[26]$E_n(z)$ can be related to $E_1(z)$ through the recursion relation

$$(n-1)E_n(z)=e^{-z}-zE_{n-1}(z)\,,\quad n=2,3,\ldots\,.$$

Then, $E_1(-|z|)\equiv -\mathrm{Ei}(|z|)$ can be used to define $E_1(z)$ for $z=-|z|<0$, hence, all $E_n(-|z|)$. In turn,

$$\mathrm{Ei}(z)=P\int_{-\infty}^{z}dt\frac{e^t}{t}$$

is defined for all $-\infty<z<+\infty$; see V. Kourganoff, *Basic Methods in Transfer Problems* (Oxford Univ. Press, Oxford, England, 1952), p. 254.

[27]Because of the large-s_1 saturation assumption in Eq. (9), this validity to the order of inverse logarithms is also characteristic of the popular Poisson distribution for σ_n generated in multiperipheral models, see e.g., Eq. (3.17) of Ref. 7.

[28]G. Mach, Phys. Rev. 154, 1617 (1967); H. Kastrup, *ibid.* 147, 1131 (1966).

[29]The case for which

$$A_n=A_n^{(1)}(\rho)+\delta A_n(\rho,Q^2)\,,\quad \lim_{Q^2\to\infty}\delta A_n\to 0\,,$$

when

$$\bar{n}=\bar{n}(\rho)+\delta n\,,\quad \lim_{Q^2\to\infty}\delta n\to 0\,,$$

is inconsistent with $A=A(\rho)$, as in the discussion surrounding Eq. (81).

[30](a) S. Drell *et al.*, Phys. Rev. Letters 22, 744 (1969); (b) H. Abarbanel *et al.*, *ibid.* 22, 500 (1969). In Ref. 30(b), a nonscaling form $A\sim(1/Q^2)(s/Q^2)^{\alpha(g^2)}$ is obtained. This form for A is that derived with point vertices from AFS [Ref. 7, p. 912, Eq. (4.12)] by assuming that the Q^2 dependence *is* the off-mass-shell hadron, t_0 dependence. It would follow that

$$\bar{n}/g^2=\frac{1}{A}\frac{\partial A}{\partial g^2}=\frac{\partial\alpha}{\partial g^2}\ln\rho\,.$$

With the inclusion of photon spin, still only one structure function scales [and $\sigma_L/\sigma_T\sim Q^2 f(\rho)$]; see G. Altarelli and H. Rubinstein, Phys. Rev. 187, 2111 (1969).

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PHYSICAL REVIEW D VOLUME 4, NUMBER 10 15 NOVEMBER 1971

Distribution of the Spins of the Resonances in the Dual-Resonance Model

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(Received 28 June 1971)

We use quantum-statistical mechanics to calculate the distribution of the spins of the resonances in the dual-resonance model at a fixed high energy squared (s). The resonances are concentrated in the region of spin $\lesssim\sqrt{s}$, with the distribution eventually falling off exponentially. The profile function of the Veneziano partial-wave amplitudes is mainly controlled by the spin distribution of the resonances, and not by their individual couplings. This spin distribution is largely similar to that in the statistical model of nuclei.

I. INTRODUCTION

One of the most characteristic features of the dual-resonance model[1] is the very rapid growth of the total level density $n(s)$ with increasing energy squared (s).[2] For large s, it has the asymptotic form[3,4]

$$n(s)ds\sim s^{b/2}e^{a\sqrt{s}}ds\,, \tag{1.1}$$

where

$$b=-\frac{D+3}{2}\,,\quad a=2\pi\left(\frac{D\alpha'}{6}\right)^{1/2}\,,$$

and $D=4$ in the usual Veneziano model,[1] but may be larger.[5] Thus,[4] for

$$D=4,5,6,7,\ldots,$$
$$b=-\tfrac{7}{2},\,-4,\,-\tfrac{9}{2},\,-5,\ldots,$$
$$a^{-1}=180,\ 174,\ 159,\ 147,\ldots\ \mathrm{MeV}.$$

The statistical bootstrap model for hadrons[6] also gives a level density of the form (1.1), with $a^{-1}=160$ MeV (Ref. 7) and $b<-\frac{7}{2}$ (Ref. 8). The similarity of these two results may suggest that the statistical bootstrap and dual-resonance models are closely related.[9,10] It is therefore interesting to see if the two models agree in other details of their predictions.

