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Analytic Structure of the Triple-Regge Vertex*

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We study the triple-Regge, helicity-pole and related asymptotic behaviors of the six-line amplitude using the dual-resonance model (DRM) as a guide. We show that the triple-Regge vertex in the (DRM) can be expressed as a sum of four terms, which we interpret as exhibiting the allowed tree-like configurations of singularities in the asymptotic channel invariants. These four terms have counterparts in the helicity-pole limit where there exist several distinct terms with differing powers of the asymptotic variable, i.e., several different helicity poles. As an application we investigate the interesting property of the (DRM) that the triple-Regge or helicity-pole contribution to single-particle production at zero momentum transfer vanishes for trajectory intercepts near unity. We trace this effect to a nonsense zero. This suggests a mechanism for fulfilling one of the unitarity requirements for general six-line amplitudes. We discuss the possible generality of our results.

I. INTRODUCTION

Interest in the triple-Regge vertex[1,2] was stimulated recently by the realization[3,4] that it has a practical application to the study of single-particle inclusive reactions. The application is based upon an expression, derived from unitarity for the six-line connected part, which relates a discontinuity[5,6] of the forward three-particle scattering amplitude for the reaction

$$a+b+x\rightarrow a+b+x \tag{1.1}$$

to the cross section for producing particle x in the reaction

$$a+b\rightarrow x+\text{anything} \tag{1.2}$$

(see Fig. 1). In the asymptotic limit corresponding to large squared energies $s=(p_a+p_b)^2$, large squared missing mass $M^2=(p_a+p_b-p_x)^2$, low inelasticity M^2/s, and fixed momentum transfer squared $t=(p_b+p_x)^2$, the triple-Regge analysis of the three-particle scattering amplitude suggests that the production cross section has the form

$$\frac{d\sigma}{dt\,d(M^2/s)}\sim\frac{1}{s}G_{aa}(M^2)^{\alpha_0}\left|\frac{s}{M^2}\right|^{2\alpha(t)}$$
$$\times f(t)\Gamma(-\alpha(t))^2|\xi(t)|^2G_{bx}(t)^2, \tag{1.3}$$

where α_0 is the $t=0$ intercept of the leading vacuum trajectory and $\alpha(t)$ is the leading trajectory coupling to the $b\bar{x}$ channel. Assuming factorizable residues, G_{aa} and $G_{bx}(t)$ are the two-particle–single-Regge vertex factors, $\xi(t)$ the signature factor for the trajectory $\alpha(t)$, and $f(t)$ is the missing-mass discontinuity of the triple-Regge vertex function.

The derivation of Eq. (1.3), starting with the assumption of triple-Regge behavior for the six-line amplitude, involves some interesting subtleties, which we discuss here. The conventional triple-O(2, 1) analysis[2,7,8] does not apply to the forward elastic three-particle amplitude. For one thing the triple-O(2, 1) limit requires that the invariants $(p_b-p_b')^2$ and $(p_x-p_x')^2$ be asymptotically large, where p_a', p_b', and p_x' are the momenta of the "outgoing" a, b, and x in (1.1). However, these invariants are zero in the forward configuration. In fact the helicity-pole limit of Jones, Low, and Young[9] provides a more suitable framework in which to discuss the asymptotic limit that leads to the result (1.3) for the production cross section. In this analysis the asymptotic behavior of the amplitude is determined by the location of poles in complex helicity as well as complex angular momentum.[10] We argue below, however, that the result of the triple-Regge analysis, naively continued to the forward configuration, agrees with the result of the helicity-pole analysis in the application to the production cross section.

We define in detail in Sec. II the triple-Regge asymptotic limit (TR limit) and helicity-pole asymptotic limit (HP limit) for the general scalar six-line amplitude in terms of both the cosines of scattering angles and the channel invariants. After decomposing the six-line amplitude with respect to the O(3)× O(3)×O(3) group, we discuss its asymptotic behavior. Of course such an analysis is suitable for describing the leading asymptotic behavior of the scattering amplitude only at the particle poles corresponding to the recurrences of the highest Regge trajectory. To discuss the asymptotic behavior away from the poles, it is necessary to resort to some sort of multiple Sommerfeld-Watson transformation or to an O(2, 1)-group decomposition.[2,7,8] We shall be especially interested in the continuation away from the poles and we shall study the explicit form it takes in the dual-resonance model[11] (DRM). It is interesting that some features of the DRM result appear already in the O(3) analysis of Sec. II.

In Secs. III and IV, we study the behavior of the DRM six-line amplitude B_6 in the triple-Regge and helicity-pole limits. We show in Sec. III that the triple-Regge vertex decomposes into a sum of four terms, which have a direct interpretation in terms of the allowed arrangement of singularities in the asymptotic channel invariants.[12] There are exactly four tree diagrams that can be constructed according to the criterion that each internal line denotes an asymptotic channel invariant having poles in the B_6 function (see Fig. 3, to be discussed in Sec. III). Each term in the vertex corresponds to one such tree diagram. Only one term contributes to the missing-mass discontinuity, which in the forward configuration gives the production cross section. The particle–double-Regge vertex of the B_5 function decomposes in an analogous way into a sum of two terms.[12,13] We discuss the B_5 function in Appendix A, since it is of great use as a guide to understanding the B_6 function.

The four-term expansion for the triple-Regge vertex has its counterpart in the helicity-pole asymptotic behavior, which we discuss in Sec. IV. In addition to a term contributing to discontinuities in M^2, giving the familiar expression (1.3) in the forward configuration, there are terms which have no singularities in M^2 in leading order.[14]

In the triple-Regge or helicity-pole case, if we examine the pole structure of the various terms in the variables t_0, t_1, and t_2, the squared masses of the Regge lines, another interesting property emerges. Each term has one or more "spurious" poles not associated with poles in the original B_6 function. However, the poles cancel among the various terms so that they do not give rise to unwanted singularities in the sum.[15]

In the DRM the function $f(t)$ in Eq. (1.3) is just[16,17]

$$f(t)=1/\Gamma(1+\alpha_0(0)-2\alpha(t))\ . \qquad (1.4)$$

An interesting feature of this function is that it has a zero at $2\alpha(t)-\alpha_0(0)=1$.[18] Thus, if we were to use the function (1.4) to describe the triple-Pomeranchukon vertex, the contribution to the production cross section at $t=0$ would be proportional to $1-\alpha_P(0)$, if the intercept of the Pomeranchukon trajectory is near 1.

The connection between a small quantity $1-\alpha_P(0)$ and a small triple-Pomeranchukon contribution to production cross sections was stressed recently by Abarbanel *et al.*[19] If $\alpha_P(0)=1$ and $f(0)\neq 0$, unitarity is violated. We have traced the result $f(0) \propto 1-\alpha_P(0)$ in the DRM to the presence of a nonsense zero. We show in Sec. V that the asymptotic limit leading to Eq. (1.3) selects, so to speak, the maximum helicity flip in the coupling of the trajectory α_0 to the channel described by the two Regge trajectories $\alpha(t)$. If only sense couplings are allowed, as in the DRM, then the triple-Pomeranchukon coupling at $t=0$ vanishes if $\alpha_P(0)=1$, since it involves coupling angular momentum 1 to helicity 2. A nonsense zero on a trajectory implies the absence of a fixed pole with singular residue at that angular momentum. As a further check in Sec. V we show that inserting a fixed pole can in fact eliminate the zero.

The DRM therefore suggests two possible general mechanisms for a vanishing triple-Pomeranchukon contribution to the production cross section. (i) The first is the trivial mechanism in which the full triple vertex has an over-all zero. In this case nonsense wrong-signature fixed poles are not ruled out. (ii) In the second the full triple vertex is not required to be intrinsically small but nonsense fixed poles with singular residues are excluded.

We conclude in Sec. VI with a discussion of a possible generalization of these results, making use of an extension of the Steinmann relation.[5,20] According to our extension, in multi-Regge asymptotic limits, amplitudes are assumed to have no simultaneous discontinuities in overlapping *asymptotic* channel invariants, whether they have positive or negative energy. We show in Appendix B that with a few general assumptions the double-Regge–single-particle vertex decomposes into two terms in a way analogous to the double-Regge vertex in the DRM.

II. ASYMPTOTIC LIMITS OF THE SIX-LINE AMPLITUDE

A. Introduction

Two steps are necessary in applying Regge theory to multiparticle scattering amplitudes. First, it is necessary to define an asymptotic limit, second, to specify the behavior of the amplitude in this limit.

The group-theoretical analysis of Toller and others[21] provides a natural basis for extending the definition of multiple partial-wave amplitudes from integral to complex angular momenta. Thus, it is most natural to define the Regge asymptotic limit in terms of limits on the appropriate group variables. A limit so defined is easily translated in terms of the traditional Mandelstam or channel invariants, although defining the limit in terms of the latter is not always well motivated. On the other hand, the asymptotic behavior of the scattering amplitude is most naturally expressed in terms of the channel invariants, rather than the group invariants. The reasons for this are twofold. First, if one uses real subgroups of the Lorentz group (the current method of choice) in constructing a group parametrization for the partial-wave decomposition, one must use different subgroups for different configurations of the momenta. [Thus in 2-2 amplitudes one uses O(3) for $t > 0$, and O(2, 1) for $t < 0$.] Kinematical singularities arise in the partial-wave amplitudes when momenta reach a critical configuration at which the group structure changes. Expressing the asymptotic behavior in terms of channel invariants permits a smooth connection among the various subgroup regimes. A second reason for using channel invariants for expressing asymptotic behavior is one which we discuss below (Secs. III and VI). We shall interpret the asymptotic phase in the channel invariants as reflecting the presence of physical threshold and particle poles in the channels to which they correspond. With a particular assumption about the allowed arrangement of these singularities in the asymptotic behavior, we obtain a definite statement about the asymptotic phases, which is not easily motivated from group theory alone.

In Sec. II we define various Regge asymptotic limits of the scalar six-point amplitude[22] shown in Fig. 2 by referring to a triple-O(3)-group decomposition of the amplitude. This decomposition is described in Sec. II B. Such a decomposition is the natural choice for describing a scattering process in which scalar particles A and A' form a resonance 0 of spin J_0 which subsequently decays into two resonances $\bar{1}$ and $\bar{2}$ of spins J_1 and J_2, which in turn decay into scalar particles B, B' and C, C', respectively. We choose the triple-O(3) framework for several reasons. It is one with which most readers are familiar. It serves as an adequate basis for defining and discussing the various asymptotic limits. It is necessary to begin with a discussion of the triple-Regge vertex at integral angular momenta in order to discuss the continuation of the vertex to complex angular momenta (and particularly to obtain the helicity dependence of the vertex). Indeed we shall find that the expressions for integral angular momenta given below already exhibit some of the basic structure of the DRM expressions for the triple-Regge and helicity-pole limits. However, our use of O(3) means that we can rigorously discuss only the leading asymptotic behavior of the amplitude at the poles corresponding to resonances on the leading Regge trajectory. For a discussion of the triple-O(2, 1) decomposition, we refer the reader to Refs. 2, 7, and 8.

In Secs. II C and II D we define the triple-Regge (TR) and helicity-pole (HP) asymptotic limits in terms of the angles in the O(3) decomposition and in terms of the channel invariants. We discuss the asymptotic behavior of the amplitude near the three poles corresponding to the resonances on the leading trajectories of the triple-Regge expansion. The asymptotic limit which leads to the expression (1.3) for the production cross section is a special case of the HP limit. In Sec. II E we discuss the relationship between the TR and HP limits and the circumstances under which the asymptotic behavior of the amplitude in one limit determines the behavior in the other.

B. Triple-O(3) Decomposition

We begin by defining the O(3)-group variables in terms of the components of the momenta in the rest frame of particle 0 (see Fig. 2). We introduce a symmetric notation to aid in reading the formulas of Secs. III and IV. We use a special notation for clarity in discussing particle production in Sec. V. The correspondence is found by comparing Figs. 1 and 2.

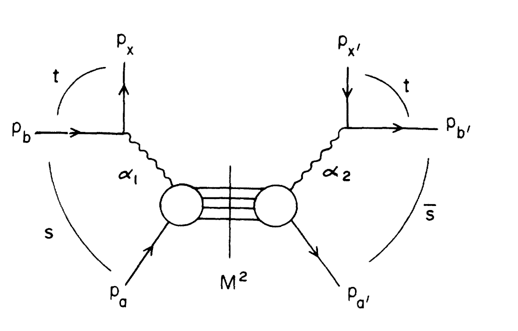

FIG. 1. Diagram showing the missing-mass discontinuity and notation for single-particle production.

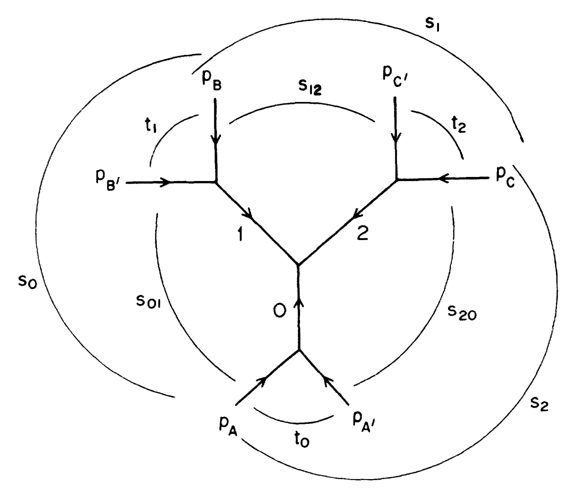

FIG. 2. Momentum diagram for the six-line amplitude showing the definition of channel invariants.

$$
\begin{aligned}
p_A &= (E_A, 0, k_A \sin\theta_0, k_A \cos\theta_0),\\
p_{A'} &= (E_{A'}, 0, -k_A \sin\theta_0, -k_A \cos\theta_0),\\
p_1 &= p_B + p_{B'} = (E_1, 0, 0, k),\\
p_2 &= p_C + p_{C'} = (E_2, 0, 0, -k),\\
p_B &= ((E_1 E_B + k k_B \cos\theta_1)/\sqrt{t_1}\,, k_B \sin\theta_1 \sin\phi_1, k_B \sin\theta_1 \cos\phi_1, (E_1 k_B \cos\theta_1 + k E_B)/\sqrt{t_1}\,),\\
p_C &= ((E_2 E_C - k k_C \cos\theta_2)/\sqrt{t_2}\,, k_C \sin\theta_2 \sin\phi_2, k_C \sin\theta_2 \cos\phi_2, (E_2 k_C \cos\theta_2 - k E_C)/\sqrt{t_2}\,).
\end{aligned}
\tag{2.1}
$$

We also define the channel invariants

$$
\begin{aligned}
s_{01} &= (p_A + p_{B'})^2,\\
s_{12} &= (p_B + p_{C'})^2,\\
s_{20} &= (p_C + p_{A'})^2,\\
s_0 &= (p_B + p_{C'} + p_A)^2 = (p_1 + p_A)^2,\\
s_1 &= (p_C + p_{C'} + p_B)^2 = (p_2 + p_B)^2,\\
s_2 &= (p_A + p_{A'} + p_C)^2 = (p_0 + p_C)^2,\\
t_0 &= (p_A + p_{A'})^2 = p_0^{\,2},\\
t_1 &= (p_B + p_{B'})^2 = p_1^{\,2},\\
t_2 &= (p_C + p_{C'})^2 = p_2^{\,2},
\end{aligned}
\tag{2.2}
$$

and the related quantities

$$
\eta_{01} = \frac{s_{01}}{s_0 s_1}, \quad
\eta_{12} = \frac{s_{12}}{s_1 s_2}, \quad
\eta_{20} = \frac{s_{20}}{s_2 s_0}.
\tag{2.3}
$$

The angle θ_0 is the usual center-of-mass (c.m.) system scattering angle for the process $A + A' \to \bar{1} + \bar{2}$. In the c.m. system for particle $\bar{1}$, θ_1 and ϕ_1 are the polar angles for particle B referred to a coordinate system in which particle $\bar{2}$ moves in the negative z direction and particle A in the y-z plane. Angles θ_2 and ϕ_2 are correspondingly defined in the c.m. system of particle $\bar{2}$. The energy and momentum components depend upon the particle masses $m_A^{\,2}$, $m_{A'}^{\,2}$, etc. and t_0, t_1, and t_2. For example,

$$
\begin{aligned}
k_A &= \lambda^{1/2}(t_0, m_A^{\,2}, m_{A'}^{\,2})/2\sqrt{t_0}\,,\\
E_A &= (m_A^{\,2} - m_{A'}^{\,2} + t_0)/2\sqrt{t_0}\,,\\
k &= \lambda^{1/2}(t_0, t_1, t_2)/2\sqrt{t_0}\,,\\
E_1 &= (t_2 - t_1 - t_0)/2\sqrt{t_0}\,,
\end{aligned}
\tag{2.4}
$$

where $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca$.

