}\xi'^t{}_{[1]}\bar{\gamma}_{[s4]}+\delta_{kn}(m)^{-2}\xi'^t{}_{[1]}\bar{\gamma}_{[s4]}-\delta_{mn}(k)^{-2}\xi'^t{}_{[1]}\bar{\gamma}_{[s4]}-\delta_{km}(t)^{-2}\xi'^n{}_{[1]}\bar{\gamma}_{[s4]}-\delta_{kn}(t)^{-2}\xi'^m{}_{[1]}\bar{\gamma}_{[s4]}]$$

$$+({}^{(p)}e^D/c^2l^2)\epsilon_{stk}{}^{(p)}[-\tfrac{1}{8}(mnk)^0{}_{[1]}\bar{\gamma}_{[st]}-\tfrac{1}{8}\delta_{km}(n)^0{}_{[1]}\bar{\gamma}_{[st]}-\tfrac{1}{8}\delta_{kn}(m)^0{}_{[1]}\bar{\gamma}_{[st]}-\tfrac{1}{8}\delta_{mn}(k)^0{}_{[1]}\bar{\gamma}_{[st]}]$$

$$+({}^{(p)}e^M/c^3)^{(p)}[\tfrac{1}{2}(mnsrl)^0{}_{[1]}\bar{\gamma}_{[s4],rl}+\tfrac{1}{12}(mns)^0\nabla^2{}_{[1]}\bar{\gamma}_{[s4]}$$

$$+\tfrac{1}{12}(msr)^0{}_{[1]}\bar{\gamma}_{[n4],sr}+\tfrac{1}{12}(nsr)^0{}_{[1]}\bar{\gamma}_{[m4],sr}-\tfrac{1}{12}(msr)^0{}_{[1]}\bar{\gamma}_{[s4],nr}-\tfrac{1}{12}(nsr)^0{}_{[1]}\bar{\gamma}_{[s4],mr}$$

$$+\tfrac{1}{3}\delta_{mn}(srl)^0{}_{[1]}\bar{\gamma}_{[s4],rl}+\tfrac{1}{6}(m)^0\nabla^2{}_{[1]}\bar{\gamma}_{[n4]}+\tfrac{1}{6}(n)^0\nabla^2{}_{[1]}\bar{\gamma}_{[m4]}-\tfrac{1}{12}\delta_{mn}(s)^0\nabla^2{}_{[1]}\bar{\gamma}_{[s4]}+\tfrac{7}{12}(s)^0{}_{[1]}\bar{\gamma}_{[m4],ns}$$

$$+\tfrac{7}{12}(s)^0\,{}_{[1]}\bar{\gamma}_{[n4],ms}-\tfrac{1}{6}(s)^0\,{}_{[1]}\bar{\gamma}_{[s4],mn}-\tfrac{3}{4}(mnsrl)^0\xi''^r\,{}_{[1]}\bar{\gamma}_{[s4],l}$$

$$-\tfrac{1}{4}(mns)^0\xi''^r\,{}_{[1]}\bar{\gamma}_{[s4],r}-\tfrac{1}{4}(mnr)^0\xi''^s\,{}_{[1]}\bar{\gamma}_{[s4],r}+\tfrac{1}{4}(msr)^0\xi''^n\,{}_{[1]}\bar{\gamma}_{[s4],r}$$

$$+\tfrac{1}{4}(nsr)^0\xi''^m\,{}_{[1]}\bar{\gamma}_{[s4],r}-\tfrac{1}{4}\delta_{mn}(rst)^0\xi''^r\,{}_{[1]}\bar{\gamma}_{[s4],t}-\tfrac{1}{4}\delta_{mn}(s)^0\xi''^r\,{}_{[1]}\bar{\gamma}_{[s4],r}-\tfrac{1}{4}\delta_{mn}(r)^0\xi''^s\,{}_{[1]}\bar{\gamma}_{[s4],r}$$

$$-\tfrac{1}{4}(r)^0\xi''^r\,{}_{[1]}\bar{\gamma}_{[m4],n}-\tfrac{1}{4}(r)^0\xi''^r\,{}_{[1]}\bar{\gamma}_{[n4],m}-\tfrac{1}{4}(mns)^0\,{}_{[1]}\bar{\gamma}_{[s4],44}$$

$$-\tfrac{1}{4}\delta_{mn}(s)^0\,{}_{[1]}\bar{\gamma}_{[s4],44}+\tfrac{1}{4}(mrs)^0\,{}_{[1]}\bar{\gamma}_{[sn],4r}+\tfrac{1}{4}(nrs)^0\,{}_{[1]}\bar{\gamma}_{[sm],4r}$$

$$+\tfrac{3}{4}(s)^0\,{}_{[1]}\bar{\gamma}_{[sm],4n}+\tfrac{3}{4}(s)^0\,{}_{[1]}\bar{\gamma}_{[sn],4m}+\tfrac{9}{8}(mnsrl)^0\xi''^r\xi''^l\,{}_{[1]}\bar{\gamma}_{[s4]}$$

$$-\tfrac{3}{8}(mns)^0\xi''^r\xi''^r\,{}_{[1]}\bar{\gamma}_{[s4]}+\tfrac{3}{4}(mnr)^0\xi''^r\xi''^s\,{}_{[1]}\bar{\gamma}_{[s4]}-\tfrac{3}{4}(msr)^0\xi''^r\xi''^n\,{}_{[1]}\bar{\gamma}_{[s4]}-\tfrac{3}{4}(nsr)^0\xi''^r\xi''^m\,{}_{[1]}\bar{\gamma}_{[s4]}$$

$$-\tfrac{3}{8}(mrl)^0\xi''^r\xi''^l\,{}_{[1]}\bar{\gamma}_{[n4]}-\tfrac{3}{8}(nrl)^0\xi''^r\xi''^l\,{}_{[1]}\bar{\gamma}_{[m4]}+\tfrac{3}{8}\delta_{mn}(srl)^0\xi''^r\xi''^l\,{}_{[1]}\bar{\gamma}_{[s4]}-\tfrac{3}{4}(m)^0\xi''^s\xi''^n\,{}_{[1]}\bar{\gamma}_{[s4]}$$

$$-\tfrac{3}{4}(n)^0\xi''^s\xi''^m\,{}_{[1]}\bar{\gamma}_{[s4]}+\tfrac{3}{4}(s)^0\xi''^m\xi''^n\,{}_{[1]}\bar{\gamma}_{[s4]}+\tfrac{3}{8}(m)^0\xi''^r\xi''^r\,{}_{[1]}\bar{\gamma}_{[n4]}+\tfrac{3}{8}(n)^0\xi''^r\xi''^r\,{}_{[1]}\bar{\gamma}_{[m4]}$$

$$-\tfrac{3}{8}\delta_{mn}(s)^0\xi''^r\xi''^r\,{}_{[1]}\bar{\gamma}_{[s4]}-\tfrac{3}{4}(r)^0\xi''^r\xi''^m\,{}_{[1]}\bar{\gamma}_{[n4]}-\tfrac{3}{4}(r)^0\xi''^r\xi''^n\,{}_{[1]}\bar{\gamma}_{[m4]}+\tfrac{3}{4}\delta_{mn}(r)^0\xi''^r\xi''^s\,{}_{[1]}\bar{\gamma}_{[s4]}$$

$$-\tfrac{1}{2}(mns)^0\xi'''^t\,{}_{[1]}\bar{\gamma}_{[st]}-\tfrac{1}{2}(m)^0\xi'''^s\,{}_{[1]}\bar{\gamma}_{[sn]}-\tfrac{1}{2}(n)^0\xi'''^s\,{}_{[1]}\bar{\gamma}_{[sm]}$$

$$-\tfrac{1}{2}\delta_{mn}(s)^0\xi'''^t\,{}_{[1]}\bar{\gamma}_{[st]}+\tfrac{1}{2}(s)^0\xi'''^m\,{}_{[1]}\bar{\gamma}_{[sn]}+\tfrac{1}{2}(s)^0\xi'''^n\,{}_{[1]}\bar{\gamma}_{[sm]}-\tfrac{1}{2}(mns)^0\xi''^t\,{}_{[1]}\bar{\gamma}_{[st],4}$$

$$-\tfrac{1}{4}(mrs)^0\xi''^r\,{}_{[1]}\bar{\gamma}_{[sn],4}-\tfrac{1}{4}(nrs)^0\xi''^r\,{}_{[1]}\bar{\gamma}_{[sm],4}-\tfrac{3}{4}(m)^0\xi''^s\,{}_{[1]}\bar{\gamma}_{[sn],4}-\tfrac{3}{4}(n)^0\xi''^s\,{}_{[1]}\bar{\gamma}_{[sm],4}$$

$$-\tfrac{1}{2}\delta_{mn}(s)^0\xi''^t\,{}_{[1]}\bar{\gamma}_{[st],4}+\tfrac{5}{4}(s)^0\xi''^m\,{}_{[1]}\bar{\gamma}_{[sn],4}+\tfrac{5}{4}(s)^0\xi''^n\,{}_{[1]}\bar{\gamma}_{[sm],4}$$