In this paper, we consider the distribution of angular momentum (l) among the resonances of the dual-resonance model at a definite mass $\sqrt{s}$, as the first step in a comparison with the same distribution in the statistical bootstrap. Some results for the dual-resonance model have already been obtained, but not the complete distribution. By applying the Darwin-Fowler method of statistical mechanics to the harmonic oscillators of the Veneziano model,[11] Corgnier and D'Adda[12] found that the mean and standard deviation of the square of the resonance spins behave like

$$\langle \vec{l}^2 \rangle \equiv 3\langle l_z^2 \rangle \sim \tfrac{3}{2}s \tag{1.2}$$

and

$$[\langle (\vec{l}^2)^2 \rangle - \langle \vec{l}^2 \rangle^2]^{1/2} \sim \sqrt{3}\, s, \tag{1.3}$$

respectively.

Fritzsch[13] has also considered this problem using statistical methods, which have also been applied to the Veneziano amplitude by Chang, Freund, and Nambu.[14] By taking into account the Bose nature of the quanta, we calculate the distribution of l_z and of l for the factorizing resonant states of the Veneziano model. Our method counts the timelike (spinless) excitations correctly; it does include the ghost states, and does not allow for the linear dependences of some of the states,[3] but it is unlikely that these will greatly influence our main conclusions.[15]

II. CALCULATION OF THE ANGULAR MOMENTUM DISTRIBUTION

A. General Method of the Operator Formalism

The resonances of the Veneziano model are the eigenstates of an operator H (the "Hamiltonian"), with eigenvalue $\alpha(s)$, an integer. H is built from an infinite set of four-vector harmonic oscillators[11]:

$$\begin{aligned} H &= \sum_{\mu=0}^{3} \sum_{k=1}^{\infty} k a_\mu^{(k)+} a_\mu^{(k)} \\ &= \sum_{\mu=0}^{3} \sum_{k=1}^{\infty} k \mathfrak{N}_\mu^{(k)}. \end{aligned} \tag{2.1}$$

We label the states $a_\mu^{(k)+}|0\rangle$ to be diagonal in L_z, the z component of the angular momentum:

$$\mu = +,\ -,\ z, \text{ or } 0,$$

where

$$a_\pm^{(k)} = \frac{a_x^{(k)} \pm i a_y^{(k)}}{\sqrt{2}} .$$

The operator L_z is then[12]

$$\begin{aligned} L_z &= i \sum_{k=1}^{\infty} (a_x^{(k)+} a_y^{(k)} - a_y^{(k)+} a_x^{(k)}) \\ &= \sum_{k=1}^{\infty} (\mathfrak{N}_+^{(k)} - \mathfrak{N}_-^{(k)}). \end{aligned} \tag{2.2}$$

The $\mathfrak{N}_\mu^{(k)}$ are the occupation-number operators for the oscillators.

More completely, H [Eq. (2.1)] should be replaced by[16]

$$H = H + a^{(0)+} a^{(0)} \tag{2.3}$$

including an extra scalar (spin-zero) oscillator mode, except when $\alpha(0) = 1$. The scalar mode does not change the main features of the l distribution, and so we shall use (2.1) instead.

B. The Statistical Method

The oscillator excitations that label the resonances can be regarded as forming a statistical ensemble, whose Hamiltonian is the operator H. Then $\alpha(s)$ is like the energy, and the statistical average of a quantity at energy $\alpha(s) = N$ can be evaluated by replacing the sum over states at a fixed energy N by a sum over states of *all* energies, provided that we include a weight of $e^{-H/T}/\mathrm{tr}(e^{-H/T})$ in performing the average. This is the normal device of statistical mechanics, in passing from the microcanonical to the canonical (Gibbs) ensemble.

The temperature T is defined by the requirement that the average of H coincides with N:

$$N = \frac{\mathrm{tr}(H e^{-H/T})}{\mathrm{tr}(e^{-H/T})}, \tag{2.4}$$

where the trace is over all of the oscillator states. This gives

$$\begin{aligned} N &= -\frac{\partial}{\partial(1/T)} \ln\left\{ \prod_{k=1}^{\infty} \sum_{\{n_\mu^{(k)}=0\}}^{\infty} \exp\left[-\frac{k}{T}(n_+^{(k)} + n_-^{(k)} + n_z^{(k)} + n_0^{(k)}) \right] \right\} \\ &= 4 \sum_{k=1}^{\infty} \frac{\partial}{\partial(1/T)} \left\{ \ln\left[1 - \exp\left(-\frac{k}{T}\right)\right] \right\} \\ &\sim \tfrac{2}{3}\pi^2 T^2 \quad \text{as} \quad T \to \infty. \end{aligned} \tag{2.5}$$

So

$$T \sim (3/2\pi^2)^{1/2} \sqrt{N} . \tag{2.6}$$

If the scalar mode [Eq. (2.3)] is included, the product in Eq. (2.5) is multiplied by $[1 - \exp(-1/T)]^{-1}$, and then

$$T \sim \left(\frac{3}{2\pi^2}\right)^{1/2} \sqrt{N} + \frac{1}{8}\left(\frac{3}{2\pi^2}\right)^{3/2} \frac{1}{\sqrt{N}},$$

so the effect on T is negligible asymptotically.