The scattering amplitude is defined completely by the eight Lorentz invariants t_0, t_1, t_2, θ_0, θ_1, θ_2, ϕ_1, and ϕ_2. The triple partial-wave decomposition

$$
A(t_0, t_1, t_2, \theta_0, \theta_1, \phi_1, \theta_2, \phi_2)
= \sum_{J_0, J_1, J_2, \lambda_1, \lambda_2} A^{J_0\,J_1 J_2}_{\lambda_1 \lambda_2}(t_0, t_1, t_2)\, d^{J_0}_{0,\lambda_1-\lambda_2}(\theta_0)
\times d^{J_1}_{\lambda_1,0}(\theta_1)\, d^{J_2}_{\lambda_2,0}(\theta_2)\, e^{-i(\lambda_1\phi_1 - \lambda_2\phi_2)}
\tag{2.5}
$$

defines the partial-wave amplitude. With the above conventions for the angles θ_0, θ_1, ϕ_1, θ_2, and ϕ_2, the residue of $A^{J_0 J_1 J_2}_{\lambda_1\lambda_2}$ at physical particle poles in t_1 and t_2 is proportional to the usual Jacob-Wick partial-wave helicity amplitude for the process $A + A' \to \bar{1} + \bar{2}$, where particles $\bar{1}$ and $\bar{2}$ have spins J_1 and J_2 and helicities λ_1 and λ_2 in the AA' c.m. system, respectively.

Near the three-pole point

$$
t_0 = m_0^{\,2}, \quad t_1 = m_1^{\,2}, \quad t_2 = m_2^{\,2},
\tag{2.6}
$$

corresponding to particles of spin J_0, J_1, and J_2, respectively, the scattering amplitude has the form

$$
\frac{1}{(t_0 - m_0^{\,2})(t_1 - m_1^{\,2})(t_2 - m_2^{\,2})}
\times \sum_{\lambda_0, \lambda_1, \lambda_2} \beta_{\lambda_0 \lambda_1 \lambda_2}\, \delta_{\lambda_0, \lambda_1 - \lambda_2}\, d^{J_0}_{0,\lambda_0}(\theta_0)
\times d^{J_1}_{\lambda_1,0}(\theta_1)\, d^{J_2}_{\lambda_2,0}(\theta_2)\, e^{-i(\lambda_1\phi_1 - \lambda_2\phi_2)}.
\tag{2.7}
$$

Let us reexpress this residue in terms of the channel invariants. Residues of poles in scalar amplitudes are Lorentz-invariant polynomials in components of momenta chosen from both "incoming" and "outgoing" clusters coupling to the pole. The various s invariants in (2.2) form a complete set of such "overlapping" channel invariants.[23] All other bilinear overlapping invariants are linearly related to these and the t's. Therefore, the residue is in general a polynomial in the s invariants in (2.2). Because the s invariants are polynomials in the cosines and sines of the angles, this statement is compatible with Eq. (2.7) provided the coefficients of the polynomials are adjusted and the degree of the polynomial is chosen appropriately. If we write the polynomial residue

$$R=\sum_{n_0,n_1,n_2}\sum_{n_{01},n_{12},n_{20}} C_{n_0,n_1,n_2;n_{01},n_{12},n_{20}} \times (s_{01})^{n_{01}}(s_{12})^{n_{12}}(s_{20})^{n_{20}} \times (s_0)^{n_0}(s_1)^{n_1}(s_2)^{n_2}\,, \tag{2.8}$$

a quick comparison with (2.7) shows that if

$$\begin{aligned} &n_{01}+n_0+n_{20}\le J_0\,,\\ &n_{12}+n_1+n_{01}\le J_1\,,\\ &n_{20}+n_2+n_{12}\le J_2\,, \end{aligned} \tag{2.9}$$

then the maximum power of the cosθ's and sinθ's is not exceeded. In fact it is also guaranteed that powers of cosϕ never exceed those in (2.7).

C. Triple-Regge Asymptotic Limit

The conventional triple-Regge asymptotic limit (TR limit) is defined as follows:

$$\begin{aligned} &\cos\theta_0,\ \cos\theta_1,\ \cos\theta_2\to\infty\,;\\ &t_0,\ t_1,\ t_2,\ \phi_1,\ \phi_2 \text{ fixed.} \end{aligned} \tag{2.10}$$

To leading order in $\cos\theta_1$ and $\cos\theta_2$, the channel invariants are

$$\begin{aligned} s_{01}&\sim -2p_A\cdot p_B\sim \frac{2\lambda^{1/2}(t_0,m_A{}^2,m_{A'}{}^2)}{2\sqrt{t_0}}\ \frac{\lambda^{1/2}(t_1,m_B{}^2,m_{B'}{}^2)}{2\sqrt{t_1}}\left[\frac{t_2-t_0-t_1}{2\sqrt{t_0t_1}}+\cos\phi_1\right]\cos\theta_0\cos\theta_1\,,\\ s_{12}&\sim -2p_B\cdot p_C\sim \frac{2\lambda^{1/2}(t,m_B{}^2,m_{B'}{}^2)}{2\sqrt{t_1}}\ \frac{\lambda^{1/2}(t_2,m_C{}^2,m_{C'}{}^2)}{2\sqrt{t_2}}\left[\frac{t_0-t_1-t_2}{2\sqrt{t_1t_2}}+\cos(\phi_1-\phi_2)\right]\cos\theta_1\cos\theta_2\,,\\ s_{20}&\sim -2p_A\cdot p_C\sim \frac{2\lambda^{1/2}(t_0,m_A{}^2,m_{A'}{}^2)}{2\sqrt{t_0}}\ \frac{\lambda^{1/2}(t_2,m_C{}^2,m_{C'}{}^2)}{2\sqrt{t_2}}\left[\frac{t_1-t_2-t_0}{2\sqrt{t_0t_2}}+\cos\phi_2\right]\cos\theta_0\cos\theta_2\,,\\ s_0&\sim 2p_1\cdot p_A\sim \frac{-2\lambda^{1/2}(t_0,m_A{}^2,m_{A'}{}^2)}{2\sqrt{t_0}}\ \frac{\lambda^{1/2}(t_0,t_1,t_2)}{2\sqrt{t_0}}\cos\theta_0\,,\\ s_1&\sim 2p_2\cdot p_B\sim \frac{-2\lambda^{1/2}(t_0,m_B{}^2,m_{B'}{}^2)}{2\sqrt{t_1}}\ \frac{\lambda^{1/2}(t_0,t_1,t_2)}{2\sqrt{t_1}}\cos\theta_1\,,\\ s_2&\sim 2p_0\cdot p_C\sim \frac{-2\lambda^{1/2}(t_0,m_C{}^2,m_{C'}{}^2)}{2\sqrt{t_2}}\ \frac{\lambda^{1/2}(t_0,t_1,t_2)}{2\sqrt{t_2}}\cos\theta_2\,. \end{aligned} \tag{2.11}$$

The η's are

$$\begin{aligned} \eta_{01}&\sim\frac{t_2-t_0-t_1+2\sqrt{t_0t_1}\,\cos\phi_1}{\lambda(t_0,t_1,t_2)}\,,\\ \eta_{12}&\sim\frac{t_0-t_1-t_2+2\sqrt{t_1t_2}\,\cos(\phi_1-\phi_2)}{\lambda(t_0,t_1,t_2)}\,,\\ \eta_{20}&\sim\frac{t_1-t_0-t_2+2\sqrt{t_0t_2}\,\cos\phi_2}{\lambda(t_0,t_1,t_2)}\,. \end{aligned} \tag{2.12}$$

These results are easily obtained from (2.2), (2.3), and (2.4). Thus, to leading order in the cosines, the TR limit is reached by taking

$$\begin{aligned} \text{TR}:\ &s_{01},\ s_{12},\ s_{20},\ s_0,\ s_1,\ s_2\to\infty\,;\\ &\eta_{01},\ \eta_{12},\ \eta_{20},\ t_0,\ t_1,\ t_2 \text{ fixed.} \end{aligned} \tag{2.13}$$

Note that the three η's are not independent. Any five of the six momentum vectors defining the amplitude must be linearly related, since the momentum space is four-dimensional. This fact is automatically accommodated in the O(3)-group parametrization, but imposes a special nonlinear constraint on the channel invariants. In the TR asymptotic limit the constraint can be obtained simply by solving (2.12) for ϕ_1 and ϕ_2 in terms of η_{01} and η_{20}, and then substituting the result into the expression for η_{12}.

The TR behavior of the amplitude near the three poles (2.6) can now be reexpressed in terms of the invariants. The polynomial residue (2.8) can be written as a polynomial in the η's and s_0, s_1, s_2:

$$R=\sum C_{n_0,n_1,n_2;n_{01},n_{12},n_{20}}(\eta_{01})^{n_{01}}(\eta_{12})^{n_{12}}(\eta_{20})^{n_{20}} \times (s_0)^{n_0+n_{01}+n_{20}}(s_1)^{n_1+n_{01}+n_{12}}(s_2)^{n_2+n_{12}+n_{20}}. \tag{2.14}$$

Because of the special relationship between the channel invariants and the cosines in this limit [see Eqs. (2.11) and (2.12)], the polynomial (2.14) still corresponds to a polynomial in the sines and cosines as it should. Taking the TR limit (2.13) selects the maximum powers of s_0, s_1, and s_2. From the constraint (2.9) we see that these are just J_0, J_1, and J_2, respectively. Therefore,

$$R=\gamma(\eta_{01},\eta_{12},\eta_{20})(s_0)^{J_0}(s_1)^{J_1}(s_2)^{J_2}\,, \tag{2.15}$$

where

$$\gamma(\eta_{01},\eta_{12},\eta_{20})=\sum_{n_{01},n_{12},n_{20}} C_{J_0-n_{01}-n_{20},\,J_1-n_{01}-n_{12},\,J_2-n_{20}-n_{12};\,n_{01},n_{12},n_{20}}(\eta_{01})^{n_{01}}(\eta_{12})^{n_{12}}(\eta_{20})^{n_{20}}\,. \tag{2.16}$$

The sum is constrained by (2.9) so that

$$n_{01}+n_{20}\leq J_0\,,\quad n_{12}+n_{01}\leq J_1\,,\quad n_{20}+n_{12}\leq J_2\,. \qquad (2.17)$$

We will find it useful to rewrite the polynomial (2.16) so as to exhibit the maximum powers of the η's. There are four different expressions depending on the relative size of the J's. If for all i, j, k,

$$J_i+J_j\geq J_k\,, \qquad (2.18)$$

then we define

$$i_0=J_0-n_{01}-n_{20}\,,\quad i_1=J_1-n_{12}-n_{01}\,,\quad i_2=J_2-n_{20}-n_{12}\,, \qquad (2.19)$$

and write

$$\gamma=\sum_{i_0,i_1,i_2}C_{i_0,i_1,i_2}(\eta_{01})^{(J_0+J_1-J_2-i_0-i_1+i_2)/2}\times(\eta_{12})^{(J_1+J_2-J_0-i_1-i_2+i_0)/2}\times(\eta_{20})^{(J_2+J_0-J_1-i_2-i_0+i_1)/2}\,, \qquad (2.20)$$

where the sum over the i's begins at zero and runs over all positive values for which none of the η's has a negative or nonintegral power. If $J_1+J_2\leq J_0$, then we write

$$\gamma=\sum_{i_1,i_2,n_{12}}C^0_{i_1,i_2,n_{12}}(\eta_{01})^{J_1-i_1}(\eta_{20})^{J_2-i_2}\left(\frac{\eta_{20}\eta_{01}}{\eta_{12}}\right)^{-n_{12}}, \qquad (2.21)$$

where i_1 and i_2 are defined as before. The third and fourth expressions are similarly defined for the cases $J_0+J_2\leq J_1$ and $J_0+J_1\leq J_2$.

The existence of these four expressions is closely related to a four-term expansion at general nonintegral angular momentum J_i, as we shall see in Sec. III.

D. Helicity-Pole Asymptotic Limit

The helicity-pole asymptotic limit (HP limit) is defined in terms of the O(3) variables as follows:

$$\cos\theta_0,\ \cos\phi_1,\ \cos\phi_2\to\infty\,, \qquad (2.22)$$
$$\cos\theta_1,\ \cos\theta_2,\ t_0,\ t_1,\ t_2,\ \cos\phi_1/\cos\phi_2 \text{ fixed.}$$

The restriction $\cos\phi_1\propto\cos\phi_2$ can be lifted to define a more general asymptotic limit. However, for the application to the forward elastic 3-to-3 amplitude, the limit (2.22) is sufficient. With this restriction it is possible to keep $\cos(\phi_1-\phi_2)$ fixed. As a consequence, in terms of the channel invariants the HP limit is

$$\text{HP}:\ s_{01},\ s_{20},\ s_0,\ \eta_{01},\ \eta_{20}\to\infty\,;$$
$$s_1,\ s_2,\ s_{12},\ t_0,\ t_1,\ t_2,\ \eta_{12},\ \text{fixed};$$
$$\eta_{01}/\eta_{20},\ s_{01}/s_{20},\ \text{fixed}. \qquad (2.23)$$

This limit can be applied to the forward amplitude (Fig. 1) by putting

$$t_0=s_{12}=0\,,\quad t_1=t_2=t\,,\quad s_1=s_2=m_b{}^2\,,\quad s_{01}=s\,,\quad s_{20}=\bar{s}\,,\quad s_0=M^2\,, \qquad (2.24)$$

where the variables on the right are defined and this application is discussed further in Sec. V.