$$+\tfrac{3}{2}(mnsr)^{-1}\,{}_{[1]}\bar{\gamma}_{[s4],r}+\tfrac{1}{2}\delta_{mn}(sr)^{-1}\,{}_{[1]}\bar{\gamma}_{[s4],r}+\tfrac{1}{2}|\vec{\mathbf{r}}|^{-1}\,{}_{[1]}\bar{\gamma}_{[m4],n}$$

$$+\tfrac{1}{2}|\vec{\mathbf{r}}|^{-1}\,{}_{[1]}\bar{\gamma}_{[n4],m}-\tfrac{3}{2}(mnsr)^{-1}\xi''^r\,{}_{[1]}\bar{\gamma}_{[s4]}-\tfrac{1}{2}(mn)^{-1}\xi''^s\,{}_{[1]}\bar{\gamma}_{[s4]}$$

$$+\tfrac{1}{2}(ms)^{-1}\xi''^n\,{}_{[1]}\bar{\gamma}_{[s4]}+\tfrac{1}{2}(ns)^{-1}\xi''^m\,{}_{[1]}\bar{\gamma}_{[s4]}+\tfrac{1}{2}(mr)^{-1}\xi''^r\,{}_{[1]}\bar{\gamma}_{[n4]}$$

$$+\tfrac{1}{2}(nr)^{-1}\xi''^r\,{}_{[1]}\bar{\gamma}_{[m4]}-\tfrac{1}{2}\delta_{mn}(sr)^{-1}\xi''^r\,{}_{[1]}\bar{\gamma}_{[s4]}-\tfrac{3}{2}(mnsr)^{-1}\xi'^r\,{}_{[1]}\bar{\gamma}_{[s4],4}$$

$$+\tfrac{1}{2}(ms)^{-1}\xi'^n\,{}_{[1]}\bar{\gamma}_{[s4],4}+\tfrac{1}{2}(ns)^{-1}\xi'^m\,{}_{[1]}\bar{\gamma}_{[s4],4}+\tfrac{1}{2}(mr)^{-1}\xi'^r\,{}_{[1]}\bar{\gamma}_{[n4],4}$$

$$+\tfrac{1}{2}(nr)^{-1}\xi'^r\,{}_{[1]}\bar{\gamma}_{[m4],4}-\tfrac{1}{2}\delta_{mn}(sr)^{-1}\xi'^r\,{}_{[1]}\bar{\gamma}_{[s4],4}$$

$$+\tfrac{3}{2}(mnsr)^{-1}\xi'^t\,{}_{[1]}\bar{\gamma}_{[st],r}+\tfrac{1}{2}(ms)^{-1}\xi'^t\,{}_{[1]}\bar{\gamma}_{[sn],t}+\tfrac{1}{2}(ns)^{-1}\xi'^t\,{}_{[1]}\bar{\gamma}_{[sm],t}+\tfrac{1}{2}\delta_{mn}(sr)^{-1}\xi'^t\,{}_{[1]}\bar{\gamma}_{[st],r}$$

$$+\tfrac{1}{2}(ms)^{-1}\,{}_{[1]}\bar{\gamma}_{[sn],4}+\tfrac{1}{2}(ns)^{-1}\,{}_{[1]}\bar{\gamma}_{[sm],4}+\tfrac{1}{2}|\vec{\mathbf{r}}|^{-1}\xi''^m\,{}_{[1]}\bar{\gamma}_{[n4]}$$

$$+\tfrac{1}{2}|\vec{\mathbf{r}}|^{-1}\xi''^n\,{}_{[1]}\bar{\gamma}_{[m4]}-\tfrac{1}{2}\delta_{mn}|\vec{\mathbf{r}}|^{-1}\xi''^s\,{}_{[1]}\bar{\gamma}_{[s4]}+\tfrac{1}{2}|\vec{\mathbf{r}}|^{-1}\xi'^m\,{}_{[1]}\bar{\gamma}_{[n4],4}$$

$$+\tfrac{1}{2}|\vec{\mathbf{r}}|^{-1}\xi'^n\,{}_{[1]}\bar{\gamma}_{[m4],4}-\tfrac{1}{2}|\vec{\mathbf{r}}|^{-1}\xi'^s\,{}_{[1]}\bar{\gamma}_{[sm],n}-\tfrac{1}{2}|\vec{\mathbf{r}}|^{-1}\xi'^s\,{}_{[1]}\bar{\gamma}_{[sn],m}$$

$$-3(mnsr)^{-1}\xi'^r\xi''^t\,{}_{[1]}\bar{\gamma}_{[st]}-\tfrac{3}{2}(mnsr)^{-1}\xi''^r\xi'^t\,{}_{[1]}\bar{\gamma}_{[st]}+\tfrac{1}{2}(mn)^{-1}\xi'^s\xi''^t\,{}_{[1]}\bar{\gamma}_{[st]}+\tfrac{1}{2}(ms)^{-1}\xi'^t\xi''^n\,{}_{[1]}\bar{\gamma}_{[st]}$$

$$+\tfrac{1}{2}(ns)^{-1}\xi'^t\xi''^m\,{}_{[1]}\bar{\gamma}_{[st]}+(ms)^{-1}\xi''^t\xi'^n\,{}_{[1]}\bar{\gamma}_{[st]}+(ns)^{-1}\xi''^t\xi'^m\,{}_{[1]}\bar{\gamma}_{[st]}-(mr)^{-1}\xi'^r\xi''^s\,{}_{[1]}\bar{\gamma}_{[sn]}$$

$$-(nr)^{-1}\xi'^r\xi''^s\,{}_{[1]}\bar{\gamma}_{[sm]}-\delta_{mn}(sr)^{-1}\xi'^r\xi''^t\,{}_{[1]}\bar{\gamma}_{[st]}-\tfrac{1}{2}(mr)^{-1}\xi''^r\xi'^s\,{}_{[1]}\bar{\gamma}_{[sn]}-\tfrac{1}{2}(nr)^{-1}\xi''^r\xi'^s\,{}_{[1]}\bar{\gamma}_{[sm]}$$

$$-\tfrac{1}{2}\delta_{mn}(sr)^{-1}\xi''^r\xi'^t\,{}_{[1]}\bar{\gamma}_{[st]}+(sr)^{-1}\xi'^r\xi''^m\,{}_{[1]}\bar{\gamma}_{[sn]}$$

$$+(sr)^{-1}\xi'^r\xi''^n\,{}_{[1]}\bar{\gamma}_{[sm]}+\tfrac{1}{2}(sr)^{-1}\xi''^r\xi'^m\,{}_{[1]}\bar{\gamma}_{[sn]}+\tfrac{1}{2}(sr)^{-1}\xi''^r\xi'^n\,{}_{[1]}\bar{\gamma}_{[sm]}$$

$$-\tfrac{1}{2}|\vec{r}|^{-1}\xi'^{s}\xi''^{m}\,{}_{[1]}\bar{\gamma}_{[sn]}-\tfrac{1}{2}|\vec{r}|^{-1}\xi'^{s}\xi''^{n}\,{}_{[1]}\bar{\gamma}_{[sm]}+\tfrac{1}{2}|\vec{r}|^{-1}\xi''^{s}\xi'^{m}\,{}_{[1]}\bar{\gamma}_{[sn]}+\tfrac{1}{2}|\vec{r}|^{-1}\xi''^{s}\xi'^{n}\,{}_{[1]}\bar{\gamma}_{[sm]}$$
$$+\tfrac{1}{2}\delta_{mn}|\vec{r}|^{-1}\xi'^{s}\xi''^{t}\,{}_{[1]}\bar{\gamma}_{[st]}+3(mns)^{-2}\,{}_{[1]}\bar{\gamma}_{[s4]}-(m)^{-2}\,{}_{[1]}\bar{\gamma}_{[n4]}$$
$$-(n)^{-2}\,{}_{[1]}\bar{\gamma}_{[m4]}+\delta_{mn}(s)^{-2}\,{}_{[1]}\bar{\gamma}_{[s4]}-\tfrac{15}{2}(mnsrl)^{-2}\xi'^{r}\xi'^{l}\,{}_{[1]}\bar{\gamma}_{[s4]}$$
$$+\tfrac{3}{2}(mns)^{-2}\xi'^{r}\xi'^{r}\,{}_{[1]}\bar{\gamma}_{[s4]}+3(mrs)^{-2}\xi'^{r}\xi'^{m}\,{}_{[1]}\bar{\gamma}_{[s4]}+3(nrs)^{-2}\xi'^{r}\xi'^{m}\,{}_{[1]}\bar{\gamma}_{[s4]}+\tfrac{3}{2}(mrl)^{-2}\xi'^{r}\xi'^{l}\,{}_{[1]}\bar{\gamma}_{[n4]}$$
$$+\tfrac{3}{2}(nrl)^{-2}\xi'^{r}\xi'^{l}\,{}_{[1]}\bar{\gamma}_{[m4]}-\tfrac{3}{2}\delta_{mn}(srl)^{-2}\xi'^{r}\xi'^{l}\,{}_{[1]}\bar{\gamma}_{[s4]}-\tfrac{1}{2}(m)^{-2}\xi'^{r}\xi'^{r}\,{}_{[1]}\bar{\gamma}_{[n4]}-\tfrac{1}{2}(n)^{-2}\xi'^{r}\xi'^{r}\,{}_{[1]}\bar{\gamma}_{[m4]}$$
$$+\tfrac{1}{2}\delta_{mn}(s)^{-2}\xi'^{r}\xi'^{r}\,{}_{[1]}\bar{\gamma}_{[s4]}-(s)^{-2}\xi'^{m}\xi'^{n}\,{}_{[1]}\bar{\gamma}_{[s4]}+(m)^{-2}\xi'^{s}\,{}_{[1]}\bar{\gamma}_{[sn]}+(n)^{-2}\xi'^{s}\,{}_{[1]}\bar{\gamma}_{[sm]}+\delta_{mn}(s)^{-2}\xi'^{t}\,{}_{[1]}\bar{\gamma}_{[st]}$$
$$+3(mns)^{-2}\xi'^{t}\,{}_{[1]}\bar{\gamma}_{[st]}-(s)^{-2}\xi'^{m}\,{}_{[1]}\bar{\gamma}_{[sn]}-(s)^{-2}\xi'^{n}\,{}_{[1]}\bar{\gamma}_{[sm]}\big]+{}^{(p)}_{[2]}\gamma^{+}_{(mn)}+O(\kappa^3)\,. \tag{A1c}$$