Once the correspondence between the microcanonical and canonical ensembles is established, the average of any operator A over the states with a definite energy N, $\langle A \rangle_N$, can be calculated according to the formula

$$\langle A\rangle_N = \frac{\mathrm{tr}(Ae^{-H/T})}{\mathrm{tr}(e^{-H/T})}, \tag{2.7}$$

where the trace is taken over *all* the states of the system. The validity of Eq. (2.7) depends upon the system possessing a large number of degrees of freedom and so having a high degeneracy, conditions that are clearly satisfied by the Veneziano spectrum.

C. The Angular Momentum Distribution

We find the distribution in l by first obtaining that in l_z. Let the proportion of oscillator states with eigenvalue l_z of the z component of angular momentum be $\sigma_N(l_z)$ at $\alpha(s)=N$, with normalization

$$\sum_{l_z=-N}^{N} \sigma_N(l_z) = 1. \tag{2.8}$$

We first evaluate the average of the operator $e^{i\alpha L_z}$, which is equivalent to finding all moments of the distribution. By definition,

$$\langle e^{i\alpha L_z}\rangle_N = \sum_{l_z=-N}^{N} \sigma_N(l_z)e^{i\alpha L_z}. \tag{2.9}$$

By our previous considerations [Eqs. (2.1), (2.2), and (2.7)],

$$\langle e^{i\alpha L_z}\rangle_N = \frac{\mathrm{tr}(e^{i\alpha L_z}e^{-H/T})}{\mathrm{tr}(e^{-H/T})}$$

$$= \prod_{k=1}^{\infty}\left(\frac{\sum_{n_+=0}^{\infty}\sum_{n_-=0}^{\infty} e^{n_+(-k/T+i\alpha)}e^{n_-(-k/T-i\alpha)}}{(1-e^{-k/T})^{-2}}\right)$$

$$= \prod_{k=1}^{\infty} \frac{(1-e^{-k/T})^2}{(1-e^{-k/T+i\alpha})(1-e^{-k/T-i\alpha})}, \tag{2.10}$$

so

$$\langle e^{i\alpha L_z}\rangle_N = G(\alpha,T)e^{-(\pi^2/3)T}, \tag{2.11}$$

where[17]

$$G(\alpha,T) = \prod_{k=1}^{\infty}\frac{1}{1-2e^{-k/T}\cos\alpha + e^{-2k/T}}.$$

The distribution function in l_z is then found by inverting the Fourier series, Eq. (2.9). With Eq. (2.11),

$$\sigma_N(l_z) = \frac{e^{-(\pi^2/3)T}}{2\pi}\int_{-\pi}^{\pi} d\alpha\, e^{-i\alpha l_z}G(\alpha,T), \tag{2.12}$$

which can be evaluated numerically, or analytically in approximate form. In the latter case, we replace $G(\alpha,T)$ by[18]

$$\tilde{G}(\alpha,T) = C(T)\prod_{k=1}^{\infty}\frac{1}{1+(\alpha T/k)^2}$$

$$= C(T)\frac{\pi\alpha T}{\sinh(\pi\alpha T)},$$

where $C(T)$ is a function of T only.[18] Then Eq. (2.9) becomes a Fourier integral and the integral equation (2.12) is replaced by

$$\tilde{\sigma}_N(l_z) = \int_{-\infty}^{\infty} d\alpha\,\tilde{G}(\alpha,T)e^{i\alpha l_z}e^{-(\pi^2/3)T}. \tag{2.13}$$

This integral can be evaluated by contour integration, distorting the contour to encircle the lower half α plane, and explicitly performing the sum over the pole residues. This gives

$$\sigma_N(l_z) \approx \frac{1}{4T}\mathrm{sech}^2\left(\frac{l_z}{2T}\right), \tag{2.14}$$

where the coefficient has been fixed by normalizing $\sigma_N(l_z)$ to be in accord with Eq. (2.8). In Fig. 1, we compare the spin distribution obtained using the analytic expression Eq. (2.14) with that found by a numerical evaluation of Eq. (2.12). It is seen that Eq. (2.14) provides an excellent asymptotic approximation, so that in the following we shall use analytic expressions obtained from Eq. (2.14).