In the HP limit the channel invariants are related to the cosines as follows:

$$s_{01}\sim\frac{2\lambda^{1/2}(t_0,m_A{}^2,m_{A'}{}^2)}{2\sqrt{t_0}}\,\frac{\lambda^{1/2}(t_1,m_B{}^2,m_{B'}{}^2)}{2\sqrt{t_1}}\times\cos\theta_0\cos\theta_1\cos\phi_1\,,$$
$$s_{20}\sim\frac{2\lambda^{1/2}(t_0,m_A{}^2,m_{A'}{}^2)}{2\sqrt{t_0}}\,\frac{\lambda^{1/2}(t_2,m_C{}^2,m_{C'}{}^2)}{2\sqrt{t_2}}\times\cos\theta_0\cos\theta_2\cos\phi_2\,, \qquad (2.25)$$
$$s_0\sim\frac{-2\lambda^{1/2}(t_0,m_A{}^2,m_{A'}{}^2)}{2\sqrt{t_0}}\,\frac{\lambda^{1/2}(t_0,t_1,t_2)}{2\sqrt{t_0}}\cos\theta_0\,,$$
$$\eta_{01}\sim-\frac{\lambda^{1/2}(t_1,m_B{}^2,m_{B'}{}^2)/2\sqrt{t_1}}{\lambda^{1/2}(t_0,t_1,t_2)/2\sqrt{t_0}}\,\frac{\cos\theta_1\cos\phi_1}{s_1}\,,$$
$$\eta_{20}\sim-\frac{\lambda^{1/2}(t_2,m_C{}^2,m_{C'}{}^2)/2\sqrt{t_2}}{\lambda^{1/2}(t_0,t_1,t_2)/2\sqrt{t_0}}\,\frac{\cos\theta_2\cos\phi_2}{s_2}\,.$$

Since s_{01}, s_{20}, and s_0 are proportional to cosines and sines in this limit, the polynomial residue (2.8) can now be written

$$R=\sum C_{n_0,n_1,n_2;n_{01},n_{12},n_{20}}(s_{01}/s_0)^{n_{01}}(s_{20}/s_0)^{n_{20}}\times(s_{12})^{n_{12}}(s_0)^{n_0+n_{01}+n_{20}}(s_1)^{n_1}(s_2)^{n_2}\,. \qquad (2.26)$$

The HP limit selects the maximum powers of s_0 and s_{01}/s_0. Since s_{01}/s_{20} is fixed in this limit, the leading term is the one with the maximum value of $n_0+n_{01}+n_{20}$ and $n_{01}+n_{20}$ subject to the constraints (2.9). If $J_1+J_2\leq J_0$, the leading asymptotic term is just

$$C_{J_0-J_1-J_2,0,0;J_1,0,J_2}(s_{01}/s_0)^{J_1}(s_{20}/s_0)^{J_2}(s_0)^{J_0}\,. \qquad (2.27)$$

If $J_1+J_2\geq J_0$, it is a polynomial

$$\sum_{n_1,n_2}\sum_{n_{12},n_{01}}C_{0,n_1,n_2:n_{01},J_0-n_{01},n_{12}}\times(s_{01}/s_0)^{n_{01}}(s_{20}/s_0)^{J_0-n_{01}}\times(s_0)^{J_0}(s_{12})^{n_{12}}(s_1)^{n_1}(s_2)^{n_2}\,, \qquad (2.28)$$

with the constraints

$$\begin{aligned} &n_1 + n_{12} + n_{01} \leq J_1 , \\ &n_2 + n_{12} + (J_0 - n_{01}) \leq J_2 , \\ &n_{01} \leq J_0 . \end{aligned} \tag{2.29}$$

Thus in the HP limit there are *two* types of contributions to the asymptotic behavior depending on the relative size of J_0 and $J_1 + J_2$. The second term has no dependence on s_0 $(= M^2)$.

The existence of these two distinct contributions has a close correspondence to the two-term expansion of the DRM in the helicity-pole limit for arbitrary nonintegral angular momenta as we shall see in Sec. IV.

E. Relationship Between the Limits

Although they are quite different limits, the TR and HP limits can be related to a common asymptotic limit. This helps to draw a connection between them. Starting with the TR limit (2.13), we take η_{01} and η_{20} to infinity so that η_{12} is fixed. This defines a hybrid limit, the TR-HP limit:

$$\begin{aligned} \text{TR-HP}: \; &s_{01}, s_{12}, s_{20}, s_0, s_1, s_2, \eta_{01}, \eta_{20} \to \infty ; \\ &\eta_{12}, t_0, t_1, t_2, s_{01}/s_{20}, \;\eta_{01}/\eta_{20}, \; \text{fixed.} \end{aligned} \tag{2.30}$$

Assuming the order of limits can be interchanged, one obtains the same limit starting with the HP limit (2.23) and taking s_1, s_2, and s_{12} to infinity with η_{12} fixed. In fact the order of limits can be interchanged in the DRM, giving the same result.

Comparing the HP behavior (2.27) and (2.28) with the TR behavior (2.20), (2.21), and its analogs, we see that they both lead to the same TR-HP expression. For $J_1 + J_2 \leq J_0$ Eq. (2.21) yields, in the limit $\eta_{01}, \eta_{20} \to \infty$ and η_{01}/η_{20}, η_{12} fixed, the expression

$$C^0_{000}\,(s_0)^{J_0} (s_{01}/s_0)^{J_1} (s_{20}/s_0)^{J_2} . \tag{2.31}$$

This is the same as the expression (2.27), since $C^0_{000} = C_{J_0 - J_1 - J_2, 0, 0 ; J_1, 0, J_2}$. The other three TR expressions [(2.20) and the analogs to (2.21)] correspond to the second HP expression (2.28) for $J_1 + J_2 \geq J_0$. All expressions yield

$$\begin{aligned} \sum_{n_{12}, n_{01}} &C_{0, J_1 - n_{01} - n_{12}, J_2 - n_{12} - J_0 + n_{01} ; n_{01}, n_{12}, J_0 - n_{01}} \\ &\times (s_{01}/s_0 s_1)^{n_{01}} (s_{20}/s_0 s_2)^{J_0 - n_{01}} (s_{12}/s_1 s_2)^{n_{12}} \\ &\times (s_0)^{J_0} (s_1)^{J_1} (s_2)^{J_2} , \end{aligned} \tag{2.32}$$

where

$$\begin{aligned} &n_{12} + n_{01} \leq J_1 , \\ &n_{12} + (J_0 - n_{01}) \leq J_2 , \\ &n_{01} \leq J_0 . \end{aligned} \tag{2.33}$$

It is interesting that Eq. (2.31), obtained by taking the HP limit of the TR behavior agrees with the HP behavior (2.27) without any additional manipulation. The TR-HP behavior applies for asymptotic s_1, s_2, and s_{12}. However, the leading term for $J_1 + J_2 \leq J_0$ is independent of these variables in the HP limit. When continued away from the poles in t_0, t_1, and t_2, it is the only term contributing to the discontinuity in $s_0 = M^2$ as we shall see in Sec. V. This explains why the TR analysis gives the same result as the HP analysis for the production cross section.

III. TRIPLE-REGGE VERTEX IN THE DUAL-RESONANCE MODEL

In this section we study the structure of the triple-Regge vertex in the DRM. We derive an expression for the vertex which exhibits its singularities in the η_{ij}. These singularities are shown to have a natural interpretation in terms of the singularity structure in the asymptotic invariants. We use this expression to discuss the threshold $[\lambda(t_0, t_1, t_2) = 0]$ behavior of the triple-Regge vertex and amplitude.

Mathematically speaking, the work in this and the following section can be regarded as a study of a certain generalization of the well-known hypergeometric functions to functions of several variables. For comparison, in Appendix A we have discussed these well-known functions in a manner corresponding step-by-step to that used below for their generalizations. At each step one can verify that the results here can be reduced to those of Appendix A by taking the residue of the pole at $\alpha_2 = 0$.

We first determine the triple-Regge vertex from the asymptotic limit of the six-point generalized beta function, B_6. A convenient expression for B_6 is

$$\begin{aligned} B_6 = \int_0^1 \int_0^1 \int_0^1 &dx_0 dx_1 dx_2 \, x_1^{-\alpha_1 - 1} (1 - x_1)^{-\alpha(s_{01}) - 1} x_0^{-\alpha(s_0) - 1} (1 - x_0)^{-\alpha_0 - 1} x_2^{-\alpha_2 - 1} (1 - x_2)^{-\alpha(s_{20}) - 1} \\ &\times (1 - x_0 x_1)^{-\alpha(s_1) + \alpha(s_{01}) + \alpha_0} (1 - x_0 x_2)^{-\alpha(s_2) + \alpha(s_{20}) + \alpha_0} (1 - x_0 x_1 x_2)^{-\alpha(s_{12}) + \alpha(s_1) + \alpha(s_2) - \alpha_0} , \end{aligned} \tag{3.1}$$

where $\alpha_i = \alpha(t_i)$. We first take the single asymptotic limit, $\cos\theta_0 \to \infty$ $(s_0, s_{01}, s_{20} \to \infty)$ making the usual exponentiation substitution $x_0 = 1 + y_0/\alpha(s_0)$.

$$B_6 \sim (-s_0)^{\alpha_0} \int_0^1 \int_0^1 dx_1 dx_2\, x_1^{-\alpha_1 - 1}(1-x_1)^{-\alpha(s_1)+\alpha_0-1}\; x_2^{-\alpha_2-1}(1-x_2)^{-\alpha(s_2)+\alpha_0-1}(1-x_1x_2)^{-\alpha(s_{12})+\alpha(s_1)+\alpha(s_2)-\alpha_0}$$
$$\times \int_0^\infty dy_0\, y_0^{-\alpha_0-1} \exp\left(-y_0 - \frac{s_{01}}{s_0}\frac{x_1}{1-x_1}y_0 - \frac{s_{20}}{s_0}\frac{x_2}{1-x_2}y_0\right). \tag{3.2}$$

Equation (3.2) is the starting point for both the triple-Regge and helicity-pole limits. In the triple-Regge limit, $\cos\theta_1$ and $\cos\theta_2 \to \infty$, we find[1,2]

$$B_6 \sim (-s_0)^{\alpha_0}(-s_1)^{\alpha_1}(-s_2)^{\alpha_2} \int_0^\infty \int_0^\infty \int_0^\infty dy_0 dy_1 dy_2\, y_0^{-\alpha_0-1} y_1^{-\alpha_1-1} y_2^{-\alpha_2-1} \exp(-y_0 - y_1 - y_2 + \eta_{01}y_0y_1 + \eta_{12}y_1y_2 + \eta_{20}y_2y_0)$$
$$\equiv (-s_0)^{\alpha_0}(-s_1)^{\alpha_1}(-s_2)^{\alpha_2}\, V(\alpha_0, \alpha_1, \alpha_2; \eta_{01}, \eta_{12}, \eta_{20}). \tag{3.3}$$

The integral representation (3.3) for V is defined for $\eta_{ij} < 0$. In order to continue to $\eta_{ij} > 0$, we need a different integral representation. To obtain such a representation we use successively the Mellin-Barnes integral[24]

$$\Gamma(-\alpha)(1-z)^\alpha = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dr\, \Gamma(-\alpha + r)\Gamma(-r)(-z)^r\,, \tag{3.4}$$

to obtain

$$V = \int_0^\infty \int_0^\infty dy_1 dy_2\, y_1^{-\alpha_1-1} y_2^{-\alpha_2-1} \exp(-y_1 - y_2 + \eta_{12}y_1y_2)\, \Gamma(-\alpha_0)(1 - \eta_{01}y_1 - \eta_{20}y_2)^{\alpha_0}$$
$$= \int_0^\infty \int_0^\infty dy_1 dy_2\, y_1^{-\alpha_1-1} y_2^{-\alpha_2-1} \exp(-y_1 - y_2 + \eta_{12}y_1y_2) \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dt\, \Gamma(-\alpha_0 + t)\Gamma(-t)(-\eta_{01})^t y_1^t \left(1 + \frac{\eta_{20}}{\eta_{01}}\frac{y_2}{y_1}\right)^t$$
$$= \int_0^\infty \int_0^\infty dy_1 dy_2\, y_1^{-\alpha_1-1} y_2^{-\alpha_2-1} \exp(-y_1 - y_2 + \eta_{12}y_1y_2)$$
$$\times \left(\frac{1}{2\pi i}\right)^2 \int_{-i\infty}^{i\infty} dp \int_{-i\infty}^{i\infty} dr\, \Gamma(-\alpha_0 + r + p)\Gamma(-p)\Gamma(-r) y_1^p y_2^r (-\eta_{01})^p(-\eta_{20})^r \quad (\text{setting } p = t - r)$$
$$= \left(\frac{1}{2\pi i}\right)^2 \int_0^\infty dy_2 \int_{-i\infty}^{i\infty} dp \int_{-i\infty}^{i\infty} dr\, y_2^{-\alpha_2 + r - 1} e^{-y_2}\, \Gamma(-\alpha_0 + r + p)\Gamma(-p)\Gamma(-r)\Gamma(-\alpha_1 + p)(1 - \eta_{12}y_2)^{\alpha_1 - p}(-\eta_{01})^p(-\eta_{20})^r$$
$$= \left(\frac{1}{2\pi i}\right)^3 \int_{-i\infty}^{i\infty} dp \int_{-i\infty}^{i\infty} dq \int_{-i\infty}^{i\infty} dr\, \Gamma(-\alpha_0 + r + p)\Gamma(-\alpha_1 + p + q)\Gamma(-\alpha_2 + q + r)\Gamma(-p)\Gamma(-q)\Gamma(-r)(-\eta_{01})^p(-\eta_{12})^q(-\eta_{20})^r. \tag{3.5}$$

The interchanges of integrals made in obtaining (3.5) are allowed for certain ranges of the α_i. Continuation to all α_i can then be made if the contours are distorted away from the singularities in the integrand. This leads to the prescription that the contours in (3.5) are to be taken so that the singularities in the first three Γ functions lie to the left of all contours whereas the singularities in the last three Γ functions lie to the right.