We are using the notation

$$\begin{aligned}&{}^{(p)}(s_1s_2\cdots s_i)^n=|\vec{r}|^{n\,(p)}(|\vec{r}|_{,s_1}|\vec{r}|_{,s_2}\cdots|\vec{r}|_{,s_i}),\\&{}^{(p)}\xi'^{\mu}=d({}^{(p)}\xi^{\mu})/cdt,\quad {}^{(p)}\xi''^{\mu}=d^2({}^{(p)}\xi^{\mu})/c^2dt^2,\quad {}^{(p)}\xi'''^{\mu}=d^3({}^{(p)}\xi^{\mu})/c^3dt^3,\end{aligned} \tag{A2}$$

in (A1). The field ${}^{(p)}_{[1]}\bar{\gamma}_{[\mu\nu]}$ is the nonsingular part of ${}_{[1]}\gamma_{[\mu\nu]}$ evaluated at the position of the pth particle. The field ${}^{(p)}_{[2]}\gamma^{+}_{(\mu\nu)}$ contains only those terms in ${}_{[2]}\gamma^{I}_{(\mu\nu)}$ which vanish as ${}^{(p)}|\vec{r}|\to 0$ or whose form is such that they will not contribute to ${}_{[2]}\gamma^{I}_{(\mu\nu)}{}^{,\nu}$ in a rest system of the pth particle. We are grouping these terms together into the field ${}^{(p)}_{[2]}\gamma^{+}_{(\mu\nu)}$ as a detailed knowledge of them is not needed in our analysis.

From the above solution to Eqs. (1.45) we find, in a rest system of the pth particle and keeping only those terms which become infinite as ${}^{(p)}|\vec{r}|\to 0$,

$${}_{[2]}\gamma^{I}_{(\mu\nu)}{}^{,\nu}={}^{(p)}_{[2]}C^{I}_{\mu}\,{}^{(p)}|\vec{r}|^{-1}+O(\kappa^3), \tag{A3}$$

where

$$\begin{aligned}&{}^{(p)}_{[2]}C^{I}_{0}=0,\\&{}^{(p)}_{[2]}C^{I}_{m}=-(1/c^2)\epsilon_{stm}\,{}^{(p)}[-\tfrac{1}{2}e^{E}\Box^2\,{}_{[1]}\bar{\gamma}_{[st]}-\tfrac{1}{2}(e^{D}/l^2)\,{}_{[1]}\bar{\gamma}_{[st]}]-(1/c^2)\,{}^{(p)}[(e^{M}/c)\Box^2\,{}_{[1]}\bar{\gamma}_{[0m]}]\,.\end{aligned} \tag{A4}$$

We note here that the field ${}^{(p)}_{[2]}\gamma^{+}_{(\mu\nu)}$ does not contribute to ${}^{(p)}_{[2]}C^{I}_{\mu}$. If we make use of the transformation properties of the various terms in (A3) and (A4) and compare (A3) with (5.59), we find that in an arbitrary inertial system,

$${}_{[2]}\gamma^{I}_{(\mu\nu)}{}^{,\nu}=\sum_{p=1}^{N}{}^{(p)}a_{\rm ret}\,{}^{(p)}[{}_{[2]}C^{I}_{\mu}(ru)^{-1}]_{\rm ret}+\sum_{p=1}^{N}{}^{(p)}a_{\rm adv}\,{}^{(p)}[{}_{[2]}C^{I}_{\mu}(ru)^{-1}]_{\rm adv}+O(\kappa^3), \tag{A5}$$

where

$$\begin{aligned}&{}^{(p)}_{[2]}C^{I}_{\mu}=-(1/c^2)\,{}^{(p)}[e^{E}\Box^2\,{}_{[1]}\bar{\gamma}^{*}_{[\mu\nu]}u^{\nu}+(e^{D}/l^2)\bar{\gamma}^{*}_{[\mu\nu]}u^{\nu}-(e^{M}/c)\Box^2\,{}_{[1]}\bar{\gamma}_{[\mu\nu]}u^{\nu}],\\&{}^{(p)}_{[1]}\bar{\gamma}^{*}_{[\mu\nu]}=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\,{}^{(p)}_{[1]}\bar{\gamma}^{[\rho\sigma]}\,.\end{aligned} \tag{A6}$$

The field ${}^{(p)}_{[1]}\bar{\gamma}_{[\mu\nu]}$ can be broken up as follows:

$${}^{(p)}_{[1]}\bar{\gamma}_{[\mu\nu]}={}^{(p)}_{[1]}\tilde{\gamma}_{[\mu\nu]}+{}^{(p)}_{[1]}\gamma^{\rm ext}_{[\mu\nu]}, \tag{A7}$$

where ${}^{(p)}_{[1]}\tilde{\gamma}_{[\mu\nu]}$ is that part of ${}^{(p)}_{[1]}\gamma_{[\mu\nu]}$,

$${}^{(p)}_{[1]}\gamma_{[\mu\nu]}={}^{(p)}_{[1]}\gamma^{E}_{[\mu\nu]}+{}^{(p)}_{[1]}\gamma^{M}_{[\mu\nu]}, \tag{A8}$$

which is nonsingular along the world line of the pth particle, and

$${}^{(p)}_{[1]}\gamma^{\rm ext}_{[\mu\nu]}=\sum_{p'\neq p}{}^{(p')}_{[1]}\gamma_{[\mu\nu]}\,. \tag{A9}$$

The field ${}^{(p)}_{[1]}\tilde{\gamma}_{[\mu\nu]}$ in (A7) can itself be broken up into two parts,

$${}^{(p)}_{[1]}\tilde{\gamma}_{[\mu\nu]}={}^{(p)}_{[1]}\tilde{\gamma}^{H}_{[\mu\nu]}+{}^{(p)}_{[1]}\tilde{\gamma}^{I}_{[\mu\nu]}, \tag{A10}$$

where ${}^{(p)}_{[1]}\tilde{\gamma}^{H}_{[\mu\nu]}$ is that part of ${}^{(p)}_{[1]}\gamma^{H}_{[\mu\nu]}$,

$$ {}^{(p)}_{[1]}\gamma^{H}_{[\mu\nu]} = {}^{(p)}_{[1]}\gamma^{EH}_{[\mu\nu]} + {}^{(p)}_{[1]}\gamma^{MH}_{[\mu\nu]}, \tag{A11} $$

$$ {}^{(p)}_{[1]}\gamma^{EH}_{[\mu\nu]} = \epsilon_{\mu\nu\rho\sigma}\, {}^{(p)}_{[1]}\gamma^{EH\sigma,\rho}, \qquad {}^{(p)}_{[1]}\gamma^{MH}_{[\mu\nu]} = {}^{(p)}_{[1]}\gamma^{MH}_{\nu,\mu} - {}^{(p)}_{[1]}\gamma^{MH}_{\mu,\nu}, \tag{A12} $$

which is nonsingular along the world line of the pth particle, and ${}^{(p)}_{[1]}\tilde{\gamma}^{I}_{[\mu\nu]}$ is that part of ${}^{(p)}_{[1]}\gamma^{I}_{[\mu\nu]}$,

$$ {}^{(p)}_{[1]}\gamma^{I}_{[\mu\nu]} = {}^{(p)}_{[1]}\gamma^{EI}_{[\mu\nu]} + {}^{(p)}_{[1]}\gamma^{MI}_{[\mu\nu]}, \tag{A13} $$

$$ {}^{(p)}_{[1]}\gamma^{EI}_{[\mu\nu]} = \epsilon_{\mu\nu\rho\sigma}\, {}^{(p)}_{[1]}\gamma^{EI\sigma,\rho}, \qquad {}^{(p)}_{[1]}\gamma^{MI}_{[\mu\nu]} = {}^{(p)}_{[1]}\gamma^{MI}_{\nu,\mu} - {}^{(p)}_{[1]}\gamma^{MI}_{\mu,\nu}, \tag{A14} $$

which is nonsingular along the world line of the pth particle. Note that

$$ {}^{(p)}e^{E} l^2 \Box^2\, {}^{(p)}_{[1]}\tilde{\gamma}_{[\mu\nu]} = {}^{(p)}e^{D}\, {}^{(p)}_{[1]}\tilde{\gamma}^{EH}_{[\mu\nu]} + O(\kappa^3), \tag{A15} $$

where ${}^{(p)}_{[1]}\tilde{\gamma}^{EH}_{[\mu\nu]}$ is the electric part of ${}^{(p)}_{[1]}\tilde{\gamma}^{H}_{[\mu\nu]}$. The magnetic part will be denoted by ${}^{(p)}_{[1]}\tilde{\gamma}^{MH}_{[\mu\nu]}$, so we have

$$ {}^{(p)}_{[1]}\tilde{\gamma}^{H}_{[\mu\nu]} = {}^{(p)}_{[1]}\tilde{\gamma}^{EH}_{[\mu\nu]} + {}^{(p)}_{[1]}\tilde{\gamma}^{MH}_{[\mu\nu]}. \tag{A16} $$