Now let $\rho_N(l)$ be the fraction of oscillator states at $\alpha(s)=N$ with spin l, and $\bar{\rho}_N(l)$ that fraction with some definite z component of this spin, so that

$$\sum_{l=0}^{N} \rho_N(l) = 1. \tag{2.15}$$

Then $\sigma_N(l_z)$ has contributions from all resonances of spin $l \geq l_z$.

$$\sigma_N(l_z) = \sum_{l=l_z}^{N} \bar{\rho}_N(l),$$

giving

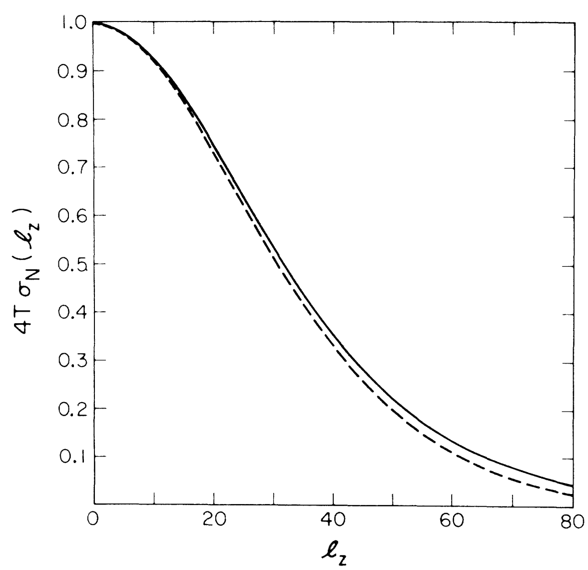

FIG. 1. Graph of the quantity $4T\sigma_N(l_z)$ as a function of l_z at a fixed high energy, giving the fraction $[\sigma_N(l_z)]$ of states with a given spin component, l_z. The solid curve is the analytical approximation, $\mathrm{sech}^2(l_z/2T)$; the dashed curve is the numerical evaluation. Here $N=2125$.

$$\bar{\rho}_N(l) = \sigma_N(l) - \sigma_N(l+1). \tag{2.16}$$

With Eq. (2.14), the spin distribution of the resonances at $\alpha(s) = N$ is given by

$$\bar{\rho}_N(l) \approx -\left.\frac{\partial \sigma_N(l_z)}{\partial l_z}\right|_{l_z = l} = \frac{1}{4T^2}\frac{\sinh(l/2T)}{\cosh^3(l/2T)}. \tag{2.17}$$

It is convenient to make use of the impact parameter $b = 2l/\sqrt{N}$, and then $4T^2\bar{\rho}_N(l)$ is a universal function of b:

$$\bar{\rho}_N(b) = \frac{1}{4T^2}\frac{\sinh cb}{\cosh^3 cb} \equiv \frac{1}{4T^2} f(b), \tag{2.18}$$

where $c = \pi/2\sqrt{6}$, and N is measured in units of $(\alpha')^{-1} \approx 1\ \mathrm{GeV}^2$. Although we use an impact parameter, we remind the reader that $\bar{\rho}_N(b)$ is the fraction of resonance which can be found at b and *not* the profile function of the partial-wave amplitudes, the latter being related to $\bar{\rho}_N(b)$ in a nontrivial way. The direct dependence of $\bar{\rho}$ on b given by Eq. (2.18) enables it to be related to this profile in the same way at all energies, and will be discussed in Sec. III. In Fig. 2 we show the b dependence of $\bar{\rho}$ (at a fixed value of N), and also of $\rho = (2l+1)\bar{\rho}$, which is the total fraction of states with spin $l = \frac{1}{2}b\sqrt{N}$.