If the contours are shifted to the right in (3.5), an asymptotic expansion for V for small η_{ij} is obtained:

$$V \sim \sum_{i,j,k=0}^{\infty} \Gamma(-\alpha_0 + k + i)\Gamma(-\alpha_1 + i + j)\Gamma(-\alpha_2 + j + k)\frac{\eta_{01}^i}{i!}\frac{\eta_{12}^j}{j!}\frac{\eta_{20}^k}{k!}. \tag{3.6}$$

Although an identical expression is obtained by expanding the last three exponentials in (3.3) we cannot prove (3.5) by passing from (3.3) to (3.6) to (3.5). This is because (3.5) is an equality whereas (3.6) is only an asymptotic expansion since the contours cannot be *closed* in the right-half planes due to the behavior of the integrand, e.g., $\sim (p\eta_{12})^p$ as $|p| \to \infty$.

A more interesting and useful representation for V is obtained by closing the contours in a different manner: roughly speaking, in the left-half planes. We can first close the p contour in the left-half plane (for all η_{12}), since the integrand behaves like $(p\eta_{12})^p$ as noted above. We obtain two contributions, one from the poles in each of the first two Γ functions in the integrand. In the first contribution we can close the q contour to the left as the integrand behaves like q^q. From closing the contour we obtain two terms. In the one from the second Γ function we can close the r contour to the left (the integrand behaves like r^r). In the one from the third Γ function we can close the r contour to the right (the integrand behaves like r^{-r}). Similar manipulations can be made for the second contribution from closing the p contour. Care must be taken in these manipulations to take proper account of the effect of moving the contour in one variable on the singularities in the other variables. The net result of the contortions outlined above is

$$V=(-\eta_{01})^{\alpha_1}(-\eta_{20})^{\alpha_2}\sum_{i,j,k=0}^{\infty}\Gamma(-\alpha_1+i+k)\Gamma(-\alpha_2+j+k)\Gamma(\alpha_1+\alpha_2-\alpha_0-i-j-2k)\frac{1}{i!j!k!}\eta_{01}^{-i}\eta_{20}^{-j}\left(\frac{\eta_{20}\eta_{01}}{\eta_{12}}\right)^{-k}$$
$$+(-\eta_{12})^{\alpha_2}(-\eta_{01})^{\alpha_0}\sum_{i,j,k=0}^{\infty}\Gamma(-\alpha_2+i+k)\Gamma(-\alpha_0+j+k)\Gamma(\alpha_2+\alpha_0-\alpha_1-i-j-2k)\frac{1}{i!j!k!}\eta_{12}^{-i}\eta_{01}^{-j}\left(\frac{\eta_{01}\eta_{12}}{\eta_{20}}\right)^{-k}$$
$$+(-\eta_{20})^{\alpha_0}(-\eta_{12})^{\alpha_1}\sum_{i,j,k=0}^{\infty}\Gamma(-\alpha_0+i+k)\Gamma(-\alpha_1+j+k)\Gamma(\alpha_0+\alpha_1-\alpha_2-i-j-2k)\frac{1}{i!j!k!}\eta_{20}^{-i}\eta_{12}^{-j}\left(\frac{\eta_{12}\eta_{20}}{\eta_{01}}\right)^{-k}$$
$$+\tfrac{1}{2}(-\eta_{01})^{(\alpha_0+\alpha_1-\alpha_2)/2}(-\eta_{12})^{(\alpha_1+\alpha_2-\alpha_0)/2}(-\eta_{20})^{(\alpha_2+\alpha_0-\alpha_1)/2}$$
$$\times\sum_{i,j,k=0}^{\infty}\Gamma(\tfrac{1}{2}(\alpha_0-\alpha_1-\alpha_2-i+j+k))\Gamma(\tfrac{1}{2}(\alpha_1-\alpha_2-\alpha_0-j+i+k))\Gamma(\tfrac{1}{2}(\alpha_2-\alpha_0-\alpha_1-k+i+j))$$
$$\times\frac{(-1)^{i+j+k}}{i!j!k!}\left(-\frac{\eta_{20}\eta_{01}}{\eta_{12}}\right)^{-i/2}\left(-\frac{\eta_{01}\eta_{12}}{\eta_{20}}\right)^{-j/2}\left(-\frac{\eta_{12}\eta_{20}}{\eta_{01}}\right)^{-k/2}. \tag{3.7}$$

Each of the triply infinite sums in (3.7) is a holomorphic function in the finite planes of the variables raised to the powers i, j, and k [e.g., η_{01}^{-1}, η_{20}^{-1}, and $(\eta_{20}\eta_{01}/\eta_{12})^{-1}$ in the first term and $(-\eta_{20}\eta_{01}/\eta_{12})^{-1/2}$, etc. in the last term]. This should seem plausible from a cursory examination, since using Stirling's formula for the Γ functions and factorials shows that the coefficients of the first three sums behave like $i^{-i}j^{-j}k^{-k}$ and the last like $i^{-i/2}j^{-j/2}k^{-k/2}$. A more careful study shows that each sum is indeed uniformly convergent for all finite values of the variables when the α_i and the $\alpha_i+\alpha_j-\alpha_k$ are nonintegral. Uniform convergence assures holomorphy.

The four terms in (3.7) correspond one-to-one to the four terms in (2.20) and (2.21) for integral angular momentum. Away from the poles the four terms coexist simultaneously in the asymptotic behavior, whereas at the poles only one or two contribute according to the relative sizes of J_0, J_1, and J_2. This correspondence is very intriguing since it suggests that in multiple angular momentum continuations the multiple Sommerfeld-Watson transforms should be taken separately on the various terms satisfying the inequalities $J_i+J_j\le J_k$, etc. discussed in Sec. II. It would be very interesting to study whether or not this feature is more general than the DRM.

The origin of the spurious singularities[25] at integral values of $\alpha_i+\alpha_j-\alpha_k$ in the individual terms in (3.7) can be understood in the continuation away from integral angular momentum of the usual nonsense zeros present in the DRM. To see this, consider the residue of the double pole at $\alpha_1=J_1$ and $\alpha_2=J_2$ in (3.7). Only the first term contributes:

$$\mathrm{Res}_{\alpha_1=J_1;\alpha_2=J_2}V=\sum_{i,j,k;\,i+k\le J_1;\,j+k\le J_2}[i!j!k!(J_1-i-k)!(J_2-j-k)!]^{-1}$$
$$\times(-1)^{i+j}\Gamma(-\alpha_0+J_1+J_2-i-j-2k)\,\eta_{01}^{J_1-i-k}\eta_{20}^{J_2-j-k}\eta_{12}^{k}. \tag{3.8}$$

From (2.7) and (2.12) we see that the maximum helicity $\lambda_0=\pm(J_1+J_2)$ receives contributions from only $i=j=k=0$. The coefficient of this term is $\Gamma(-\alpha_0+J_1+J_2)$ and thus there are no poles in α_0 for nonsense states ($J_0<|\lambda_0|=J_1+J_2$). Similar arguments apply to helicities $|\lambda_0|<J_1+J_2$. Thus the Γ functions like $\Gamma(-\alpha_0+\alpha_1+\alpha_2)$ can be regarded as natural continuations of those like $\Gamma(-\alpha_0+J_1+J_2)$ away from the integers. We note that the η_{ij} also provide in a natural way the kinematic singularities of helicity amplitudes at $t_i=0$ and threshold [see Eq. (2.12)].

The four terms in (3.7) also have a very nice physical interpretation in terms of the singularity structure in the asymptotic variables. The cuts in the η_{ij} exhibited in (3.7) are asymptotic representations of the poles in the s_i and s_{ij}. Let us study the cut structure in the s_i and s_{ij} implied by the cuts in η_{ij} by combining (3.4) and (3.7) and using the definition of η_{ij} (2.3):

$$B_6\sim(-s_0)^{\alpha_0-\alpha_1-\alpha_2}(-s_{01})^{\alpha_1}(-s_{20})^{\alpha_2}\Gamma(-\alpha_1)\Gamma(-\alpha_2)\Gamma(\alpha_1+\alpha_2-\alpha_0)$$
$$+(-s_1)^{\alpha_1-\alpha_2-\alpha_0}(-s_{12})^{\alpha_2}(-s_{01})^{\alpha_0}\Gamma(-\alpha_2)\Gamma(-\alpha_0)\Gamma(\alpha_2+\alpha_0-\alpha_1)$$
$$+(-s_2)^{\alpha_2-\alpha_0-\alpha_1}(-s_{20})^{\alpha_0}(-s_{12})^{\alpha_1}\Gamma(-\alpha_0)\Gamma(-\alpha_1)\Gamma(\alpha_0+\alpha_1-\alpha_2)$$
$$+\tfrac{1}{2}(-s_{01})^{(\alpha_0+\alpha_1-\alpha_2)/2}(-s_{12})^{(\alpha_1+\alpha_2-\alpha_0)/2}(-s_{20})^{(\alpha_2+\alpha_0-\alpha_1)/2}\Gamma(\tfrac{1}{2}(\alpha_2-\alpha_0-\alpha_1))\Gamma(\tfrac{1}{2}(\alpha_0-\alpha_1-\alpha_2))\Gamma(\tfrac{1}{2}(\alpha_1-\alpha_2-\alpha_0)). \tag{3.9}$$

The terms in (3.9) correspond to cuts in the four possible combinations of nonoverlapping asymptotic variables – see Fig. 3. These are the only multiple-cut configurations allowed since the DRM does not have poles (asymptotically cuts) in overlapping variables. Thus asymptotically B_6 splits into a *sum* of terms corresponding to the various possible tree diagrams exhibiting singularities in the *asymptotic* variables. Duality forbids such a decomposition in the nonasymptotic region since the sum over poles in one variable will produce poles in overlapping variables. In (3.9) we have written only the leading terms in the sums in (3.7). The holomorphy of the infinite sums in the specified variables assures that they do not have singularities in variables other than those which already have singularities exhibited in (3.9). [They may (and probably do) have further singularities in those variables, however.] For example, since $(-\eta_{20}\eta_{01}/\eta_{12})^{-1/2} = s_0(-s_{12}/s_{20}s_{01})^{1/2}$, etc., the last sum has no singularities in s_0, s_1, and s_2 and thus the full contribution of the last term has singularities only in s_{01}, s_{12}, and s_{20}. In Sec. VI, we will discuss why we believe the existence of a decomposition of the form (3.9) may be a general property of scattering amplitudes independent of the DRM.

To conclude this section we use (3.7) to study the threshold behavior of the triple vertex. By threshold we mean $\sqrt{t_i} = \pm\sqrt{t_j} \pm \sqrt{t_k}$, i.e., $\lambda(t_0, t_1, t_2) = 0$, at fixed values of the group variables, ϕ_i. Since $\eta_{ij}^{-1} \propto \lambda$, the terms with $i=j=k=0$ in (3.7) dominate and V has four distinct possible behaviors:

$$V \sim \lambda^{-\alpha_1-\alpha_2},\ \lambda^{-\alpha_2-\alpha_0},\ \lambda^{-\alpha_0-\alpha_1},\ \lambda^{-(\alpha_0+\alpha_1+\alpha_2)/2}, \tag{3.10}$$

with the dominant behavior depending on the relative sizes of the α_i. The behavior is quite different from the behavior suggested by Goddard and White,[8] $V \sim \lambda^0$. Since we believe the form (3.7) is general, we expect the general vertex to behave like (3.10) rather than being constant.[26] We note that the first three terms in (3.10) are certainly necessary to give the correct threshold dependence for helicity amplitudes at the resonance poles in two of the Regge trajectories. The singularities in (3.10) arise because the limit $\lambda \to 0$ has been taken for fixed ϕ_i or, equivalently, fixed helicities λ_i. The amplitude, of course, has no singularity for $\lambda \to 0$ for fixed s_i and s_{ij}.

IV. HELICITY-POLE LIMIT IN THE DUAL-RESONANCE MODEL

In this section we study the helicity-pole limit (2.23) in the DRM. The asymptotic behavior of B_6 in this limit is calculated. The result is rather more complicated than that of the triple-Regge limit discussed in Sec. III, although it has a similar structure. This is due to the fact that only three variables are asymptotic in this case rather than six in the TR case (away from threshold and the forward direction). The further asymptotic limit $s_{12} \to \infty$ could be taken on our final expression (4.2) in order to obtain the general HP limit [see discussion after (2.22)], but we shall not perform the calculation here.

Our starting point is the single-Regge limit of B_6 given in Eq. (3.2). The limit of interest (2.23) is given by the further limit $s_{01}/s_0 \to \infty$, $s_{02}/s_0 \to \infty$. To extract this limit we again make successive use of the Mellin-Barnes formula (3.4). The steps below follow those leading to Eq. (3.5) of Sec. III:

$$\begin{aligned}
B_6 \sim{}& (-s_0)^{\alpha_0} \int_0^1 \int_0^1 dx_1 dx_2\, x_1^{-\alpha_1-1}(1-x_1)^{-\alpha(s_1)+\alpha_0-1} x_2^{-\alpha_2-1}(1-x_2)^{-\alpha(s_2)+\alpha_0-1}(1-x_1x_2)^{-\alpha(s_{12})+\alpha(s_1)+\alpha(s_2)-\alpha_0} \\
&\times \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dt\, \Gamma(-\alpha_0+t)\Gamma(-t)\left(\frac{s_{01}}{s_0}\frac{x_1}{(1-x_1)} + \frac{s_{20}}{s_0}\frac{x_2}{(1-x_2)}\right)^t \\
={}& (-s_0)^{\alpha_0} \int_0^1 \int_0^1 dx_1 dx_2\, x_1^{-\alpha_1+p-1}(1-x_1)^{-\alpha(s_1)+\alpha_0-p-1} x_2^{-\alpha_2+r-1}(1-x_2)^{-\alpha(s_2)+\alpha_0-r-1}(1-x_1x_2)^{-\alpha(s_{12})+\alpha(s_1)+\alpha(s_2)-\alpha_0} \\
&\times \left(\frac{1}{2\pi i}\right)^2 \int_{-i\infty}^{i\infty} dp \int_{-i\infty}^{i\infty} dr\, \Gamma(-\alpha_0+r+p)\Gamma(-p)\Gamma(-r)\left(\frac{s_{01}}{s_0}\right)^p\left(\frac{s_{20}}{s_0}\right)^r \\
={}& (-s_0)^{\alpha_0}\left(\frac{1}{2\pi i}\right)^3 \int_{-i\infty}^{i\infty} dp \int_{-i\infty}^{i\infty} dq \int_{-i\infty}^{i\infty} dr\, \frac{\Gamma(-\alpha_1+p+q)\Gamma(-\alpha(s_1)+\alpha_0-p)}{\Gamma(-\alpha(s_1)-\alpha_1+\alpha_0+q)}\,\frac{\Gamma(-\alpha_2+q+r)\Gamma(-\alpha(s_2)+\alpha_0-r)}{\Gamma(-\alpha(s_2)-\alpha_2+\alpha_0+q)} \\
&\times \frac{\Gamma(\alpha_0+\alpha(s_{12})-\alpha(s_1)-\alpha(s_2)+q)}{\Gamma(\alpha_0+\alpha(s_{12})-\alpha(s_1)-\alpha(s_2))}\,\Gamma(-\alpha_0+r+p)\Gamma(-p)\Gamma(-q)\Gamma(-r)(-1)^q\left(\frac{s_{01}}{s_0}\right)^p\left(\frac{s_{20}}{s_0}\right)^r .
\end{aligned} \tag{4.1}$$

To make the last step we have expanded the factor $(1-x_1x_2)$ and used the integral representation of the beta function.