Note also that along the world line of the pth particle,

$$ {}^{(p)}_{[1]}\tilde{\gamma}^{EH}_{[\mu\nu]} = -\tfrac{1}{3}\, {}^{(p)}a({}^{(p)}e^{E}/c^2)\epsilon_{\mu\nu\rho\sigma}\, {}^{(p)}(\ddot{u}^{\rho}u^{\sigma} - \ddot{u}^{\sigma}u^{\rho}) + O(\kappa^2), \qquad {}^{(p)}_{[1]}\tilde{\gamma}^{MH}_{[\mu\nu]} = -\tfrac{2}{3}\, {}^{(p)}a({}^{(p)}e^{M}/c^3)\, {}^{(p)}(\ddot{u}_{\mu}u_{\nu} - \ddot{u}_{\nu}u_{\mu}) + O(\kappa^2), \tag{A17} $$

$$ {}^{(p)}_{[1]}\tilde{\gamma}^{I}_{[\mu\nu]} = O(\kappa^2), \tag{A18} $$

where

$$ {}^{(p)}a = {}^{(p)}a_{\text{adv}} - {}^{(p)}a_{\text{ret}}. \tag{A19} $$

Through the use of (A7)–(A19) we can put ${}^{(p)}_{[2]}C^{I}_{\mu}$ in a form which will prove to be more convenient than that form given in (A6). Using (A7)–(A19) in (A6) we find that

$$ {}^{(p)}_{[2]}C^{I}_{\mu} = -(1/c^2)\, {}^{(p)}[e^{E}\Box^2\, {}_{[1]}\gamma^{*\text{ext}}_{[\mu\nu]}u^{\nu} + (e^{D}/l^2)\, {}_{[1]}\gamma^{*\text{ext}}_{[\mu\nu]}u^{\nu} + \tfrac{4}{3}a(e^{E}e^{D}/c^2 l^2)(\ddot{u}_{\mu} + \dot{u}_{\rho}\dot{u}^{\rho}u_{\mu}) - (e^{M}/c)\Box^2\, {}_{[1]}\gamma^{\text{ext}}_{[\mu\nu]}u^{\nu}], \tag{A20} $$

$$ {}^{(p)}_{[1]}\gamma^{*\text{ext}}_{[\mu\nu]} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\, {}^{(p)}_{[1]}\gamma^{\text{ext}\,[\rho\sigma]}. $$

To the solution (A1) we can always add a term ${}_{[2]}\gamma^{\dagger}_{(\mu\nu)}$ of the form

$$ {}_{[2]}\gamma^{\dagger}_{(\mu\nu)} = (\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma} - \eta_{\mu\nu}\eta_{\rho\sigma})\, {}_{[2]}\psi^{\rho,\sigma}, \tag{A21} $$

where

$$ \Box^2\, {}_{[2]}\psi^{\rho} = O(\kappa^3). \tag{A22} $$

Since the field ${}_{[2]}\gamma^{\dagger}_{(\mu\nu)}$ satisfies the equations

$$ \Box^2\, {}_{[2]}\gamma^{\dagger}_{(\mu\nu)} = O(\kappa^3), \quad {}_{[2]}\gamma^{\dagger}_{(\mu\nu)}{}^{,\nu} = O(\kappa^3), \tag{A23} $$

it will not contribute to the equations of mass, motion, and spin to second order. The fact that we can always add a term of the form (A21) to the solution (A1) is related to the gauge invariance of the field equations.

APPENDIX B: ON SOLUTIONS TO THE FIELD EQUATIONS $\Box^2\, {}_{[4]}\gamma^{I}_{(\mu\nu)} = {}_{[4]}t_{\mu\nu} + O(\kappa^5)$

We shall study the solutions to Eqs. (1.63) for which ${}_{[4]}\gamma^{I}_{(\mu\nu)}{}^{,\nu}$ takes the form (I5.59). In doing this we shall be concerned only with fourth-order accuracy. If all quantities in the equations are expressed as functions of t instead of as functions of ${}^{(p)}\tau$ or x^4, we can expand both sides of Eqs. (1.63) in a power series in c^{-1} and solve the equations step by step with respect to powers of c^{-1}. We find, in the neighborhood of the pth particle,

$$ {}_{[4]}\gamma^{I}_{(00)} = ({}^{(p)}m^{G}/c^{2})^{(p)}[-\tfrac{1}{4}(rst)^{0}\,{}_{[2]}\overline{\gamma}_{(st),r} + \tfrac{7}{8}(r)^{0}\,{}_{[2]}\overline{\gamma}_{(00),r} + \tfrac{1}{8}(r)^{0}\,{}_{[2]}\overline{\gamma}_{(ss),r} - \tfrac{1}{2}(st)^{-1}\,{}_{[2]}\overline{\gamma}_{(st)} + \tfrac{3}{4}|\vec{r}|^{-1}\,{}_{[2]}\overline{\gamma}_{(00)} + \tfrac{1}{4}|\vec{r}|^{-1}\,{}_{[2]}\overline{\gamma}_{(ss)} - (rs)^{-1}\xi'^{r}\,{}_{[2]}\overline{\gamma}_{(0s)} + 2|\vec{r}|^{-1}\xi'^{s}\,{}_{[2]}\overline{\gamma}_{(0s)}] + {}^{(p)}_{[4]}\gamma^{+}_{(00)} + O(\kappa^{5}), \tag{B1a} $$

$$ {}_{[4]}\gamma^{I}_{(0m)} = ({}^{(p)}m^{G}/c^{2})^{(p)}[-\tfrac{1}{2}(s)^{0}\,{}_{[2]}\overline{\gamma}_{(4s),m} + \tfrac{1}{2}(r)^{0}\,{}_{[2]}\overline{\gamma}_{(4m),r} + \tfrac{3}{8}(m)^{0}\,{}_{[2]}\overline{\gamma}_{(44),4} - \tfrac{1}{8}(m)^{0}\,{}_{[2]}\overline{\gamma}_{(ss),4} + \tfrac{1}{2}(s)^{0}\,{}_{[2]}\overline{\gamma}_{(sm),4} + \tfrac{1}{2}(st)^{-1}\xi'^{m}\,{}_{[2]}\overline{\gamma}_{(st)} - \tfrac{3}{4}|\vec{r}|^{-1}\xi'^{m}\,{}_{[2]}\overline{\gamma}_{(44)} - \tfrac{1}{4}|\vec{r}|^{-1}\xi'^{m}\,{}_{[2]}\overline{\gamma}_{(ss)}] + {}^{(p)}_{[4]}\gamma^{+}_{(0m)} + O(\kappa^{5}), \tag{B1b} $$

$$ {}_{[4]}\gamma^{I}_{(mn)} = ({}^{(p)}m^{G}/c^{2})^{(p)}[-\tfrac{1}{8}(m)^{0}\,{}_{[2]}\overline{\gamma}_{(44),n} - \tfrac{1}{8}(n)^{0}\,{}_{[2]}\overline{\gamma}_{(44),m} + \tfrac{1}{8}\delta_{mn}(s)^{0}\,{}_{[2]}\overline{\gamma}_{(44),s} - \tfrac{1}{8}(m)^{0}\,{}_{[2]}\overline{\gamma}_{(ss),n} - \tfrac{1}{8}(n)^{0}\,{}_{[2]}\overline{\gamma}_{(ss),m} + \tfrac{1}{8}\delta_{mn}(r)^{0}\,{}_{[2]}\overline{\gamma}_{(ss),r} + \tfrac{1}{2}(m)^{0}\,{}_{[2]}\overline{\gamma}_{(4n),4} + \tfrac{1}{2}(n)^{0}\,{}_{[2]}\overline{\gamma}_{(4m),4} - \tfrac{1}{2}\delta_{mn}(s)^{0}\,{}_{[2]}\overline{\gamma}_{(4s),4}] + {}^{(p)}_{[4]}\gamma^{+}_{(mn)} + O(\kappa^{5}). \tag{B1c} $$

The notation used in this appendix is similar to that used in Appendix A. In particular, the field ${}^{(p)}_{[4]}\gamma^{+}_{(\mu\nu)}$ contains those terms which vanish as ${}^{(p)}|\vec{r}|\to 0$ or whose form is such that they cannot contribute to ${}_{[4]}\gamma^{I}_{(\mu\nu)}{}^{,\nu}$ in a rest system of the pth particle.