D. Details of the Distribution

For small values of l, $\bar{\rho}_N(l)$ increases linearly with l:

$$\bar{\rho}_N(l) \approx \frac{l}{8T^3} \quad \text{for } l \ll T \quad (\text{but } l \neq 0), \tag{2.19}$$

and then rises to a maximum at

$$\left.\begin{matrix} l_0 = 0.51\sqrt{N} \\ \text{or} \quad b = 0.20\ \mathrm{F} \end{matrix}\right\} \text{for } \bar{\rho} \text{ and } \left.\begin{matrix} 0.77\sqrt{N} \\ 0.30\ \mathrm{F} \end{matrix}\right\} \text{for } \rho.$$

For larger values of l, the distribution falls exponentially,

$$\bar{\rho}_N(l) \sim \frac{1}{4T^2} e^{-l/2T} \quad \text{for } l \gg T, \tag{2.20}$$

and not as a Gaussian.[13]

With our normalization, Eq. (2.15), the total number of spin-l resonances at $\alpha(s) = N$ is determined by the total level density [Eq. (1.1)], and so increases very rapidly with N. There is a strong clustering of resonances below a parabola displaced by $\Delta l \sim \sqrt{N}$ away from the peak at $l_0 \sim \sqrt{N}$. In particular, there is a relative linear depletion of low-spin resonances.

In Fig. 3 we show the behavior of the cumulative distribution function $I(b)$, which gives the proportion of the states whose spins lie below $l = \frac{1}{2}b\sqrt{N}$, from Eq. (2.18),

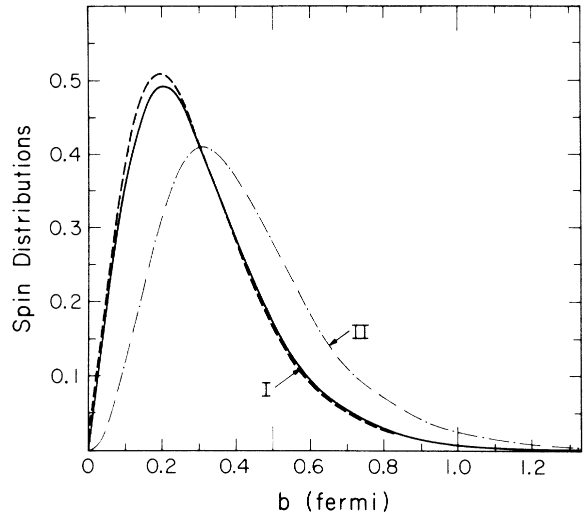

FIG. 2. Curves I are $8Tc^2\bar{\rho}$ plotted against the impact parameter b, giving the fraction of resonances at b. The solid curve is the analytical approximation, $(\pi/\sqrt{6})f(b)$; the dashed curve is the numerical calculation, with $N = 2125$. Curve II is $8cT\rho$, giving the total fraction of states at b, showing only the approximate form, $\frac{1}{12}\pi^2 b f(b)$.

$$I(b) = \int_0^l dl' \rho_N(l') \approx \frac{\pi^2}{12}\int_0^b db' b' f(b') = \frac{\sinh(2cb) - 2cb}{\cosh(2cb) + 1}. \tag{2.21}$$

The leading trajectory ($l = N$) has $b = 2\sqrt{N}$, so we see that $I(b)$ approaches unity rather rapidly when N is large. For example, 90% of all the states have a spin below

$$l = 1.8\sqrt{N},$$

which corresponds to an impact parameter of

$$b = 0.73\ \mathrm{F}.$$

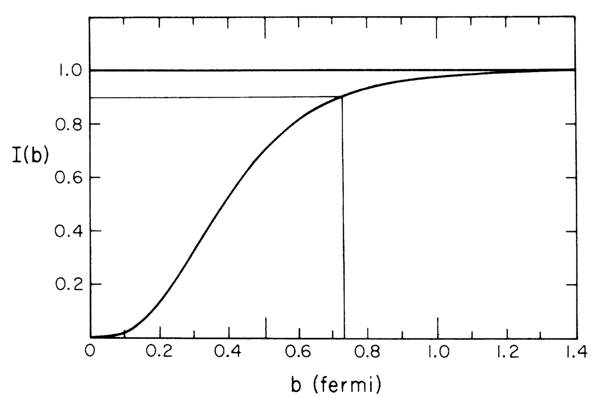

FIG. 3. Graph of the cumulative fraction of states, $I(b)$, that occur below impact parameter b, as a function of b. The 90% level is indicated.

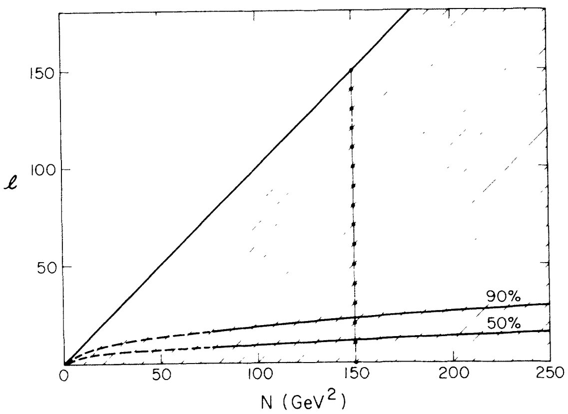

FIG. 4. Parabolic contours enclosing 50% and 90% of the states on the Chew-Frautschi plot of the Veneziano amplitude. The asymptotic contours have been continued as dashed lines down to low energy. Every tenth Regge trajectory, and their resonances at one energy, are shown.