The greater complexity of the forward limit is easily appreciated by comparing (4.1) with (3.5). However, we can still obtain an expression which exhibits the leading behavior as s_{01}/s_0, $s_{20}/s_0 \to \infty$ in much the same way as we obtained (3.7). We follow the same sequence of contour closings as there, but at each step we group together all contributions not analogous to the first term in (3.7). We thus obtain

$$
\begin{aligned}
B_6 \sim\, & (-s_0)^{\alpha_0}\left(\frac{s_{01}}{s_0}\right)^{\alpha_1}\left(\frac{s_{20}}{s_0}\right)^{\alpha_2}\sum_{i,j,k=0}^{\infty}\Gamma(-\alpha_1+i+k)\Gamma(-\alpha_2+j+k)\Gamma(\alpha_1+\alpha_2-\alpha_0-i-j-2k)\\
&\times\frac{\Gamma(-\alpha(s_1)-\alpha_1+\alpha_0+i+k)\Gamma(-\alpha(s_2)-\alpha_2+\alpha_0+j+k)}{\Gamma(-\alpha(s_1)-\alpha_1+\alpha_0+k)\Gamma(-\alpha(s_2)-\alpha_2+\alpha_0+k)}\\
&\times\frac{\Gamma(\alpha_0+\alpha(s_{12})-\alpha(s_1)-\alpha(s_2)+k)}{\Gamma(\alpha_0+\alpha(s_{12})-\alpha(s_1)-\alpha(s_2))\,i!\,j!\,k!}\left(-\frac{s_0}{s_{01}}\right)^{i+k}\left(-\frac{s_0}{s_{20}}\right)^{j+k}\\
&+\sum_{i=0}^{\infty}\int_0^1\int_0^1 dx_1dx_2\, x_1^{-\alpha_1-1}(1-x_1)^{-\alpha(s_1)-1}x_2^{-\alpha_2-1}(1-x_2)^{-\alpha(s_2)-1}(1-x_1x_2)^{-\alpha(s_{12})+\alpha(s_1)+\alpha(s_2)-\alpha_0}\\
&\times[(-s_{01})x_1(1-x_2)+(-s_{20})x_2(1-x_1)]^{\alpha_0}\frac{1}{i!}\Gamma(-\alpha_0+i)\left[\left(-\frac{s_{01}}{s_0}\right)\frac{x_1}{(1-x_1)}+\left(-\frac{s_{20}}{s_0}\right)\frac{x_2}{(1-x_2)}\right]^{-i},
\end{aligned}
\tag{4.2}
$$

where in the second term we have reintroduced the integral representation of the beta function. We expect the sums above to have singularities in s_0/s_{01} and s_0/s_{20} (in contrast to the sums of Sec. III), since the contour closings can only be carried out for $|s_0/s_{01}|, |s_0/s_{20}| < 1$. This is the case in the double-Regge special case discussed in Appendix A.

Nonleading terms have been kept in (4.2) so that the result can be compared to the triple-Regge result. Indeed, in the subsequent limit s_1, s_2, $s_{12} \to \infty$ with the η_{ij} fixed, (4.2) does go over into (3.3) with (3.7). The limit calculated here essentially corresponds to the triple-Regge limit with $\eta_{01}, \eta_{20} \to \infty$, i.e., the TR-HP limit (2.30). The comparison of the limits is particularly easy to make in the case $s_{01} \propto s_{20}$: The two terms in (4.2) are then easily seen to arise from the singularities in t from $\Gamma(-\alpha_0+t)$ and from singularities of the integral from $x_1 \approx x_2 \approx 0$; a similar analysis applies to the second equation of (3.5). We remark that the sum over i cannot be done in (4.2) to give a factor

$$
\Gamma(-\alpha_0)\left[1+\left(\frac{s_{01}}{s_0}\frac{x_1}{1-x_1}+\frac{s_{20}}{s_0}\frac{x_2}{1-x_2}\right)^{-1}\right]^{\alpha_0},
$$

since the series does not converge for $x_1 \approx x_2 \approx 0$; if the sum could be done one would recover the full amplitude B_6.

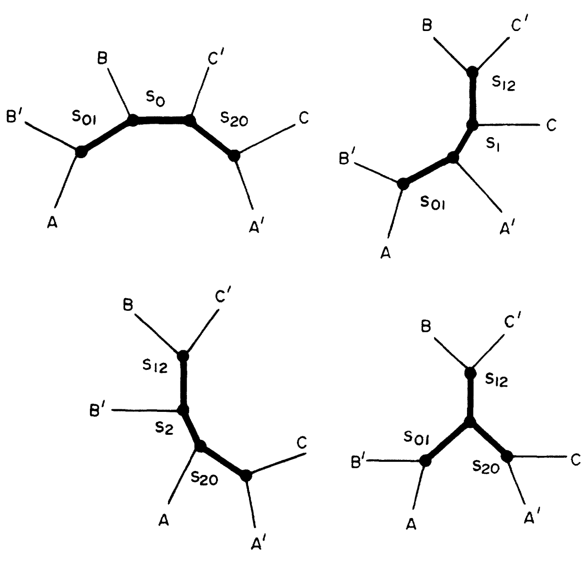

FIG. 3. Diagrammatic representation of asymptotic singularity structure of triple-Regge vertex.

We now discuss the interpretation of (4.2). It is impossible to define a "helicity-pole vertex" from (4.2.) for two reasons. First, there is more than one power of the asymptotic variables s_{01}/s_0 and s_{20}/s_0 present so there are several different singularities in the helicity plane[14] (specifically, poles at $\alpha_1+\alpha_2$ and α_0 for $s_{01} \propto s_{20}$). It would be more reasonable to define a vertex for each different singularity (asymptotic power). Second, and more important, the dependence on the external lines does not factor out in the second term in (4.2).[27] As we discuss in greater detail below, this term has singularities in the nonasymptotic variables s_1, s_2, and s_{12} which overlap the clusters attached to α_1 and α_2. Thus, for example, different "vertices" would be obtained for different numbers of lines in these clusters. However, in leading order only the first term has a discontinuity in s_0 and its dependence on the clusters factorizes off. As we will see in Sec. V, this has the important consequence that it does make sense to talk of a factored vertex in the application to inclusive cross sections since it involves this discontinuity.

The first term in (4.2) has simultaneous poles at $\alpha_1 = J_1$ and $\alpha_2 = J_2$. It also has a set of spurious singularities at integral $\alpha_0 - \alpha_1 - \alpha_2$. However, it has no singularities in s_1, s_2, and s_{12} (at least for a finite number of terms in the sums), since the three ratios of Γ functions yield polynomials. The absence of singularities in these variables is, of course, necessary since this term has a cut in the overlapping variable s_0.

The second term has a structure similar to a B_5 function and contains the poles in s_1, s_2, and s_{12}. The leading term in (4.2) ($i=0$) has poles from the various regions of integration as follows:

$$\begin{aligned} &x_1 \approx 0, \quad \alpha_1 = J_1, \\ &x_2 \approx 0, \quad \alpha_2 = J_2, \\ &x_1 \approx x_2 \approx 0, \quad \alpha_0 - \alpha_1 - \alpha_2 = -N, \\ &x_1 \approx 1, \quad \alpha(s_1) = N_1, \\ &x_2 \approx 1, \quad \alpha(s_2) = N_2, \\ &x_1 \approx x_2 \approx 1, \quad \alpha(s_{12}) = N_{12}\,. \end{aligned} \tag{4.3}$$

One sees that the first three contributions given in (4.3) correspond to the second, third, and fourth terms in (3.7) or (3.8): Indeed, they behave like $(-s_{01})^{\alpha_0}$, $(-s_{20})^{\alpha_0}$, and $(-s_{01})^{(\alpha_0+\alpha_1-\alpha_2)/2} \times (-s_{20})^{(\alpha_0+\alpha_2-\alpha_1)/2}$, respectively. In the limit we are interested in $s_{01} \propto s_{20}$ and all three contributions behave like ${s_{01}}^{\alpha_0} \propto {s_{20}}^{\alpha_0}$. The only "spurious" singularity (that in $\alpha_0 - \alpha_1 - \alpha_2$) cancels against the corresponding singularity in the first term of (4.2). It is important to note that this term does not have a simultaneous singularity at $\alpha_1 = J_1$ and $\alpha_2 = J_2$ and thus does not contribute to the resonance scattering amplitude defined by that limit.

Finally, as has been mentioned in passing above, the relationship between the triple-Regge and helicity-pole limits discussed in Sec. II E can be explicitly verified in the DRM using (3.7), (4.2), and various intermediate expressions.

V. APPLICATION TO SINGLE-PARTICLE PRODUCTION

Consider the inclusive process

$$a + b \to x + \text{anything},$$

where the total energy squared of a and b is s, the mass squared of anything is M^2, and the invariant momentum transfer squared between b and x is t. The cross section for this process is given by the M^2 discontinuity of the six-line amplitude shown in Fig. 1.[5,6,28] We recognize the high-energy limit near the boundary of the phase space of x (M^2, s/M^2, $\bar{s}/M^2 \to \infty$) as a forward helicity-pole limit of this amplitude[4] as defined in Sec. II, Eqs. (2.23) and (2.24). The DRM result is thus obtained by taking the M^2 ($= s_0$) discontinuity of (4.2),[16,17]

$$\frac{d\sigma}{dtd(M^2/s)} = \frac{1}{s}\,(M^2)^{\alpha_0}\left(\frac{s}{M^2}\right)^{\alpha_1(t)}\left(\frac{\bar{s}}{M^2}\right)^{\alpha_2(t)} f(t) \times \xi_1(t)\bar{\xi}_2(t)\Gamma(-\alpha_1(t))\,\Gamma(-\alpha_2(t))\,, \tag{5.1}$$

where

$$f(t) = 1/\Gamma(-\alpha_1(t) - \alpha_2(t) + \alpha_0 + 1)\,,$$

and we have suppressed kinematic factors. The signature factors $\xi_1(t)$ and $\xi_2(t)$ come from interchanging p_b and p_x and $p_{b'}$ and $p_{x'}$.

The interesting observation has recently been made[19] that $f(t)$ in (5.1) *must* vanish at $t=0$, if all three trajectories α_i have unit intercepts as might be expected if they are Pomeranchukons. This result is basically the consequence of unitarity in the "direct channel" (e.g., $a+b \to a+b$).

A particularly simple demonstration of this result is based on the fact that the integral of (5.1) over a restricted range of phase space must be less than the total cross section:

$$\int_D\!\!\int \frac{d\sigma}{dtd(M^2/s)}\,dtd(M^2/s) = \langle n(D)\rangle\,\sigma, \tag{5.2}$$

where $\langle n(D)\rangle$ is the average number of particles found in the volume D of phase space. The average number of particles carrying more than half the total energy is less than one by energy conservation.[29] The fraction of the total energy carried off by particle x in the lab system is

$$E_x/E_{\text{tot}} \approx 1 - M^2/s \tag{5.3}$$

for large M^2 and s. Therefore,

$$\int_{M_0^2/s}^{1/N} d(M^2/s) \int_{-\infty}^{0} dt\,\frac{d\sigma}{dtd(M^2/s)} < \sigma\,, \tag{5.4}$$

where $N > 2$ and M_0^2 is a constant. If we put

$$\alpha_i(t) = 1 + \alpha' t$$

and

$$f(t)\Gamma(-\alpha_1(t))\Gamma(-\alpha_2(t))\,\xi_1(t)\bar{\xi}_2(t) = A e^{bt} \tag{5.5}$$

in (5.1) and substitute the result into (5.4) we get

$$\frac{A}{2\alpha'}\ln\left[\frac{b + 2\alpha'\ln(s/M_0^2)}{b + 2\alpha'\ln N}\right] < \sigma\,. \tag{5.6}$$

If the total cross section is also dominated by the Pomeranchukon, $\sigma \sim$ const and (5.6) cannot be satisfied for asymptotic s.

It seems remarkable that the DRM result (5.1) indeed has this zero.[18] Let us discuss its origin in the model. The zero comes from $[\Gamma(-\alpha_1 - \alpha_2 + \alpha_0 + 1)]^{-1}$ which arises from the factor $\Gamma(\alpha_1 + \alpha_2 - \alpha_0)$ in (4.2) and $\sin\pi(\alpha_0 - \alpha_1 - \alpha_2)$ coming from the discontinuity in M^2. As we discussed in Sec. III around Eq. (3.8), the factor $\Gamma(\alpha_1 + \alpha_2 - \alpha_0)$ is the continuation away from integral angular momentum of the nonsense zeros. Thus the zero in the DRM reflects the absence of nonsense poles at $J_0=J_1+J_2-n$ ($n=1,2,3,\dots$) in the helicity amplitude with $|\lambda_0| = J_1+J_2$.

This mechanism for producing a zero in $f(t)$ is, of course, more general than the DRM. Consider the amplitude of Fig. 1 for $\alpha_1=J_1$ and $\alpha_2=J_2$. It will be a polynomial in

$$\begin{aligned}
s/M^2 &\sim A+(B+C\cos\phi_1)\cos\theta_1\,,\\
\bar{s}/M^2 &\sim \bar{A}+(\bar{B}+\bar{C}\cos\phi_2)\cos\theta_2\,,\\
s_1 &= A_1+B_1\cos\theta_1\,,\\
s_2 &= A_2+B_2\cos\theta_2\,,\\
s_{12} &= A_{12}+B_{12}\cos\theta_1+C_{12}\cos\theta_2\\
&\quad +[D_{12}+E_{12}\cos(\phi_1-\phi_2)]\cos\theta_1\cos\theta_2\,,
\end{aligned}\tag{5.7}$$

where we have taken $\cos\theta_0\to\infty$ [see (2.25)]. Thus $(s/M^2)^{\lambda_1}$ can contribute to helicity amplitudes for the J_1 angular momentum with helicity up to λ_1 and, by angular momentum conservation ($\lambda_0=\lambda_1-\lambda_2$), makes a contribution of up to λ_1 units in the helicity for the J_0 angular momentum. On the other hand, ${s_{12}}^{\lambda}$ makes equal contributions to λ_0. Hence the amplitude with maximal helicities ($|\lambda_1|=J_1$, $|\lambda_2|=J_2$, $|\lambda_0|=J_1+J_2$) is proportional to the term in the polynomial with $(s/M^2)^{J_1}(\bar{s}/M^2)^{J_2}$, that is, the dominant term for $s/M^2, \bar{s}/M^2\to\infty$. In other words, the helicity-pole limit picks out the amplitude with maximal helicity for J_0. Thus we expect the behavior

$$\Gamma(J_1+J_2-\alpha_0)(M^2)^{\alpha_0}(s/M^2)^{J_1}(\bar{s}/M^2)^{J_2}\,,\tag{5.8}$$

assuming the Regge trajectory α_0 chooses sense. The natural continuation of (5.8) away from integers then produces the zero in the discontinuity. This is only intended to be a plausibility argument, since we do not know how to make a rigorous continuation from integral angular momentum. In particular, we have obtained only one of the two terms in (4.2) [or of the four terms in (3.7)]. For reasons not presently understood, fortunately this is precisely the term which has the desired discontinuity in M^2.