The field ${}^{(p)}_{[2]}\overline{\gamma}_{(\mu\nu)}$ in (B1) is the nonsingular part of ${}_{[2]}\gamma_{(\mu\nu)}$ evaluated at the position of the pth particle.

We find from (B1), in a rest system of the pth particle and keeping only those terms which become infinite as ${}^{(p)}|\vec{r}|\to 0$,

$$ {}_{[4]}\gamma_{(\mu\nu)}{}^{,\nu} = {}^{(p)}_{[4]}C^{I}_{\mu}\,{}^{(p)}|\vec{r}|^{-1} + O(\kappa^{5}), \tag{B2} $$

where

$$ \begin{aligned} {}^{(p)}_{[4]}C^{I}_{0} &= 0, \\ {}^{(p)}_{[4]}C^{I}_{s} &= -(1/c^{2})\,{}^{(p)}[m^{G}(-\tfrac{1}{4}{}_{[2]}\overline{\gamma}_{(44),s} - \tfrac{1}{4}{}_{[2]}\overline{\gamma}_{(tt),s} + {}_{[2]}\overline{\gamma}_{(4s),4})]. \end{aligned} \tag{B3} $$

If we make use of the transformation properties of the various terms in (B2) and (B3) and compare (B2) with (I5.59), we see that

$$ {}_{[4]}\gamma^{I}_{(\mu\nu)}{}^{,\nu} = \sum_{p=1}^{N} {}^{(p)}a_{\text{ret}}\,{}^{(p)}[{}_{[4]}C^{I}_{\mu}(ru)^{-1}]_{\text{ret}} + \sum_{p=1}^{N} {}^{(p)}a_{\text{adv}}\,{}^{(p)}[{}_{[4]}C^{I}_{\mu}(ru)^{-1}]_{\text{adv}} + O(\kappa^{5}), \tag{B4} $$

where

$$ {}^{(p)}_{[4]}C^{I}_{\mu} = -(1/c^{2})\,{}^{(p)}[m^{G}(\tfrac{1}{4}\eta^{\rho\sigma}\,{}_{[2]}\overline{\gamma}_{(\rho\sigma),\mu} - \tfrac{1}{2}{}_{[2]}\overline{\gamma}_{(\rho\sigma),\mu}u^{\rho}u^{\sigma} + {}_{[2]}\overline{\gamma}_{(\mu\rho),\sigma}u^{\rho}u^{\sigma} - \tfrac{1}{4}\eta^{\rho\sigma}\,{}_{[2]}\overline{\gamma}_{(\rho\sigma),\tau}u^{\tau}u_{\mu} - \tfrac{1}{2}\,{}_{[2]}\overline{\gamma}_{(\rho\sigma),\tau}u^{\rho}u^{\sigma}u^{\tau}u_{\mu})]. \tag{B5} $$

The field ${}^{(p)}_{[2]}\overline{\gamma}_{(\mu\nu)}$ can be broken up in the following manner:

$$ {}^{(p)}_{[2]}\overline{\gamma}_{(\mu\nu)} = {}^{(p)}_{[2]}\tilde{\gamma}_{(\mu\nu)} + {}^{(p)}_{[2]}\gamma^{\text{ext}}_{(\mu\nu)}, \tag{B6} $$

where ${}^{(p)}_{[2]}\tilde{\gamma}_{(\mu\nu)}$ is that part of ${}^{(p)}_{[2]}\gamma_{(\mu\nu)}$ which is nonsingular along the world line of the pth particle and

$$ {}^{(p)}_{[2]}\gamma^{\text{ext}}_{(\mu\nu)} = \sum_{p'\neq p} {}^{(p')}_{[2]}\gamma_{(\mu\nu)}. \tag{B7} $$

If we make use of the fact that along the world line of the pth particle,

$$ {}^{(p)}_{[2]}\tilde{\gamma}_{(\mu\nu),\rho} = O(\kappa^{3}), \tag{B8} $$

then we find that the ${}^{(p)}_{[4]}C^{I}_{\mu}$ can be put in the form

$$ {}^{(p)}_{[4]}C^{I}_{\mu} = -(1/c^{2})\,{}^{(p)}[m^{G}(\tfrac{1}{4}\eta^{\rho\sigma}\,{}_{[2]}\gamma^{\text{ext}}_{(\rho\sigma),\mu} - \tfrac{1}{2}\,{}_{[2]}\gamma^{\text{ext}}_{(\rho\sigma),\mu}u^{\rho}u^{\sigma} + {}_{[2]}\gamma^{\text{ext}}_{(\mu\rho),\sigma}u^{\rho}u^{\sigma} - \tfrac{1}{4}\eta^{\rho\sigma}\,{}_{[2]}\gamma^{\text{ext}}_{(\rho\sigma),\tau}u^{\tau}u_{\mu} - \tfrac{1}{2}\,{}_{[2]}\gamma^{\text{ext}}_{(\rho\sigma),\tau}u^{\rho}u^{\sigma}u^{\tau}u_{\mu})]. \tag{B9} $$

To the solutions (B1) we can always add a term ${}_{[4]}\gamma^{\dagger}_{(\mu\nu)}$ of the form

$$ {}_{[4]}\gamma^{\dagger}_{(\mu\nu)} = (\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma} - \eta_{\mu\nu}\eta_{\rho\sigma})\,{}_{[4]}\psi^{\rho,\sigma}, \tag{B10}$$

where

$$\Box^2\,{}_{[4]}\psi^{\rho} = O(\kappa^5). \tag{B11}$$

Since such terms will not contribute to ${}_{[4]}\gamma^{I}_{(\mu\nu)}{}^{,\nu}$, they cannot affect the equations of mass, motion, and spin to fourth order.

APPENDIX C: A RELATIONSHIP BETWEEN l AND THE SIZE OF A CHARGED GENERALIZED MAXWELLIAN PARTICLE

We can regard the field $g_{\mu\nu}$ over a perfectly isolated region of the continuum containing only generalized Maxwellian particles as broken up in the following manner:

$$g_{\mu\nu} = \eta_{\mu\nu} + g^{L}_{\mu\nu} + g^{N}_{\mu\nu}, \tag{C1}$$

where $g^{L}_{\mu\nu}$ contains those terms of $g_{\mu\nu}$ which are linear in the quantities ${}^{(p)}e^D$, ${}^{(p)}e^E$, ${}^{(p)}m^G$, and ${}^{(p)}S_{\mu\nu}$, and $g^{N}_{\mu\nu}$ contains those terms which are nonlinear in these quantities. Note that in an isolated region of the continuum containing only an isolated generalized Maxwellian particle at rest we would find, neglecting any spin the particle may have,

$$g^{L}_{[st]} = -(e^E/c^2)(|\vec{r}|^{-2} - 1/2l^2)\epsilon_{stk}|\vec{r}|_{,k},\qquad g^{L}_{[s4]} = -g^{L}_{[4s]} = 0, \tag{C2}$$

$$g^{L}_{(st)} = -\delta_{st}(m^G/2c^2)|\vec{r}|^{-1},\quad g^{L}_{(s4)} = g^{L}_{(4s)} = 0,\qquad g^{L}_{(44)} = -(m^G/2c^2)|\vec{r}|^{-1}. \tag{C3}$$

We shall assume that for the particles in any region of the continuum we investigate,

$$|e^E|/c^2l^2 \ll 1. \tag{C4}$$

We shall show later that (C4) seems to hold for generalized Maxwellian particles of physical interest.

The gravitational radius r_G of a generalized Maxwellian particle will be defined as the distance from the center of the particle to those points where, when the particle is isolated and observed in a rest system,

$$|g^{L}_{(st)}| = \delta_{st},\quad |g^{L}_{(44)}| = 1. \tag{C5}$$

This distance seems to be a reasonable measure of the size, with respect to the gravitational field, of a generalized Maxwellian particle, as within this distance from the center of such a particle the field $g^{L}_{(\mu\nu)}$ is so strong that the approximation method we have been using, involving the expansion of the field $g_{\mu\nu}$ in a power series in κ, is almost certainly not valid. At distances much greater than r_G from the center of such a particle, we will in general find, as can be seen from (C5) and the distance dependence of $g^{L}_{(\mu\nu)}$ in (C3), that

$$|g^{L}_{(\mu\nu)}| \ll 1. \tag{C6}$$

The expansion of the field $g_{\mu\nu}$ in a power series in κ is presumably valid at such distances from the center of a generalized Maxwellian particle provided the charge and spin associated with the particle are not too large.