Half of all the resonances occur in a ring between

$$l = 0.69\sqrt{N} \text{ and } 1.42\sqrt{N}$$

or

$$b = 0.27 \text{ to } 0.56 \text{ F},$$

centered around

$$b = 0.42 \text{ F}.$$

In Fig. 4 we present the Chew-Frautschi plot with the parabolic contours which enclose 50% and 90% of the resonances. The concentration of resonances at low spin is evident.

The average spin of the resonances given by Eq. (2.18) is

$$\langle l\rangle = \int_0^N dl\rho(l)$$

giving[19]

$$\langle l\rangle \approx 1.1\sqrt{N}$$

or

$$\langle b\rangle \approx 0.43 \text{ F}$$

for the average radius.

Finally, we can use our approximation to evaluate the moments of the distribution. For example, Eq. (2.14) gives

$$\begin{aligned}\langle L_z^2\rangle &= \int_{-N}^{N} \sigma_N(l_z) l_z^2 dl_z \\ &\approx \frac{1}{4T}\int_{-N}^{N} dl_z l_z^2 \operatorname{sech}^2\left(\frac{l_z}{2T}\right) \\ &\approx \tfrac{1}{2}N \text{ as } N\to\infty,\end{aligned}$$

as previously found by Corgnier and D'Adda.

E. Average Number of Excited Quanta

For later use, we give the average number of oscillator quanta excited at $\alpha(s)=N$. It was obtained by Fritzsch,[13] from the Bose distribution, and this eventually also follows from our Eq. (2.7), with the operator A equal to

$$\mathfrak{N} = \sum_{k=1}^{\infty} (\mathfrak{N}_+^{(k)} + \mathfrak{N}_-^{(k)} + \mathfrak{N}_z^{(k)} + \mathfrak{N}_0^{(k)}),$$

the total-number operator. The average is

$$\begin{aligned}\langle\mathfrak{N}\rangle_N &= 4\sum_{k=1}^{\infty}\frac{1}{e^{k/T}-1} \\ &\sim 4T\ln T \text{ as } T\to\infty.\end{aligned} \tag{2.22}$$

III. DISCUSSION

A. Partial-Wave Couplings

The form of the spectrum $\rho_N(l)$ has implications for the couplings of the external particles to the individual internal resonances in the lth partial wave of a two-body scattering amplitude. For simplicity, we take the $2\to2$ scattering amplitude $-B(-\alpha(s), -\alpha(t))$, for which the residue of the resonance poles in the lth partial wave at $\alpha(s)=N$ is

$$\beta_N(l) = \frac{1}{2\alpha'}\int_{-1}^{1}\frac{[\alpha(t)+1][\alpha(t)+2]\cdots[\alpha(t)+N]}{N!}P_l(z_s)dz_s. \tag{3.1}$$

The behavior of $\beta_N(l)$ is rather complicated, and has been investigated (as $N\to\infty$) by Nambu and Frampton[20] for the Lovelace amplitude[21] for

$$\pi^+\pi^-\to\pi^+\pi^- \text{ and } \frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))}.$$

We adapt their results to our amplitude for scalar bosons. For small l, ($l \ll N$), $\beta_N(l)$ is independent of l, and varies mainly as a power of N:

$$\begin{aligned}&\beta_N(l)\sim \text{const}\frac{N^d}{(\ln N)^\epsilon}, \\ &d = \max\{\alpha(0)-1,\ -3-2\alpha(0)\},\end{aligned} \tag{3.2}$$

and $\epsilon = 1$ or 2.[20] At fixed N, this form remains approximately, up to $l\sim\sqrt{N}\ln N$, and then $\beta_N(l)$ decreases exponentially.[20] This cutoff in l is similar to what we find for $\bar\rho_N(l)$, within a logarithmic factor[22] [see Fig. 2 and Eq. (2.18)], and is a consequence of the fact that the strong interactions have a finite range.