The mechanism we have been discussing for producing the zero should be distinguished from the more trivial mechanism of having the full triple-Pomeranchukon vertex vanish at $t_i=0$ for all helicities, i.e., an "over-all" zero in the vertex. Our mechanism only requires that the $|\lambda_1|=|\lambda_2|=1$, $|\lambda_0|=2$ part vanish.

In general, under what conditions does one expect the presence of nonsense zeros? In the case of four-line amplitudes, the absence of wrong-signature nonsense zeros requires singular Regge residues. These singular residues are usually correlated with the existence of fixed poles with compensating singular residues and the presence of third double-spectral functions.[30] Thus

$$\begin{aligned}
a_J^{\pm}(t) &= \frac{\beta(t)}{J-\alpha(t)}-\frac{\beta'(t)}{J-J_0}\\
&\underset{t=t_0}{\approx}\frac{\gamma}{[J-\alpha(t)][J_0-\alpha(t)]}-\frac{\gamma}{[J-J_0][J_0-\alpha(t)]}\\
&=\frac{-\gamma}{[J-\alpha(t)][J-J_0]}\,,
\end{aligned}$$

where $\alpha(t_0)=J_0$, implies

$$A^{\pm}(s,t)\underset{s\to\infty}{\approx}\frac{\gamma}{J_0-\alpha(t)}(-s)^{\alpha(t)}-\frac{\gamma}{J_0-\alpha(t)}(-s)^{J_0}$$

and

$$\mathrm{Disc}_s A(s,t)\propto \mathrm{Disc}_s A^{\pm}(s,t)\sim\gamma s^{\alpha(t)}=\text{finite}\,.$$

Furthermore, unitarity requires that the fixed poles must be masked by moving Regge cuts, but we shall neglect this complication. Assuming by analogy a similar situation for the six-line amplitude, where the angular momentum plane structure is largely unknown, we are led to the suggestion that the *vanishing* (or smallness) *of the triple Pomeranchukon contribution to single particle production is directly correlated with the absence* (or weakness) *of a fixed pole with singular residue at* $J_0=1$.

In order to show that this suggestion is not an impossibility, we suggest a possible behavior for the signatured six-line amplitude in the neighborhood of $\alpha_i=1$ ($s=\bar{s}$):

$$A^{+}\approx\frac{(M^2)^{\alpha_0}(s/M^2)^{\alpha_1}(s/M^2)^{\alpha_2}}{(\alpha_0-\alpha_1-\alpha_2+1)(\alpha_1-1)(\alpha_2-1)}-\frac{(M^2)^{-1}s^{\alpha_1}s^{\alpha_2}}{(\alpha_0-\alpha_1-\alpha_2+1)(\alpha_1-1)(\alpha_2-1)}\,.\tag{5.9}$$

This should be compared with the DRM result obtained from (4.2),

$$B_6\approx\frac{(M^2)^{\alpha_0}(s/M^2)^{\alpha_1}(s/M^2)^{\alpha_2}}{(\alpha_0-\alpha_1-\alpha_2)(\alpha_1-1)(\alpha_2-1)}-\frac{(M^2)^{\alpha_0}(s/M^2)^{\alpha_0}}{(\alpha_0-\alpha_1-\alpha_2)}\left[\frac{1}{(\alpha_0-1)(\alpha_1-1)}+\frac{1}{(\alpha_0-1)(\alpha_2-1)}\right].\tag{5.10}$$

Note the first term of (5.9) has a singular residue leading to a finite $f(0)$. The second term has the behavior corresponding to a fixed pole at $J_0=1$. Equation (5.9) has been constructed to satisfy the requirement that the pole at $\alpha_0 = \alpha_1 = \alpha_2 = 1$ be absent since $|\lambda_0| = 2$. This behavior should be compared with the DRM behavior (5.10). The singularity at $\alpha_0 = \alpha_1 + \alpha_2$ cancels between the two terms again except at $\alpha_1 = \alpha_2 = 1$. However, in this case it yields the allowed pole at $\alpha_0 = 2$.

Note added in proof. Equation (5.9) differs from that of the original manuscript. Both are acceptable according to the criteria discussed here.

We expect that, if the Pomeranchukon is a Regge trajectory with pure Toller quantum number $M = 0$, the fixed pole will be absent. This is a rigorous statement for four-line amplitudes as can be seen as follows. $M = 0$ means that only s-channel helicity flip zero contributes at $t = 0$. For unequal masses the crossing angles are 0 or π and so the t-channel helicity flip is also zero. For equal masses the crossing angles are $\frac{1}{2}\pi$ so, in general, most t-channel helicities will give nonvanishing contributions to the asymptotic amplitude.[31] However, if the s-channel amplitude is helicity *independent,* the t-channel amplitude will also be pure helicity-flip zero.[32] In this case there are no nonvanishing contributions for nonsense t-channel helicities. This implies there are no singular Regge residues and no fixed poles with singular residues at $J = 1$. The experimental evidence for the $M = 0$ nature of the Pomeranchukon is weak at present since there are no beams available with polarized high-spin particles. However, it has recently been suggested that the M content can be tested nicely in the inclusive cross section for two-particle production.[33]

The suggested relation of $f(0) = 0$ to the absence of fixed poles in the t_0-channel angular momentum plane is essentially a translation of the direct-channel unitarity requirement into crossed-channel language. In this respect it is very much like the Froissart bound. Direct-channel unitarity is used to derive the bound which can then be stated as a requirement on the crossed-channel angular momentum plane structure $\alpha(0) \leq 1$ (for poles). Since crossed-channel unitarity plays a vital role in determining the allowed nature of these angular momentum plane singularities, this is, in effect, a mutual constraint between direct-channel and crossed-channel unitarity. This is particularly true in the case considered here, since the existence of fixed poles is closely related by unitarity to the existence of moving Regge cuts, etc.[30]

One must turn to models to gain insight into whether or not $f(0) = 0$. The crossed-channel structure assumed in the model will be crucial in the answer. As we have discussed above, the usual DRM has $f(0) = 0$ because the model has nonsense zeros and thus no fixed poles with singular residues. The Virasoro-Shapiro DRM[34] has no wrong-signature zeros and thus fixed poles and we *expect* that it will not give $f(0) = 0$. (This could be obtained only through a multiplicative zero, which would make the full amplitude zero at $t_i = 0$, an unphysical result.)

Abarbanel *et al.*[19] have calculated $f(t)$ directly from the Amati-Bertocchi-Fubini-Stanghellini-Tonin (ABFST) multiperipheral model.[35] They find that $f(0)$ does not vanish as required by unitarity if the kernel strength is increased so that $\alpha_P(0) = 1$. Since the Pomeranchuk trajectory in their calculation is not generated self-consistently (i.e., the Pomeranchuk trajectory in the input elastic amplitude is not constrained to be the same as the Pomeranchuk trajectory in the output absorptive part), the resulting calculation need not, of course, be consistent with direct-channel unitarity. Indeed, without such a constraint, the model also predicts a tachyon pole at $J_0 = 0$, which again violates direct-channel unitarity. Even if the model were self-consistent so that direct-channel unitarity is satisfied, the crossed-channel angular momentum structure may not be consistent with crossed-channel unitarity.

Finally, we note that the earliest model for $f(t)$, that of Gribov and collaborators,[36] had $f(0) = 0$. An S-matrix development of this model has recently been given by Bronzan.[37]

We conclude this section by clearing up several possible sources of confusion. The asymptotic limit for single-particle production is often described as a "triple-Regge limit".[4] However, strictly speaking, it is a helicity-pole limit, since only three channel invariants become asymptotic, not six as in the usual triple-Regge limit. A possible source of confusion arises from the fact that vanishing kinematic factors[38] multiply the $\cos\theta_i$ in the expressions (2.11) for s_1, s_2, and s_{12}, so that these channel invariants remain finite even for large $\cos\theta_i$. Thus, if one were to define the triple-Regge limit by asymptotic $\cos\theta_i$, the single-particle-production limit would indeed be a triple-Regge limit. However, the Regge limit should in general be defined as a limit where certain channel invariants are asymptotic not where certain cosines are asymptotic. This is because the mapping between the channel invariants and the cosines is singular at some points and we expect the amplitude to be smooth in the channel invariants and not the cosines. The point of interest, $\lambda(t_0, t_1, t_2) = 0$, is such a singular point (like $t = 0$ in unequal-mass 2-2 scattering). The Regge limit should still be defined by (2.13) at this point.

Nevertheless, in the single-particle-production application where the discontinuity in M^2 is taken, the triple-Regge expression is the same as the helicity-pole expression in the DRM and it thus may be reasonable to speak of the behavior as "triple-Regge behavior". We believe this result may be general. Before the M^2 discontinuity is taken, the HP expression (4.2) exhibits poles in s_1, s_2, and s_{12}, whereas the TR expression has asymptotic Regge behavior in these invariants. However, the part of the amplitude that contributes to the M^2 discontinuity is a polynomial in s_1, s_2, and s_{12}, since pole residues must be polynomials in overlapping invariants. If, as in the DRM, the helicity poles lie as high as possible, giving the behavior $(s/M^2)^{\alpha_1}(s/M^2)^{\alpha_2}$, and if the HP and TR limits are consistent with the same hybrid TR-HP limit (2.30), then this polynomial must be a constant, and the leading TR and HP behavior must agree.[39] If the HP expression agrees with the TR expression for the M^2 discontinuity, we expect that the dependence on the external clusters attached to α_1 and α_2 will factor out. It then makes sense to speak of a vertex as far as this application is concerned.

We mention two other sources of possible confusion. First, the formula (5.1) does not imply that at $\alpha_0 = 1$ there must be a complete decoupling from states of angular momentum J_1 and J_2 with $J_1 + J_2 > 1$. Only the nonsense helicities, $|\lambda_0| > 1$, decouple. One can easily see from (3.7) or (4.2) that higher-order terms in the sums give sense couplings. Second, we have argued that the helicities of the trajectories α_1 and α_2 at $t=0$ are one for unit intercept. This may appear mysterious in the light of the fact that an $M=0$ object (like the Pomeranchukon or leading trajectories in the DRM) should contribute only to helicity zero at $t=0$. The resolution of this apparent contradiction lies in the fact that these statements are made about helicities in two different Lorentz frames. The helicity one discussed here is in a frame where p_0, p_1, and p_2 are collinear [see Eq. (2.1)] and

$$s/M^2 \propto \cos\theta_1 \cos\phi_1 , \tag{5.11}$$

whereas the usual helicity is measured in a frame where p_b and p_x are collinear (t-channel center-of-mass frame), where

$$s/M^2 \propto \cos\theta_1 \tag{5.12}$$

and θ_1 is now the t-channel center-of-mass scattering angle. In (5.11), s/M^2 clearly corresponds to $J_1 = 1$, $|\lambda_1| = 1$, whereas in (5.12) it corresponds to $J_1 = 1$, $|\lambda_1| = 0$. We note that the crossing angle between the t and s channel is zero in this case since $M^2 \gg m_a^2$.

VI. CONCLUSION

To what extent are the results from the DRM discussed in the above a general feature of six-point amplitudes? Essentially, two ingredients are necessary in order to obtain the four-term decomposition of the triple-Regge vertex. First is the assumption of simultaneous Regge asymptotic behavior in the variables s_0, s_1, and s_2, yielding the expression

$$\gamma(\eta_{01}, \eta_{12}, \eta_{20})(-s_0)^{\alpha_0}(-s_1)^{\alpha_1}(-s_2)^{\alpha_2} . \tag{6.1}$$

Second is the tree-structure requirement that the singularities in the asymptotic channel invariants be separated in such a way that simultaneous discontinuities in overlapping asymptotic channel invariants do not occur. This statement includes a separation of right- and left-hand singularities in asymptotic channel invariants. Thus the behavior of γ in the η's is constrained so as to cancel phases in s_0, s_1, and s_2 in the appropriate pattern. This is a natural feature of the dual-resonance model, since simultaneous poles occur only in "nonoverlapping" channel invariants. That it may also be a general property of scattering amplitudes is suggested by Stapp's recent elucidation[5] of the Steinmann relation for multiparticle amplitudes.[20] It stipulates that there be no simultaneous discontinuity in overlapping channel invariants that are above their lowest threshold in the physical region. The asymptotic tree-structure assumption need not conflict with the existence of double discontinuities required by unitarity, provided it is applied only to asymptotic limits which avoid double-spectral regions. An example of the application of this assumption to the asymptotic behavior of the 2-to-3 scattering amplitude is given in Appendix B.

Simply requiring phases to cancel between the η and s dependence in (6.1) determines the asymptotic behavior of γ in the η's up to a polynomial in η. It is necessary to make an additional assumption to fix the behavior uniquely. In the DRM only sense couplings are permitted and the highest allowed helicity couples. Assuming that the highest permissible helicity occurs is evidently sufficient to produce the leading powers in the four-term decomposition (3.7).

In Sec. II E we noted that the TR and HP limits can be related to a common asymptotic limit, the TR-HP limit (2.30). This suggests a method for determining the location of the singularities in complex helicity: One takes the further limit $s_1, s_2, s_{12} \to \infty$ with η_{12} fixed on the HP limit and requires it to be consistent with the further limit $\eta_{01}, \eta_{20} \to \infty$ with η_{12} fixed on the TR limit. The four-term decomposition for the triple-Regge vertex along with this uniformity-of-limits assumption then fixes the allowed powers of s_{01}/s_0 and s_{20}/s_0, i.e., the helicity-pole positions. The general nature of the four-term decomposition would thus lead to a general structure of the HP asymptotic behavior similar to Eq. (4.2).

However, the position of the Regge poles in J_0, J_1, and J_2 is not enough to determine fully the HP asymptotic behavior, since for the reasons discussed in the above further information about the helicity couplings is necessary to completely determine the analytic structure of the triple-Regge vertex. In the DRM only sense couplings are permitted and the highest allowed helicity couples which leads to the behavior (4.2). We should emphasize that the existence of, for example, nonsense couplings could well lead to different helicity-pole locations.

The locations of dynamical poles in complex angular momentum are determined solely by the quantum numbers of the channel to which they couple. They do not depend upon the individual particles composing the channel. Thus one can think of a "Regge exchange" in a diagram as an entity independent of the vertices in the diagram. We find by contrast that the locations of helicity poles depend not only upon the quantum numbers of the channels to which they couple, but also upon other quantum numbers of the amplitude. It is therefore improper to speak of a helicity-pole exchange diagram in the usual sense of the phrase.