From (C3), (C5), and (1.78) we find that

$$r_G = 2Gm/c^2. \tag{C7}$$

We see that the gravitational radius of a generalized Maxwellian particle is proportional to the mass of the particle.

We shall define the electromagnetic radius r_E of a generalized Maxwellian particle as the distance from the center of the particle to those points where, when the particle is isolated and observed in a rest system,

$$|g^{L}_{[st]}| = \epsilon_{stk}|\vec{r}|_{,k}. \tag{C8}$$

This distance is a reasonable measure of the size, with respect to the electromagnetic field, of a generalized Maxwellian particle as within this distance from the center of such a particle the field $g^{L}_{[\mu\nu]}$ is so strong that the approximation method we have been using in this paper is almost certainly not valid. At distances much greater than r_E from the center of such a particle we will generally find, as can be seen from (C8), (C4), and the distance dependence of $g^{L}_{[\mu\nu]}$ in (C2), that

$$|g^{L}_{[\mu\nu]}| \ll 1. \tag{C9}$$

The expansion of the field $g_{\mu\nu}$ in a power series in κ is presumably valid at such distances from the center of a generalized Maxwellian particle provided the mass and spin associated with the particle are not too large.

From (C2), (C8), and (1.78) we find that

$$r_E = r_0(1 + r_0^2/4l^2)^{-1/2}, \tag{C10}$$

where

$$r_0 = (2G/c^2)^{1/4}(e^2/4\pi\epsilon_0c^2)^{1/4}l^{1/2}. \tag{C11}$$

If we make use of (C4), a condition which is equivalent to

$$r_0 \ll l, \tag{C12}$$

then to a good approximation,

$$r_E = r_0. \tag{C13}$$

We note that the electromagnetic radius of a generalized Maxwellian particle, if (C4) can be considered valid, is proportional to the square root of the localized electric charge of the particle.

If we introduce the classical electrodynamic radius r_C associated with a particle of mass m and charge e,

$$r_C = e^2/4\pi\epsilon_0 mc^2, \tag{C14}$$

Eq. (C11) can be written in the form

$$r_0 = (r_C r_G)^{1/4} l^{1/2}. \tag{C15}$$

We shall now attempt to estimate the magnitude of l. Let us assume that Einstein's theory is correct and also that generalized Maxwellian particles can approximate, except near the center of a particle, real particles. It then seems to the author that a reasonable value for the electromagnetic radius r_E associated with an elementary particle having a localized charge equal in magnitude to the electron charge would be around 10^{-13}–10^{-12} cm. If the author is correct and this is the order of magnitude of the electromagnetic radius associated with such an elementary particle then at distances of the order of magnitude of 10^{-13} cm or less from the center of such a particle the electromagnetic field is so strong that the expansion of $g_{\mu\nu}$ in a power series in κ is almost certainly not valid. This suggests that the concepts diffuse electric charge, localized electric charge, force, position, velocity, etc. are all inadequate for describing the interaction of elementary particles over distances of the order of magnitude of 10^{-13} cm or less. This seems to be empirically confirmed. In order to describe the interaction of the elementary charged particles of nature over distances of the order of 10^{-13} cm or less, a more detailed knowledge of the structure of a particle than that provided through the above concepts seems to be necessary.

If r_E is of the order of magnitude of 10^{-13}–10^{-12} cm, then the empirically observed breakdown of Maxwell-Lorentz electrodynamics at atomic interaction distances probably cannot be explained as a breakdown at such distances of the approximation procedure we have been using in this paper, the expansion of $g_{\mu\nu}$ in a power series in κ, but probably must be explained through the inadequacy over atomic interaction distances of the idealization we have been using, the use of ideal particles in a perfectly isolated region to represent real particles in an isolated region. Another possibility is that the breakdown of Maxwell-Lorentz electrodynamics at atomic interaction distances is due to the importance at such distances of the nonlinear terms in the forces between elementary particles. This will be discussed later.

From Eqs. (C10) and (C11) we see that the value of r_E discussed above must be associated with a value of l of the order of magnitude of 10^6–10^8 m. A value of l equal to 1 earth radius can perhaps be considered a reasonable lower limit on the magnitude of l. Thus we have presented one reason for believing l may lie near this lower limit. There is a second reason for believing that l may lie near this lower limit. It seems to be a necessary consequence of the assumption that generalized Maxwellian particles can approximate real particles when the particles are interacting over moderately small distances.

If we assume that generalized Maxwellian particles can approximate real particles when the particles are interacting over moderately small distances, then the second-order electromagnetic force (1.96) is expected to dominate any interaction involving charged particles which interact over moderately small distances because over such distances, Maxwell-Lorentz electrodynamics seems to hold when applied to real particles. This will place a restriction on l. The reason for this is that we must in general be concerned with the effects of higher-order terms in the forces between generalized Maxwellian particles even when the ideal particles are interacting over distances much greater than their electromagnetic radii. This is related to the fact that those terms in the second-order electromagnetic force (1.96) which would have been proportional to ${}^{(p)}e^E \cdot {}^{(p')}e^E$ vanish identically, and only terms in the second-order force proportional to ${}^{(p)}e^E \cdot {}^{(p')}e^E/l^2$ and ${}^{(p)}e^E \cdot {}^{(p')}e^E/l^4$ remain.

If the higher-order terms which would have been proportional to ${}^{(p)}e^E \cdot {}^{(p')}e^E \cdot {}^{(p'')}e^E$ do not disappear from the interaction force between generalized Maxwellian particles, then the ratio of these higher-order terms to the second-order interaction force, over small interaction distances, is expected to be of the order of magnitude of $(r_E/r)^2(l/r)^2$. The quantity r is the distance between two interacting particles. The above ratio is not necessarily small for small interaction distances. This can be seen from the fact that if l lies between 1 and 10 earth radii then this ratio is equal to unity, for particles having a charge equal in magnitude to the electron charge, at distances of the order of magnitude of 10^{-2}–10^{-1} cm.

It should be pointed out that we cannot really be sure of the order of magnitude of any of the higher-order terms in the forces acting on generalized Maxwellian particles unless we actually calculate the terms using the procedure developed earlier in this paper. If we do this there are reasons for believing that the higher-order terms in the force acting on a generalized Maxwellian particle which would have been proportional to ${}^{(p)}e^{E}\cdot{}^{(p')}e^{E}\cdot{}^{(p'')}e^{E}$ vanish identically, while those terms proportional to ${}^{(p)}e^{E}\cdot{}^{(p')}e^{E}\cdot{}^{(p'')}e^{E}\cdot{}^{(p''')}e^{E}$ do not.[10] If this is so then the ratio of these latter higher-order terms to the second-order force, over small interaction distances, becomes the significant ratio. It is to be expected that this ratio will be of the order of magnitude of $(r_E/r)^4(l/r)^2$. Note that if l lies between 1 and 10 earth radii, then this ratio is equal to unity, for particles having a charge equal in magnitude to the electron charge, at distances of the order of magnitude of 10^{-6}–10^{-5} cm.

A more detailed investigation and discussion of the higher-order terms in the forces between generalized Maxwellian particles and a discussion of the possible connection of these terms with the appearance of quantum effects will be presented in a later paper.

The lower limit on l discussed in this appendix must be considered as speculative and as only a rough order of magnitude estimate. Note that if we consider 1 earth radius to be a lower limit on l, then for generalized Maxwellian particles having a localized electric charge equal in magnitude to the electron charge,

$$|e^E|/c^2l^2 = r_0^{\,2}/l^2 < 10^{-42}, \tag{C16}$$

and both (C4) and (C12) are satisfied.

Is there an experimental means of measuring l? One might expect that a modification of the Cavendish two-sphere experiment, an experiment which has been used to test Coulomb's law to great accuracy, might provide a means for measuring l. To the best of the author's knowledge the most precise experiment of the Cavendish type was performed by Plimpton and Lawton in 1936.[11] In this experiment it was found that if the force F acting between any two elementary charges e_1 and e_2 is given through the equation

$$F = (e_1e_2/4\pi\epsilon_0)\bar{r}^{\,q}/r^{2+q}, \tag{C17}$$

then q must be less than 2×10^{-9}. The quantity r in (C17) is the distance between the two elementary charges and $\bar{r}$ is an undetermined universal constant.

A simple calculation will show that if the force F between any two elementary charges is given through the equation

$$F = (e_1e_2/4\pi\epsilon_0)(1/r^2 - 1/4l^2), \tag{C18}$$

then the Plimpton-Lawton results mean that l is greater than 4.5 miles. The force F in (C18) is the force, to lowest order, acting between two charged generalized Maxwellian particles at rest. Since we already have reasons for believing that l is greater than 4.5 miles we must regard the Plimpton-Lawton experiment to be of insufficient sensitivity to determine l. If the sensitivity of the Plimpton-Lawton experiment could be increased by a factor of 10^6, then the experiment would have a chance of detecting the effects of the diffuse electric charge surrounding an elementary particle and of determining l. Whether such sensitivity is possible today the author does not know. The author also wonders if gravity effects and effects associated with the microscopic structure of matter may not complicate the experiment when performed to such accuracy.