If we define $g_N(l)$ to be the average coupling of a spin-l resonance to the external particles, then $\beta_N(l)$ is related to $\bar{\rho}_N(l)$ by

$$n(N)\bar{\rho}_N(l)g_N^2(l)=\beta_N(l). \tag{3.3}$$

Comparing Eq. (3.2) with Eq. (2.18), we see that $\bar{\rho}_N(l)$ and $\beta_N(l)$ behave rather similarly, except for two effects which are relatively small: (a) the linear suppression of low-angular-momentun states and (b) the lack of a logarithmic dependence in the "radius" of $\bar{\rho}_N(l)$.[22]

From this comparison, it follows that the coupling $g_N(l)$ varies only rather slowly with l. It seems that the Veneziano model chooses to implement the characteristic form for its profile function almost entirely through structure in the spectrum rather than in the couplings.

It is interesting to speculate on the possibility that the unitarity corrections would act in a similar way, e.g., absorption effects would be enforced through the disappearance of the low-lying states from the spectrum.

B. The Statistical Model

In the Introduction we remarked on the similarity between the statistical bootstrap model[6] and the dual-resonance model, in their predictions for the total level density, and possibly also for the predominant decay mode of high-spin massive resonances.[10] The angular momentum distribution of the statistical bootstrap is not yet known, but there is an analogous problem in nuclear physics[23] (although it is of course not a bootstrap scheme). This was first stressed by Lovelace.[9]

The nucleons are regarded as a free Fermi gas, confined to a spherical volume of radius R, with equally spaced energy levels (spacing δ) into which the individual nucleons can be excited. The total level density is found by counting all possible combinations of excitations, and is evaluated exactly as in the Veneziano model.[3] The density of levels at excitation energy E is[23]

$$n(E)\sim E^{-1}\exp\left[2\frac{\pi}{\sqrt{6}}\left(\frac{E}{\delta}\right)^{1/2}\right] \tag{3.4}$$

and the temperature is given by

$$E=\frac{\pi^2}{6}T^2\delta\,. \tag{3.5}$$

Here, E is of course the eigenvalue of the nuclear Hamiltonian. Expressions (3.4) and (3.5) should be compared with the corresponding Veneziano case, Eqs. (1.1) and (2.6). With our remarks of Sec. II B,

$$N\to E\,,$$
$$(\alpha')^{-1}\to\delta\,,$$

and we have an exact correspondence between the two cases (when the Veneziano oscillator dimension $D=1$).

Further, the orbital angular momentum distribution of the excited nucleus is given by[22]

$$\bar{\rho}(l)=\frac{(l+\frac{1}{2})e^{-(l+1/2)^2/2\gamma T}}{[2\pi(\gamma T)^3]^{1/2}}\;. \tag{3.6}$$

This is also given by considering the addition of the components l_z as a random walk.[24] Here,

$$\hbar\gamma=\tfrac{2}{5}m_N AR^2$$

the rigid-body moment of inertia of a nucleus of A nucleons of mass m_N.

The distribution (3.6) is somewhat similar to the Veneziano case (2.17): linear for small l, and with a sharp cutoff for large l. The shape of this cutoff is, however, different in the two cases: The Veneziano case does not have the Gaussian falloff characteristic of random-walk problems, but only an exponential. In the nuclear case, the cutoff occurs at

$$l_c\sim(\gamma T)^{1/2}$$
$$\sim(AT)^{1/2}.$$

If we reinterpret this in terms of the variables of the Veneziano model, we may expect that [Eq. (2.22)] for a highly excited nucleus

$$A\to\langle\mathfrak{N}\rangle\sim\sqrt{N}\ln N,$$
$$T\to\sim\sqrt{N},$$

so that

$$l_c\to\sim\sqrt{N}(\ln N)^{1/2}.$$

This is similar to the Veneziano case, Eq. (2.20), within the logarithmic factor.[22] So there is an intriguing analogy between the results of the Veneziano and statistical kinds of model, and it would be interesting to derive results for the statistical bootstrap model of hadrons to compare with our results for the Veneziano model.

ACKNOWLEDGMENTS

One of the authors (A. S.) would like to thank Professor M. Gell-Mann for the hospitality extended to him at the California Institute of Technology, where part of this work was performed. We gratefully acknowledge useful discussions with L. Caneschi, S. Frautschi, G. Veneziano, and M. Virasoro.

*Work supported in part by the U. S. Atomic Energy Commission. Prepared under Contract No. AT(11-1)-68 for the San Francisco Operations Office, U. S. Atomic Energy Commission.

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[3]S. Fubini and G. Veneziano, Nuovo Cimento 64A, 84 (1969); K. Bardakci and S. Mandelstam, Phys. Rev. 184, 1640 (1969).