The Steinmann relation as applied here, if true, could prove to be an enormously useful tool in studying the Regge asymptotic behavior of multiparticle scattering amplitudes. Other interesting questions are raised by our study. Is it possible that in order to construct a multiple Sommerfeld-Watson transform it is necessary to distinguish several partial sums over angular momenta of the type discussed in Secs. II and III? What is the role of signature in the asymptotic singularity structure of the amplitude and what is its relation to the analytic structure of the triple vertex?[40] Does the triple-Pomeranchuk zero suggest a deeper connection between direct- and crossed-channel unitarity?

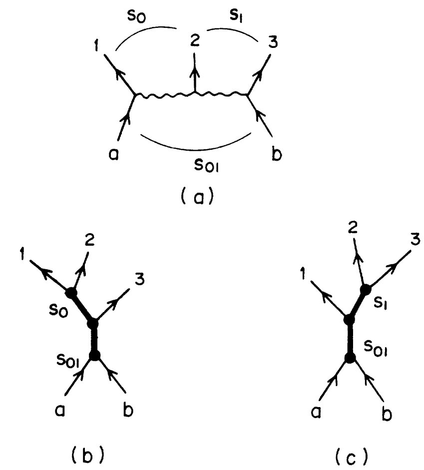

FIG. 4. (a) Diagram showing channel invariants for the 2-to-3 amplitude. (b) and (c) Diagrammatic representation of asymptotic singularity structure of double-Regge vertex.

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APPENDIX A: THE DOUBLE-REGGE VERTEX IN THE DRM

The double-Regge vertex is considerably simpler than the triple-Regge vertex; yet it exhibits the same general structure. It is therefore very useful as a guide, especially since it is simply related to the much-studied hypergeometric and confluent hypergeometric functions. We thus hope this Appendix will help provide insight into the formulas in the text. The results below also provide a useful check of the results in the text since the former should agree with the residue of the pole at $\alpha_2 = 0$ in the latter.

We first take the single-Regge limit of the five-point function [Fig. 4(a)]:

$$B_5 = \int_0^1 \int_0^1 dx_0 dx_1\, x_0^{-\alpha_0 - 1}(1-x_0)^{-\alpha(s_0)-1} x_1^{-\alpha_1-1}(1-x_1)^{-\alpha(s_1)-1}(1-x_0x_1)^{-\alpha(s_{01})+\alpha(s_0)+\alpha(s_1)}$$

$$\sim (-s_0)^{\alpha_0} \int_0^1 dx_1\, x_1^{-\alpha_1-1}(1-x_1)^{-\alpha(s_1)-1} \int_0^\infty dy_0\, y_0^{-\alpha_0-1} \exp\left(-y_0 + y_0x_1 - \frac{s_{01}}{s_0} y_0 x_1\right) \tag{A1}$$

$$= (-s_0)^{\alpha_0}\Gamma(-\alpha_0) \int_0^1 dx_1\, x_1^{-\alpha_1-1}(1-x_1)^{-\alpha(s_1)-1}\left[1-(1-s_{01}/s_0)x_1\right]^{\alpha_0} \tag{A2}$$

$$= (-s_0)^{\alpha_0} \frac{\Gamma(-\alpha_0)\Gamma(-\alpha_1)\Gamma(-\alpha(s_1))}{\Gamma(-\alpha(s_1)-\alpha_1)}\, {}_2F_1(-\alpha_0, -\alpha_1; -\alpha(s_1)-\alpha_1; (1-s_{01}/s_0)). \tag{A3}$$

Using (A1) to (A3), we can either study the double-Regge vertex by taking the further limit $s_1 \to \infty$ or the analog of the helicity-pole behavior by taking the further limit $s_{01}/s_0 \to \infty$. Rather than using (A3) and the known properties of the hypergeometric function, we study directly the integral representations (A1) and (A2).

The double-Regge vertex is obtained easily with the usual exponentiation substitution $x_1 = y_1/-\alpha(s_1)$:

$$B_5 \sim (-s_0)^{\alpha_0}(-s_1)^{\alpha_1} \int_0^\infty \int_0^\infty dy_0 dy_1\, y_0^{-\alpha_0-1}\, y_1^{-\alpha_1-1} \exp(-y_0 - y_1 + \eta_{01} y_0 y_1) \tag{A4}$$

$$= (-s_0)^{\alpha_0}(-s_1)^{\alpha_1}\Gamma(-\alpha_0) \int_0^\infty dy_1\, y_1^{-\alpha_1-1} e^{-y_1} (1 - \eta_{01} y_1)^{\alpha_0} \tag{A5}$$

$$= (-s_0)^{\alpha_0}(-s_1)^{\alpha_1}\Gamma(-\alpha_0)\Gamma(-\alpha_1)(-\eta_{01})^{\alpha_1}\Psi(-\alpha_1, \alpha_0 - \alpha_1 + 1; -1/\eta_{01}), \tag{A6}$$

where Ψ is the usual confluent hypergeometric function.[41]

Let us study the η_{01} dependence of the double-Regge vertex $V(\alpha_0, \alpha_1; \eta_{01})$, that is, the integral in (A4), using the integral representations (A4) and (A5). For example, from (A4) we can easily derive an asymptotic expansion for small η_{01} by expanding the third term in the exponent,

$$V(\alpha_0, \alpha_1; \eta_{01}) \underset{\eta_{01}\to 0}{\sim} \sum_i \frac{\Gamma(-\alpha_0 + i)\Gamma(-\alpha_1 + i)}{i!} \eta_{01}{}^i . \tag{A7}$$

However, (A7) is only an asymptotic expansion and it is desirable to exhibit the η_{01} dependence for all η_{01}. The standard trick used is to continue (A5) from $\eta_{01} < 0$ where it is defined using the Mellin-Barnes integral[24]

$$\Gamma(-\alpha)(1-z)^\alpha = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dr\, \Gamma(-\alpha + r)\Gamma(-r)(-z)^r, \tag{A8}$$

where the contour separates the poles in $\Gamma(-\alpha + r)$ from those in $\Gamma(-r)$. We find

$$V(\alpha_0, \alpha_1; \eta_{01}) = \int_0^\infty dy_1\, y_1^{-\alpha_1-1} e^{-y_1} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dr\, \Gamma(-\alpha_0 + r)\Gamma(-r)(-\eta_{01} y_1)^r$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dr\, \Gamma(-\alpha_0 + r)\Gamma(-\alpha_1 + r)\Gamma(-r)(-\eta_{01})^r, \tag{A9}$$

where the contour now separates the poles in the first two Γ functions from those in $\Gamma(-r)$.[42] If we shift the contour in (A9) to the right we obtain (A7). However, V is identically given by (A9), whereas (A7) is only an asymptotic expansion. The reason for this is easy to see: The integrand of (A9) behaves like $(\eta_{01} r)^r$ as $|r| \to \infty$ and thus a semicircle in the right-half plane will not vanish and (A7) with an equality cannot obtain. This behavior of the integrand, however, permits the closing of the contour in the left-half plane:

$$V(\alpha_0, \alpha_1; \eta_{01}) = (-\eta_{01})^{\alpha_0} \sum_{i=0}^\infty \frac{\Gamma(-\alpha_0 + i)\Gamma(\alpha_0 - \alpha_1 - i)}{i!} \eta_{01}{}^{-i} + (-\eta_{01})^{\alpha_1} \sum_{i=0}^\infty \frac{\Gamma(-\alpha_1 + i)\Gamma(\alpha_1 - \alpha_0 - i)}{i!} \eta_{01}{}^{-i} \tag{A10}$$

$$= (-\eta_{01})^{\alpha_0}\Gamma(-\alpha_0)\Gamma(\alpha_0 - \alpha_1)\Phi(-\alpha_0, \alpha_1 - \alpha_0 + 1; -1/\eta_{01})$$
$$+ (-\eta_{01})^{\alpha_1}\Gamma(-\alpha_1)\Gamma(\alpha_1 - \alpha_0)\Phi(-\alpha_1, \alpha_0 - \alpha_1 + 1; -1/\eta_{01}). \tag{A11}$$

The series in (A10) clearly define entire analytic functions of η_{01} [this also follows from the known properties of Φ (Ref. 43)], and thus V has the singularities $(-\eta_{01})^{\alpha_0}$ and $(-\eta_{01})^{\alpha_1}$ at the origin. These cuts are asymptotic representations of the poles in s_0, s_1, and s_{01} in B_5. Combining (A4) and (A10) and using the definition of η_{01}, we find

$$B_5 \sim (-s_{01})^{\alpha_0}(-s_1)^{\alpha_1 - \alpha_0}\,\Gamma(-\alpha_0)\Gamma(\alpha_0 - \alpha_1) + (-s_{01})^{\alpha_1}(-s_0)^{\alpha_0 - \alpha_1}\,\Gamma(-\alpha_1)\Gamma(\alpha_1 - \alpha_0). \tag{A12}$$

The two terms in (A12) correspond to the two possible combinations of singularities in nonoverlapping asymptotic variables – see Figs. 4(b) and 4(c).[12]

We note that each term has a singularity at $\alpha_0 = \alpha_1$ whereas the sum does not. One can interpret this "spurious" singularity as follows. The poles at $\alpha_0 = J_0$ (or $\alpha_1 = J_1$) come from only one of the two terms in (A12), i.e., the leading term is

$$B_5 \sim \frac{1}{\alpha_0 - J_0}\frac{s_{01}{}^{J_0}}{J_0!}(-s_1)^{\alpha_1 - J_0}\,\Gamma(J_0 - \alpha_1). \tag{A13}$$

The $\Gamma(J_0 - \alpha_1)$ in (A13) provides the nonsense zeros for $\alpha_1 < J_0$ appropriate to this term which corresponds to helicity J_0 for the resonance of spin J_0 [see Eq. (2.11)]. Thus $\Gamma(\alpha_0 - \alpha_1)$ can be regarded as a continuation of this behavior away from the poles. Such a spurious singularity can exist because it can be canceled by the second term in (A12).[44]

We now study the behavior analogous to the helicity-pole limit. To obtain an expansion of (A2) useful for large s_{01}/s_0, we use the Mellin-Barnes integral (A8):

$$B_5 \sim (-s_0)^{\alpha_0} \int_0^1 dx_1\, x_1^{-\alpha_1 - 1}(1-x_1)^{-\alpha(s_1)+\alpha_0-1} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dr\, \Gamma(-\alpha_0+r)\Gamma(-r)\left(\frac{s_{01}}{s_0}\right)^r \left(\frac{x_1}{1-x_1}\right)^r \tag{A14}$$

$$= (-s_0)^{\alpha_0}\frac{1}{2\pi i}\int_{i\infty}^{i\infty} dr\, \Gamma(-\alpha_0+r)\Gamma(-\alpha_1+r)\frac{\Gamma(-\alpha(s_1)+\alpha_0-r)}{\Gamma(-\alpha(s_1)-\alpha_1+\alpha_0)}\Gamma(-r)\left(\frac{s_{01}}{s_0}\right)^r, \tag{A15}$$

where the contour separates the poles in the first two Γ functions from those in the second two.[42] For sufficiently large s_{01}/s_0, the contour can be closed in the left-half plane and we obtain for (A15)

$$(-s_{01})^{\alpha_0}\sum_i \frac{\Gamma(\alpha_0-\alpha_1-i)\Gamma(-\alpha(s_1)+i)\Gamma(-\alpha_0+i)}{i!\,\Gamma(-\alpha(s_1)-\alpha_1+\alpha_0)}\left(\frac{s_0}{s_{01}}\right)^i$$
$$+(-s_{01})^{\alpha_1}(-s_0)^{\alpha_0-\alpha_1}\sum_i \frac{\Gamma(\alpha_1-\alpha_0-i)\Gamma(-\alpha(s_1)-\alpha_1+\alpha_0+i)\Gamma(-\alpha_1+i)}{i!\,\Gamma(-\alpha(s_1)-\alpha_1+\alpha_0)}\left(\frac{s_0}{s_{01}}\right)^i \tag{A16}$$

$$= (-s_{01})^{\alpha_0}\frac{\Gamma(-\alpha_0)\Gamma(-\alpha(s_1))\Gamma(\alpha_0-\alpha_1)}{\Gamma(-\alpha(s_1)-\alpha_1+\alpha_0)}\, {}_2F_1(-\alpha_0, -\alpha(s_1); 1+\alpha_1-\alpha_0; s_0/s_{01})$$
$$+(-s_{01})^{\alpha_1}(-s_0)^{\alpha_0-\alpha_1}\Gamma(-\alpha_1)\Gamma(\alpha_1-\alpha_0)\,{}_2F_1(-\alpha_1, -\alpha(s_1)-\alpha_1+\alpha_0; 1+\alpha_0-\alpha_1; s_0/s_{01}). \tag{A17}$$

Equation (A17) could have been obtained directly from (A3) using the properties of the hypergeometric function.[45]

The interpretation of the two terms in (A16) is not so simple as that of the two terms in (A10). First, since the series do not represent entire functions, the coefficients $(-s_{01})^{\alpha_0}$ and $(-s_{01})^{\alpha_0}(-s_0)^{\alpha_0-\alpha_1}$ do not exhibit the entire singularity of the function. Second, even the coefficient of the leading power as $s_{01}/s_0 \to \infty$ is much more complicated for the first term; indeed, it has the structure of a four-point function. This more complicated structure is expected: Since $\alpha(s_1)$ is not asymptotic, we should see its true pole structure.

Finally, we note that taking the limit $s_1 \to \infty$ (η_{01} fixed) on (A15), (A16), and (A17) yields (A9), (A10), and (A11), respectively.

APPENDIX B: GENERAL STUDY OF THE DOUBLE-REGGE VERTEX

We present here a general analysis of the double-Regge vertex in the scalar five-line amplitude, which reproduces in part the singularity structure of the vertex function [Eq. (A11)] discussed in Appendix A with a minimal set of assumptions. These are

(i) double-Regge asymptotic behavior with moving Regge poles,

(ii) separation of overlapping channel singularities, and

(iii) coupling of highest allowed but no higher sense helicity.

According to assumption (i) the amplitude has the asymptotic form

$$T \sim (s_0)^{\alpha_0(t_0)}(s_1)^{\alpha_1(t_1)}\, V(t_0, t_1; \eta), \tag{B1}$$

where

$$\eta = \frac{s_{01}}{s_0 s_1} \quad (=\eta_{01})$$

and the invariants are defined as shown in Fig. 4(a).

With assumption (ii) we require, first, that there be no simultaneous discontinuity in any overlapping asymptotic channel invariant. This imposes a much stronger condition on the scattering amplitude than the Steinmann relation.[5,6,20] With this assumption we first separate right- and left-hand cuts in s_0, s_1, and s_{01} in the asymptotic limit and write the part with all right-hand cuts

$$T_{RR} \sim (-s_0)^{\alpha_0}(-s_1)^{\alpha_1} V_{RR}(\eta), \tag{B2}$$

where the t's are suppressed and V_{RR} has only a right-hand cut in η. One might also think of T_{RR} as an amplitude of positive signature in s_0 and s_1.