Neglecting these complicating factors, one finds that the change in the voltage difference v between the inner and outer spheres of the experiment, when the voltage on the outer sphere changes by V, is given, to the lowest order in $(a/l)^2$, through the equation

$$v = \tfrac{1}{12}(a^2/l^2)(1 - b^2/a^2)V, \tag{C19}$$

where a is the radius of the outer sphere and b is the radius of the inner sphere. In the Plimpton-Lawton experiment a was 2.5 ft, b was 2 ft, V was 3000 V, and v was found to be less than 10^{-6} V.

APPENDIX D: INTERACTION OF GENERALIZED MAXWELLIAN PARTICLES

When generalized Maxwellian particles of charge e, mass m, and negligible spin are separated by distances of the order of magnitude of r_e or less — r_e is the greater of the two lengths r_G and r_E,

$$r_G = 2Gm/c^2, \quad r_E = r_0(1 + r_0^{\,2}/4l^2)^{-1/2}, \tag{D1}$$

where

$$r_0 = (2G/c^2)^{1/4}(e^2/4\pi\epsilon_0c^2)^{1/4}l^{1/2}, \tag{D2}$$

and will be known as the effective radius of a particle — they cannot be considered to interact even approximately as Maxwellian particles. Over such distances the approximation procedure we have been using in this paper, involving the expansion of $g_{\mu\nu}$ in a power series in κ, is almost certainly not valid. We can say nothing about interactions over such distances. This has been discussed in Appendix C.

In this appendix we shall assume that Einstein's theory is correct and also that elementary particles, atoms, and atomic ions can often be represented without too much error, at least at macroscopic distances from the center of each particle, by generalized Maxwellian particles. We shall also assume that l is an astronomical length of the order of magnitude of several earth radii. This latter assumption means, as we see from above, that the various values of r_e associated with elementary particles, atoms, and atomic ions are all microscopic lengths. This also means, as we see from (1.79)–(1.98), that the diffuse electric charge associated with such particles will in most cases produce only a negligible interaction among the particles. This can be attributed to the relatively large value of l and to the fact that, in contrast to the localized electric charge, the diffuse electric charge associated with a generalized Maxwellian particle does not produce a radiation field, a field which depends on the particle's acceleration in addition to its position and velocity at a "retarded" or "advanced" time. In general, only through cooperative effects involving many particles is it possible for the diffuse electric charge associated with elementary particles, atoms, and atomic ions to give rise to any observable effect.

In this paper we are also assuming that the distribution of positive and negative charge in the universe is such that there are no cooperative effects on a cosmic scale which preclude the existence of isolated regions. This must be so as in this paper we are assuming isolated regions exist in the universe inside of which the contribution to the electromagnetic field produced by any charge outside the region is negligible. Of course within some of these isolated regions, cooperative effects associated with particles within the region itself may allow the diffuse electric charge associated with these internal particles to produce an observable effect within the region. In other regions, because of the distribution of positive and negative charge, cooperative effects of such a magnitude presumably do not exist and all macroscopic observable electromagnetic interaction can be described through Maxwell-Lorentz electrodynamics. Maxwell-Lorentz electrodynamics of course will be valid throughout any isolated region which is sufficiently small so that all interparticle distances are negligible in comparison to l.

It should be pointed out that a region of cosmological dimensions cannot really be considered isolated and the results of our idealization (the use of ideal particles in a perfectly isolated region of the continuum to represent real particles in an isolated region of the continuum) when applied to such a region must be regarded with caution. Furthermore, it should be pointed out that although we may hope to approximate, over a finite and not too large region of the continuum, an exact nonsingular solution to the field equations by a solution in which real particles are represented by ideal particles, if we extend this solution in either a timelike or a spacelike direction, errors may accumulate, and over the extended region the solution may diverge radically from the nonsingular solution we were attempting to approximate.

Are there any observable effects of the diffuse electric charge associated with generalized Maxwellian particles which could provide a test of our work and of Einstein's relativistic field theory? Because of the presumed rather large value of l, we would in general expect observable effects of the diffuse electric charge only in astronomical phenomena. As is well known, such phenomena are often very difficult to compare with any theory because of their great complexity. However that may be, examples of astronomical phenomena, some of whose properties might be correlated with the diffuse electric charge associated with charged particles, are the magnetic fields of the sun and planets and the weak interplanetary field of the solar system.[12]

If one wishes to investigate the consequences of the equations which describe the interaction of generalized Maxwellian particles to second order, Eqs. (1.95)–(1.98), it is convenient to introduce the concept of the effective electromagnetic field produced by a generalized Maxwellian particle. We shall define the effective electromagnetic field ${}^{(p)}\hat{F}_{[\mu\nu]}$ associated with the pth particle as follows:

$$ {}^{(p)}\hat{F}_{[\mu\nu]} = {}^{(p)}F^{L}_{[\mu\nu]} + \tfrac{1}{2}{}^{(p)}F^{D}_{[\mu\nu]}. \tag{D3} $$

In terms of such fields, Eqs. (1.95)–(1.98) take the form

$$ {}^{(p)}m\,{}^{(p)}\dot{u}_\mu = {}^{(p)}F_\mu , \tag{D4} $$

$$ {}^{(p)}F_\mu = {}^{(p)}\Big[(e/c)u^\lambda \hat{F}^{\text{ext}}_{[\lambda\mu]} - \tfrac{2}{3}a(e^2/4\pi\epsilon_0 c^2)(\ddot{u}_\mu + \dot{u}_\rho \dot{u}^\rho u_\mu)\Big], \tag{D5} $$

where

$$ {}^{(p)}\hat{F}^{\text{ext}}_{[\mu\nu]} = \sum_{p' \neq p} {}^{(p')}\hat{F}_{[\mu\nu]}. \tag{D6} $$

The effective electromagnetic field associated with the pth particle can be shown to satisfy the electromagnetic field equations

$$ {}^{(p)}\hat{F}_{[\mu\nu]}{}^{,\nu} = \mu_0\,{}^{(p)}\hat{J}_\mu , \quad {}^{(p)}\hat{F}_{[\mu\nu,\rho]} = 0, \tag{D7} $$

where

$$ {}^{(p)}\hat{J}_\mu = {}^{(p)}J^L_\mu + {}^{(p)}\hat{J}^D_\mu , \tag{D8} $$

and

$$ {}^{(p)}J^L_\mu = {}^{(p)}ec\int \delta(x^\sigma - {}^{(p)}\xi^\sigma)\,{}^{(p)}u^\mu\, d({}^{(p)}\tau), \tag{D9} $$

$$ \begin{aligned} {}^{(p)}\hat{J}^D_\mu &= {}^{(p)}\hat{J}^D_{\mu\,\text{ret}} + {}^{(p)}\hat{J}^D_{\mu\,\text{adv}}, \\ {}^{(p)}\hat{J}^D_{\mu\,\text{ret(adv)}} &= {}^{(p)}a_{\text{ret(adv)}}\,{}^{(p)}\big[-(ec/8\pi l^2)u_\mu (ru)^{-1}\big]_{\text{ret(adv)}}. \end{aligned} \tag{D10} $$

The quantity ${}^{(p)}J^L_\mu$ is the current associated with the localized electric charge of the pth particle. The quantity ${}^{(p)}\hat{J}^D_\mu$ will be known as the effective electric current associated with the diffuse electric charge of the pth particle. Note that

$$^{(p)}\hat{J}^{D}_{\mu} = \tfrac{1}{2}{}^{(p)}J^{D}_{\mu}. \tag{D11}$$

From Eqs. (D4)–(D10) it is easy to calculate the effective magnetic field outside a circular current loop. If the current associated with the localized electric charge flowing in the loop is denoted by I and the area of the loop is denoted by A, we find that the effective magnetic field $\vec{B}^L$ produced by the circulating localized electric charge is given by

$$B^L_r = (\mu_0/4\pi)(IA/r^3)2\cos\theta,$$
$$B^L_\theta = (\mu_0/4\pi)(IA/r^3)\sin\theta, \quad B^L_\varphi = 0. \tag{D12}$$

We are using spherical polar coordinates r, θ, and φ in (D12) where the origin of the coordinates is at the center of the current loop and θ is measured from an axis perpendicular to the plane of the loop and running through its center. We are of course neglecting terms of higher order in a/r.

The effective magnetic field $\vec{B}^D$ produced by the diffuse electric charge associated with the particles circulating in the loop is given by

$$B^D_r = -(\mu_0/4\pi)(IA/4l^2r)2\cos\theta,$$
$$B^D_\theta = (\mu_0/4\pi)(IA/4l^2r)\sin\theta, \quad B^D_\varphi = 0. \tag{D13}$$

We are again neglecting terms of higher order in a/r. From (D12) and (D13) we find for the ratio of the magnitudes of the two fields $\vec{B}^L$ and $\vec{B}^D$ at a distance r from the center of the loop

$$B^D/B^L = r^2/4l^2. \tag{D14}$$

We conclude from (D12) and (D13) that if the magnetic field of the earth is produced by circulating currents in its interior then these currents will not only give rise to a field which falls off at large distances from the center of the earth inversely with the distance cubed, but also to a field which falls off inversely with the distance itself. From (D14) we may surmise that at distances from the center of the earth less than about $2l$ the field produced by the circulating localized electric charge inside the earth will be dominant, while at distances greater than about $2l$ the field produced by the associated circulating diffuse electric charge will dominate. Our knowledge of the magnetic field in the vicinity of the earth suggests that l cannot be less than several earth radii.