[4]K. Huang and S. Weinberg, Phys. Rev. Letters 25, 895 (1970).

[5]D is the dimension of the harmonic oscillators of the Veneziano model (Ref. 11). Unequal intercepts for the Regge trajectories can be described by increasing D above four. See P. Olesen, Nucl. Phys. B18, 459 (1970); and C. Lovelace, Ref. 9.

[6]R. Hagedorn, Nuovo Cimento Suppl. 3, 147 (1965).

[7]This is an empirical value obtained by applying the model to large-angle scattering.

[8]S. C. Frautschi, Phys. Rev. D 3, 2821 (1971), has slightly modified Hagedorn's original argument, so obtaining a bootstrap condition on b, as well as the exponential form.

[9]This has previously been pointed out by C. Lovelace, CERN Report No. CERN-TH-1123 (unpublished), presented at the Regge Pole Conference, Irvine, California, 1969; Proc. Roy. Soc. (London) 318, 321 (1970).

[10]Possible further evidence for this is provided by the fact that both models predict that high-mass resonances decay preferentially into one low-mass and one high-mass particle. For the statistical model, see S. C. Frautschi, Ref. 8; for the dual-resonance model, see Chan Hong-Mo and S. T. Tsou, Phys. Rev. D 4, 156 (1971). However, in the latter case, this is only established for the first few leading trajectories. The situation for those resonances with very low spin ($l \lesssim \sqrt{N}$) is not clear.

[11]S. Fubini, D. Gordon, and G. Veneziano, Phys. Letters 29B, 679 (1969); Y. Nambu, University of Chicago Report No. EFI 69-64, 1969 (unpublished).

[12]L. Corgnier and A. D'Adda, Nuovo Cimento 64A, 253 (1970).

[13]H. Fritzsch, Lett. Nuovo Cimento 4, 893 (1970). By applying statistical mechanics to the oscillators when regarded as a Bose gas, and treating the addition of their spins as a random walk, he obtained a Gaussian distribution for l_z, with $\langle l_z^2\rangle \sim \langle \mathfrak{N}\rangle_N$ contrary to Eq. (1.2). His result is incorrect because the individual quanta (steps in the random walk) are really indistinguishable bosons, so that the normal random-walk analysis is no longer applicable.

[14]L. N. Chang, P. G. O. Freund, and Y. Nambu, Phys. Rev. Letters 24, 628 (1970). We believe that the difference in the relation between $N=\alpha(s)$ and T used in this reference and by us is due to a different treatment of the normalization of the resonance contributions. A statistical formalism like our own has been used by Nambu, Ref. 11, to find the total level density, and in Ref. 13.

[15]In the particular case $\alpha(0)=1$ [the "Virasoro" case, see M. A. Virasoro, Phys. Rev. D 1, 2933 (1970)], Brower and Thorn have presented strong arguments that all the timelike scalars are eliminated [see R. C. Brower and C. B. Thorn, CERN Report No. CERN-TH-1293, 1971 (unpublished)]. In this case the coefficient 4 that appears in our formulas should be changed to 3, and our discussion continues to be valid.

[16]D. Amati, C. Bouchiat, and J. L. Gervais, Lett. Nuovo Cimento 2, 399 (1969).

[17]This is closely related to the ϑ_1 function. See E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge Univ. Press, Cambridge, England, 1965), Chap. 21.

[18]The approximation involves setting

$$1-2e^{-k/T}\cos\alpha+e^{-2k/T}\approx\alpha^2+\frac{k^2}{T^2}\quad\text{for }\frac{k}{T}\ll 1\,,\ \alpha\approx 0$$
$$\approx 1\quad\text{for }\frac{k}{T}\gg 1\,.$$

The function

$$C(T)=\prod_{k=1}^{k_{\max}}\left(\frac{T}{k}\right)^2,\quad\text{where }k_{\max}\propto T\,.$$

[19]This behavior was first observed to follow from local duality alone by M. Kugler, Phys. Rev. Letters 21, 570 (1968).

[20]Y. Nambu and P. Frampton, in *Quanta*, edited by P. G. O. Freund, C. J. Goebel, and Y. Nambu (Univ. of Chicago Press, Chicago, Illinois, 1969), p. 403.

[21]C. Lovelace, Phys. Letters 28B, 265 (1968).

[22]The difference in the $\ln N$ factor may be due to the approximations made.

[23]H. A. Bethe, Phys. Rev. 50, 332 (1936); Rev. Mod. Phys. 9, 69 (1937).

[24]T. Ericson, Advan. Phys. 4, 425 (1960).