With assumption (ii) we require, second, that pole and branch-point singularities in channel invariants be asymptotically represented by a pure power behavior such as that implied by the Regge powers in (B1), and that there be *no other* kind of singularity in the asymptotic invariants. (Strictly speaking, the branch-point singularities implied by the Regge power behavior do not represent singularities in the usual complex-variable sense, but describe the behavior of the function outside a wedge drawn along the appropriate positive or negative real axis.) Putting together both parts of assumption (ii), we then find that T_{RR} can be expressed as a sum of two terms

$$T_{RR} \sim (-s_0)^{\beta_0}(-s_{01})^{\gamma_0} V_0(\eta) + (-s_1)^{\beta_1}(-s_{01})^{\gamma_1} V_1(\eta)\,, \tag{B3}$$

where V_0 and V_1 have no branch-point singularities in η and are entire in η except, perhaps, at the origin and infinity. The first term exhibits simultaneous singularities in s_0 and s_{01} and the second in s_1 and s_{01}, as illustrated in Figs. 4(b) and 4(c). Note that even though it is allowed by assumption (ii), it is not possible to represent in the form (B2) a simultaneous discontinuity in s_{01} and s_2, the third Dalitz subenergy, for general values of α_0 and α_1. A term $(-s_{01})^{\beta}(-s_2)^{\gamma}$ would correspond to $(-s_0)^{\alpha}(-s_1)^{\alpha}(-\eta)^{\alpha}$ with $\alpha = \beta+\gamma$ since $s_{01} \sim s_2$ in the double-Regge limit. This forces $\alpha_0 = \alpha_1$.

Comparing (B3) with (B2), we see that

$$V_{RR}(\eta) = (-\eta)^{\alpha_1} V_0(\eta) + (-\eta)^{\alpha_0} V_1(\eta) \tag{B4}$$

so that

$$T_{RR} \sim (-s_0)^{\alpha_0-\alpha_1}(-s_{01})^{\alpha_1} V_0(\eta) + (-s_1)^{\alpha_1-\alpha_0}(-s_{01})^{\alpha_0} V_1(\eta)\,. \tag{B5}$$

We write the functions V_0 and V_1 as follows:

$$V_0(\alpha_0, \alpha_1; \eta) = \sum_{i=-\infty}^{\infty} a_i^0(\alpha_0, \alpha_1)\eta^{-i}\,,$$
$$V_1(\alpha_0, \alpha_1; \eta) = \sum_{i=-\infty}^{\infty} a_i^1(\alpha_0, \alpha_1)\eta^{-i}\,. \tag{B6}$$

Let us now consider the possible singularities of a_i^0 and a_i^1 in t_0 and t_1, i.e., in α_0 and α_1. (Here we use the assumption that the poles are moving.) Poles in t_0 and t_1 occur at the positions $\alpha_0(t_0) = J_0$ and $\alpha_1(t_1) = J_1$. Residues of poles in t_0 or α_0 must be regular in s_{01} and s_0 and residues of poles in t_1 or α_1, regular in s_{01} and s_1, since these overlap the channel invariant with the pole. Therefore, from (B5) we see that poles at $\alpha_1 = J_1$ are permitted in V_0 and at $\alpha_0 = J_0$ in V_1.

The residues of the poles at $\alpha_1 = J_1$ in V_0 yield helicity amplitudes for the four-line process with only one spinning particle. Using arguments analogous to those of Sec. II above, we can show that η is linearly related to the cosine of the Toller angle ω, which is conjugate to the helicity at the double-Regge vertex [see Eqs. (2.7) and (2.12)]. Therefore, invoking assumption (iii), the pole at $\alpha_1 = J_1$ must be a polynomial in η with maximum degree J_1. The same goes for the pole at $\alpha_0 = J_0$. We write, therefore,

$$a_i^0(\alpha_0, \alpha_1) = \begin{cases} \Gamma(i-\alpha_1)b_i^0(\alpha_0, \alpha_1) & \text{for } i \geq 0 \\ b_i^0(\alpha_0, \alpha_1) & \text{for } i < 0\,, \end{cases} \tag{B7}$$

where b_i^0 is regular at $\alpha_1 = J_1$. A similar expression holds for a_i^1.

For $\alpha_0 \neq J_0$, V_1 cannot have a pole in α_1 at J_1, since from (B5) the residue would have nonintegral powers of s_1 and s_{01}. Therefore, only V_0 contributes to poles at $\alpha_1 = J_1$ when $\alpha_0 \neq J_0$. Let us consider the residue in T_{RR} of a simultaneous pole at $\alpha_1 = J_1$ and $\alpha_0 = J_0$, obtained by taking the α_0 residue of the α_1 residue. (The order of taking the residue must not affect the result.) The result is the following:

$$\mathrm{Res}_{\alpha_0=J_0}\mathrm{Res}_{\alpha_1=J_1} T_{RR} = \mathrm{Res}_{\alpha_0=J_0}(-s_0)^{\alpha_0}(-s_1)^{J_1}\sum_{i=0}^{J_1}\frac{(-1)^i}{(J_1-i)!}\,b_i^0(\alpha_0, J_1)\eta^{J_1-i}\,. \tag{B8}$$

By assumption (iii), the maximum allowed power of η is the smaller of J_0 and J_1. Therefore, the only $b_i^0(\alpha_0, J_1)$ with poles at $\alpha_0 = J_0$ are those for which $i \geq J_1 - J_0$. If we define C_i^0 by

$$b_i^0(\alpha_0, \alpha_1) \equiv \Gamma(\alpha_1 - \alpha_0 - i)C_i^0(\alpha_0, \alpha_1)\,, \tag{B9}$$

then substituting into (B8), we get

$$\mathrm{Res}_{\alpha_0=J_0}\mathrm{Res}_{\alpha_1=J_1} T_{RR} = s_0^{J_0}s_1^{J_1}\sum_{i=p}^{J_1}\frac{1}{(J_1-i)!\,(J_0+i-J_1)!}\,C_i^0(J_0, J_1)\eta^{J_1-i}\,, \tag{B10}$$

where p is 0 or $J_1 - J_0$, whichever is larger, and assumption (iii) is satisfied, provided $C_i^0(\alpha_0, J_1)$ has no poles in α_0.

What other properties does C_i^0 have? One property follows from the requirement that the discontinuity of T_{RR} in s_0 must have no singularity in α_0. This discontinuity is

$$\mathrm{Disc}_{s_0} T_{RR} \sim \sin\pi(\alpha_0 - \alpha_1)s_0^{\alpha_0-\alpha_1}(-s_{01})^{\alpha_1} V_0\,. \tag{B11}$$

Therefore, from (B6), (B7), and (B9),

$$\sin\pi(\alpha_0-\alpha_1)\Gamma(\alpha_1-\alpha_0-i)C_i^0(\alpha_0,\alpha_1)$$

must have no poles in α_0 for any α_1. If C_i^0 had poles canceling the zeros of the first two factors above, assumption (iii) would be violated in (B10). Therefore, C_i^0 must have no singularities in α_0 for any arbitrary α_1.[46] The only singularities of C_i^0 could be fixed (α_0 independent) singularities in α_1. But these would add superfluous singularities in α_1 in T_{RR} and are therefore forbidden.[46]

Writing the full expression for V_0 and V_1, we have

$$V_0(\alpha_0,\alpha_1;\eta)=\sum_{i=0}^{\infty}\Gamma(i-\alpha_1)\Gamma(\alpha_1-\alpha_0-i)C_i^0(\alpha_0,\alpha_1)\eta^{-i}+\sum_{i=-\infty}^{-1}b_i^0(\alpha_0,\alpha_1)\eta^{-i},$$
$$V_1(\alpha_0,\alpha_1;\eta)=\sum_{i=0}^{\infty}\Gamma(i-\alpha_0)\Gamma(\alpha_0-\alpha_1-i)C_i^1(\alpha_0,\alpha_1)\eta^{-i}+\sum_{i=-\infty}^{-1}b_i^1(\alpha_0,\alpha_1)\eta^{-i}. \tag{B12}$$

Further conditions on C_i^0 and C_i^1 result from the requirement that the spurious poles at $\alpha_1=\alpha_0+n$ for nonintegral α_0 and integral n cancel in the full amplitude. This implies that

$$\frac{1}{i!}C_{i+n}^0(\alpha_0,\alpha_0+n)=\frac{1}{(i+n)!}C_i^1(\alpha_0,\alpha_0+n). \tag{B13}$$

This condition is also sufficient to assure that the residue at the simultaneous poles $\alpha_0=J_0$, $\alpha_1=J_1$ is the same as (B10), if taken in any order.

In the DRM, from (A10)

$$C_i^0(\alpha_0,\alpha_1)=\frac{1}{i!},$$
$$C_i^1(\alpha_0,\alpha_1)=\frac{1}{i!}, \tag{B14}$$

and property (B13) is, of course, explicitly satisfied. In our general study these coefficients are arbitrary except for the above-mentioned constraints. In addition, in the DRM $b_i^{0,1}\equiv 0$ for $i<0$. This property assures that V_0 and V_1 have no singularity for $\eta\to\infty$, and that the full vertex V is power-behaved in this limit. However, we do not investigate further here the generality of such a property.

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[15]Ordinarily taking the discontinuity to get Eq. (1.3) introduces a factor $\sin\pi[\alpha_0-2\alpha(t)]$, which we absorbed into $f(t)$. Unless the zeros are canceled, a negative cross section results. Veneziano has observed that the "spurious" poles are just what is required to assure that the cross section be positive definite (G. Veneziano, private communication).

[16]C. E. DeTar, Kyungsik Kang, Chung-I Tan, and J. H. Weis, Phys. Rev. D 4, 425 (1971).

[17]D. Gordon and G. Veneziano, Phys. Rev. D 3, 2116

(1971).

[18]The existence of this zero was first pointed out to one of us (C.E.D.) by J. Schwarz (private communication).

[19]H. D. I. Abarbanel, G. F. Chew, M. L. Goldberger, and L. M. Saunders, Phys. Rev. Letters 26, 937 (1971); H. D. I. Abarbanel, G. F. Chew, M. L. Goldberger, and L. M. Saunders, Princeton University report, 1971 (unpublished).

[20]H. Araki, J. Math. Phys. 2, 163 (1960). We thank C.-I Tan for stimulating discussions on extensions of the Steinmann relations (see Ref. 6).

[21]M. Toller, Nuovo Cimento 37, 631 (1965); N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. Letters 19, 614 (1967), and Phys. Rev. 163, 1572 (1967); M. Toller, Nuovo Cimento 62A, 341 (1969) and references therein.

[22]The restriction to scalar particles is made only for the sake of notational simplicity.

[23]If we were treating pseudoscalar particles, in addition to scalar particles, we would need to include invariants constructed from the four-component pseudoscalar tensor, such as $\epsilon_{\mu\nu\lambda\sigma}p_{A\mu}p_{A'\nu}p_{B\lambda}p_{B'\sigma}$. This invariant is linear in $\sin\phi_1$. Polynomials in invariants of this type introduce powers of $\sin\phi_1$, $\sin\phi_2$, and $\sin(\phi_1-\phi_2)$, but, as with s_{01}, s_{12}, and s_{20}, introduce the appropriate number of $\cos\theta$'s and $\sin\theta$'s so that the analysis of Sec. II could be easily generalized.

[24]Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 1, pp. 49 and 256.

[25]One can easily verify that these singularities cancel between the two terms and are therefore absent in V.

[26]At present it is not known whether or not the assumption of a smooth expansion in $[SL(2,C)]^3/SU(1,1)$ used in Ref. 8 is consistent with (3.10).

[27]This phenomenon has also been noted by W. Spence (private communication).

[28]Note that the phase associated with the variable $\bar{s}$ is the complex conjugate of that associated with s.

[29]For a related argument, see C. E. DeTar, D. Z. Freedman, and G. Veneziano, Phys. Rev. D 4, 906 (1971).

[30]See, for example, C. E. Jones and V. L. Teplitz, Phys. Rev. 159, 1271 (1967); S. Mandelstam and Ling-Lie Wang, *ibid.* 160, 1490 (1967).

[31]This would imply the existence of fixed poles in channels with channel spin greater than 1.

[32]This is essentially the converse of "Hara's theorem," Y. Hara, Phys. Letters 23, 696 (1966). For a nice discussion, see A. H. Mueller and T. L. Trueman, Phys. Rev. 160, 1296 (1967).

[33]D. Z. Freedman, C. E. Jones, F. E. Low, and J. E. Young, Phys. Rev. Letters 26, 1197 (1971); H. D. I. Abarbanel, Phys. Rev. D 3, 2227 (1971).

[34]M. A. Virasoro, Phys. Rev. 177, 2309 (1969); J. A. Shapiro, Phys. Letters 33B, 361 (1970).

[35]D. Amati, S. Fubini, and A. Stanghellini, Nuovo Cimento 26, 896 (1962), and L. Bertocchi, S. Fubini, and M. Tonin, *ibid.* 25, 626 (1962).

[36]V. N. Gribov and A. A. Migdal, Yadern. Fiz. 8, 1002 (1968); 8, 1213 (1968) [Soviet J. Nucl. Phys. 8, 583 (1969); 8, 703 (1969)]; Zh. Eksperim. i Teor. Fiz. 55, 1498 (1968) [Soviet Phys. JETP 28, 784 (1969)]; V. N. Gribov, Yadern. Fiz. 9, 424 (1969) [Soviet J. Nucl. Phys. 9, 246 (1969)].

[37]J. B. Bronzan, Phys. Rev. D 4, 1097 (1971).

[38]The coefficient of the asymptotic cosines in s_1 and s_2 vanishes because $\lambda^{1/2}(t_0,t_1,t_2)\to 0$ while that in s_{12} vanishes because $\phi_1\to\phi_2$.

[39]This argument is clearly general for the case where the M^2 discontinuity is dominated by poles. From the Steinmann relation it may be possible to extend the argument to general amplitudes.

[40]The relationship between the asymptotic singularity structure of multiparticle amplitudes, signature, and factorization has been recently discussed for the double-Regge–single-particle vertex by J. H. Weis, Phys. Rev. D 4, 1777 (1971).

[41]Bateman Manuscript Project, Ref. 24, p. 225.

[42]The interchange of integrals can originally be justified for certain ranges of the α and, say, $\mathrm{Re}\,r=-\frac{1}{2}$. Continuation to all α can then be made if the contours are distorted away from the singularities of the integrand. This leads to the given prescription for the position of the contours.

[43]Bateman Manuscript Project, Ref. 24, p. 248.

[44]If one tries to separate the two terms in (A12) in other ways, for example, by taking the discontinuity in s_1, one finds the result again does not have the singularity.

[45]Bateman Manuscript Project, Ref. 24, p.109.

[46]Of course, there are normal threshold cuts at positive t_0 and t_1, but we restrict our attention to values of t_0 and t_1 below the lowest threshold.