More realistic models of the circulating currents in the earth's interior can be constructed and the properties of the generated magnetic fields compared with the earth's magnetic field. This comparison would certainly not be easy to make as one would have to take into account the effects of the plasma surrounding the earth. Nevertheless a more thorough investigation of the earth's magnetic field and other magnetic fields of astronomical dimensions might provide observational evidence for Einstein's theory.

APPENDIX E: PHYSICAL APPLICATION OF LORENTZ-INVARIANT GRAVITATIONAL THEORY

We shall show that the velocity-dependent terms in Eqs. (1.108)–(1.115) are not physically meaningful when the velocities of the interacting generalized Newtonian particles are small with respect to the velocity of light and the particles are bound or nearly bound by their mutual gravitational attraction.

The equations which describe the motion and structure of interacting generalized Newtonian particles can be expanded in either a power series in κ or, if the quantities in the equations are expressed as functions of t rather than ${}^{(p)}\tau$ or x^4, in a power series in c^{-1}. We shall assume here that such power series converge. We note that the mass m^G, which is of the second order in κ, can never appear alone in the equations which describe the structure and motion of interacting generalized Newtonian particles but can only appear in combination with c^{-2} as $c^{-2}m^G$. This follows from the general form of the solutions to the homogeneous equations given in the main text. It means that the equations of motion (1.108)–(1.110), which are correct to order five in κ, will also be correct to order five in c^{-1}. It also means that the field equations (1.111)–(1.115), which are correct to order three in κ, will be correct to order three in c^{-1}. We are assuming that any charge or spin which might be associated with the generalized Newtonian particles we are investigating can be neglected.

Under the above-mentioned conditions, assuming the generalized Newtonian particles are not too near each other, generalized Newtonian particles interact to lowest order in c^{-1} as Newtonian particles. This follows from Eqs. (1.108)–(1.123) and arguments given above. It means that for slowly moving bound or nearly bound particles,

$$\tfrac{1}{2}{}^{(p)}m\,{}^{(p)}v^s\,{}^{(p)}v^s \lesssim {}^{(p)}m\,{}^{(p)}\varphi^{\text{ext}}, \tag{E1}$$

where

$$^{(p)}v^s = d({}^{(p)}\xi^s)/dt, \tag{E2}$$

or in a different notation,

$$^{(p)}\xi'^s\,{}^{(p)}\xi'^s \lesssim {}^{(p)}_{[2]}\gamma^{\text{ext}}_{(44)}, \tag{E3}$$

where

$$^{(p)}\xi'^s = d({}^{(p)}\xi^s)/cdt. \tag{E4}$$

We see that with slowly moving gravitationally bound or nearly bound particles the velocity dependent terms in Eqs. (1.108)–(1.115) are of the same order of magnitude as higher-order terms in κ which were neglected in arriving at the equations. This means that the velocity-dependent terms in Eqs. (1.108)–(1.115) should not be considered as having a physical meaning when the velocities of the interacting generalized Newtonian particles are small with respect to the velocity of light and the particles are bound or nearly bound by their mutual gravitational attraction.

If we wish to get, at each order of approximation, physically meaningful equations which describe the motion and structure of slowly moving generalized Newtonian particles it seems reasonable to assume that we should solve for these equations step by step with respect to powers of c^{-1} not κ. If we wish to find the equations of motion up to a certain order in c^{-1}, it will often be convenient to first solve for the equations up to an order in κ which is known to give equations accurate to the order in c^{-1} desired, and then to expand these equations in a power series in c^{-1} keeping only terms up to the order in c^{-1} sought. This is in fact how we arrived at Newtonian gravitational theory, Eqs. (1.117)–(1.123), which is accurate to lowest order in c^{-1}.

In summary we may say that in contrast to Lorentz-invariant electromagnetic theory, Lorentz-invariant gravitational theory is in general only physically meaningful when applied to weakly interacting ideal particles moving with velocities near the velocity of light.[13] Both theories are of course inapplicable when the ideal particles involved are very close to each other. This has been discussed in Appendix C.

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[1]C. R. Johnson, preceding paper, Phys. Rev. D 3, 295 (1971), hereafter referred to as paper I. It is assumed that the reader is familiar with paper I. The notation used in the present paper, hereafter referred to as paper II, will be the same as that used in paper I except that in paper II, when we refer to an equation in paper I, we shall place a I in front of its number. Thus in paper II Eqs. (I1.2) refers to Eqs. (1.2) of paper I.

[2]In the case where the spin is zero, Eqs. (1.69) are just the usual geodesic equations, to second order in κ, for the motion of a test particle in a gravitaional field.

[3]A relationship between the effective size of a particle and the magnitude of l is discussed in Appendix C. In Appendix C we give reasons, based on this relationship, for believing l should be an astronomical length.

[4]A more thorough discussion of the conditions under which this is true is given in Appendix D.

[5]Note that radiation reaction terms do not appear in the fourth-order equations of motion satisfied by interacting generalized Newtonian particles. This is because the acceleration of such particles is always of the second order in κ [this follows from the second-order equations of motion (1.62), equations which hold to second order in κ but contain errors of order four in κ], so that radiation reaction terms, if they are to appear at all, can only appear as physically meaningful terms in the equations of motion of sixth or higher order. To understand why this is so, remember ${}_{[4]}t_{\mu\nu}$ is proportional to the product of two masses [see Eqs. (1.64) and Eqs. (1.59)–(1.61)] and that mass in our work has been chosen to be of order two in κ. If we were to investigate the gravitational radiation associated with interacting generalized Newtonian particles we would want to investigate the radiation reaction affecting the motion. In order to do this we would have to find the equations of motion satisfied by the particles to a higher order of approximation than fourth. In this conclusion we agree with Kerr [R. P. Kerr, Nuovo Cimento 13, 492 (1959)] and disagree with Havas and Goldberg [P. Havas and J. N. Goldberg, Phys. Rev. 128, 398 (1962)], who feel that a physically meaningful radiation reaction term may appear in what we call fourth order.

[6]It is shown in Appendix E that the velocity-dependent terms in Eqs. (1.108)–(1.115) can in general only be considered physically meaningful if the velocities of the interacting particles are not small with respect to the velocity of light or the particles are not bound or nearly bound by their mutual gravitational interaction.

If the interacting generalized Newtonian particles are moving with velocities which are low with respect to the velocity of light and are bound or nearly bound by their mutual gravitational interaction the quantity which is most natural to use as an expansion parameter when seeking equations describing the motion and structure of generalized Newtonian particles is not κ but c^{-1}. If c^{-1} is used as an expansion parameter and all quantities are expressed as functions of t instead of as functions of ${}^{(p)}\tau$ or x^4, one expects to get physically meaningful equations applicable to slowly moving generalized Newtonian particles at each order of approximation. It is shown in Appendix E that to lowest order in an expansion in c^{-1} Eqs. (1.117)–(1.121) are the equations of mass and motion satisfied by such particles. We are of course assuming that charge and spin can be neglected in this lowest order of approximation.

[7]G. Bandyopadhyay, Science and Culture (Calcutta) 25, 427 (1960).

[8]A. Papapetrou, Proc. Roy. Irish Acad. A52, 69 (1948).

[9]M. Wyman, Can. J. Math. 2, 427 (1950).

[10]Subsequent investigation has shown that this is indeed what happens.

[11]Recently, more precise experiments have been performed. These experiments seem to be more sensitive than the Plimpton-Lawton experiment by a factor of about 10^3. See, G. D. Cochran and P. A. Franklin, Bull. Am. Phys. Soc. 13, 1379 (1968); D. F. Bartlett and E. A. Phillips, *ibid.* 14, 17 (1969).

[12]The author wonders whether some of the properties of recently discovered astronomical phenomena such as quasars and pulsars can be explained, at least partially, through effects associated with the diffuse electric charge which may be associated with elementary particles. Over astronomical distances greater than about $2l$, this diffuse electric charge will cause like charges

to attract each other and unlike charges to repel each other. This may give rise to some observable astronomical effects which cannot find an explanation through Maxwell-Lorentz electrodynamics.

[13]This fact does not seem to be realized in some of the earlier works on Lorentz-covariant equations of motion in Einstein's gravitational theory. See P. Havas and J. N. Goldberg, Phys. Rev. 128, 398 (1962), and S. F. Smith and P. Havas, *ibid.* 138, B495 (1965). In a paper on Lorentz-covariant equations of motion in gravitational theory, Carmeli [M. Carmeli, Nuovo Cimento 55B, 220 (1968)] arrives at some of the same conclusions we have in this appendix. His technique for finding equations of motion, however, has all the weaknesses of the other techniques mentioned in Sec. I of paper I.