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Ladder Amplitude and Its Cluster Expansion – A New Approach to Summing Diagrams

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We use the t-channel ladder diagrams in a ϕ^3 theory up to tenth order in coupling constants as an example to study the possible connection between fragmentation and pionization and to suggest a new approach to summing diagrams. We demonstrate that, at infinite energies, the nonleading logarithms are associated with the degrees of freedom in the longitudinal phase space of subsystems of two or more particles with small invariant mass, called the clusters, just as the leading logarithms are associated with the degrees of freedom in the longitudinal phase space of uncorrelated single particles. The sum of nonleading logarithms associated with each cluster also exponentiates to a power of s. The resultant s dependence of the full amplitude is the product of power dependences for individual clusters. In terms of fragmentation and pionization, we find that the ladder amplitude factors into three parts, corresponding to fragmentation of the target, pionization, and fragmentation of the projectile. Including fragmentation events only modifies the scattering amplitude by an s-independent factor; all the s dependence is contained in the pionization part.

I. INTRODUCTION

Hadron-hadron scattering at very high energies is one of the central problems in strong-interaction physics. Recently the parton model[1-3] of Feynman and the limiting-fragmentation hypothesis[4] of Benecke, Chou, Yang, and Yen have been proposed. The parton model seems to emphasize the properties of those particles ("wee" partons in Feynman's language) which have small momenta as compared with the collision energy $\sqrt{s}$ in the center-of-mass system, while the hypothesis of limiting fragmentation tends to focus attention on those particles which have finite energies in the rest system of the target or the projectile. When Lorentz-transformed to the center-of-mass system, these particles acquire energies proportional to $\sqrt{s}$. As Feynman and others pointed out, coordinate systems in which both incident particles have large (infinite at infinite energies) momenta moving opposite to each other are particularly suited for study of high-energy scatterings. It is argued that in such a frame internal motions of the virtual constituents of a physical particle are slowed down by relativistic time dilation on the one hand, and, on the other hand, because of relativistic contraction the colliding particles are expected to have almost instantaneous interactions as they pass through each other. Although this simple picture may not be completely correct, it nevertheless provides a useful basis for applying our intuition to an otherwise very complicated problem. Indeed, explicit calculations have confirmed the usefulness of this infinite-momentum technique in both extracting physical pictures and simplifying the actual computations.[5-9] Since this technique employs the physical momenta directly as natural mathematical variables, the results are especially easy to understand and to interpret in terms of the underlying physical mechanism. For instance, the eikonal picture and the $\ln s$ factors emerge very naturally from this approach.[10]

The main purpose of this paper is to use t-channel ladder diagrams up to tenth order in coupling constants as an example to study the possible connection, if any, between fragmentation and pionization at high energies, and to suggest an unconventional approach to the summing of diagrams. For these purposes we first extend the infinite-momentum technique to calculate all the nonleading terms of the type $(1/s)\ln^n s$ as $s\to\infty$. This is explained in Sec. II. We then apply the idea suggested elsewhere[7] that one should separate various terms of the amplitude into different groups corresponding to different physical situations rather than to collect together strictly according to a perturbation prescription those terms with the same order in the coupling constant and the same high-energy behavior. Therefore, pionization and fragmentation are treated separately in Sec. III and Sec. IV. In Sec. III we consider only the pionization contribution and we demonstrate that the nonleading logarithms can be associated with the degrees of freedom in the longitudinal phase space of subsystems of two or more particles which have small invariant masses and which act as single units (called the clusters), just as the leading logarithms can be associated with the degrees of freedom of the longitudinal phase space of single particles.[11] Furthermore, we present evidence which indicates that these nonleading logarithms also exponentiate to a power of s. In Secs. IV and V we study the effects of including fragmentation events and find that they modify the scattering amplitude only by an over-all factor and do not change its s dependence. An interesting connection between fragmentation and pionization is observed in this simple example of ladder diagrams. The conclusion reached in Secs. III, IV, and V reveals a very simple structure for the ladder amplitude. Namely, the ladder amplitude can be factored into three parts, corresponding to the fragmentation of the target, pionization, and the fragmentation of the projectile; all the s dependence is contained in the pionization part. In Sec. VI we discuss our results. In Appendix B we verify the results obtained in Sec. II by the conventional Feynman parameter technique.[12]

II. THE METHOD

A. Momentum Variables

In this section the infinite-momentum technique is extended so that we can extract all the logarithmic terms of the form $(1/s)\ln^n s$ as $s\to\infty$ for t-channel ladders. We shall ignore all contributions

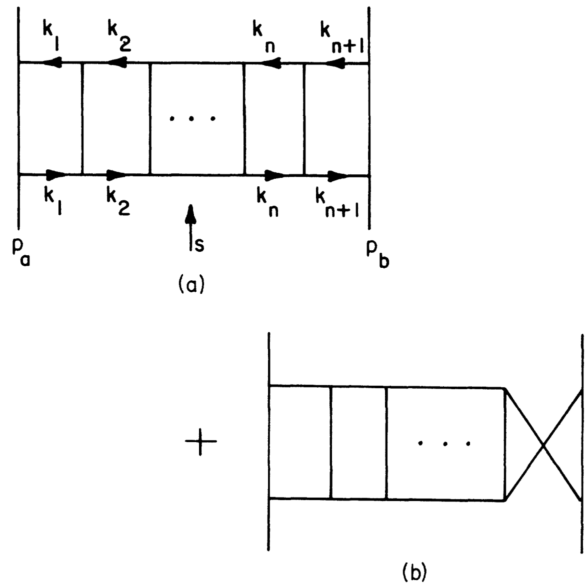

FIG. 1. (a) and (b). The general straight- and crossed-ladder diagrams with n internal rungs.

in each order of perturbation which decrease faster than $1/s$ as $s\to\infty$. To maintain the s-u crossing symmetry the crossed ladder obtained from the straight ladder with the Mandelstam variables s and u interchanged is always included. The diagrams considered in this paper are given in Fig. 1. For simplicity we shall only consider the forward elastic scattering amplitude. The sixth-order diagrams (Fig. 2) will be used as an example to illustrate the method.

First, let us fix our terminology of an n-rung amplitude. We shall refer to a ladder diagram as an n-rung ladder if it contains n internal rungs. Such a ladder is shown in Fig. 1. Thus, the amplitude of Fig. 2 is referred to as T_1, and is given by

$$T_1 = i(ig)^6\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{i^2}{(k_1^2-\mu^2+i\epsilon)^2}\frac{i^2}{(k_2^2-\mu^2+i\epsilon)^2}\frac{i}{(p_a-k_1)^2-\mu_a^2+i\epsilon}\frac{i}{(k_1-k_2)^2-\mu^2+i\epsilon}\frac{i}{(k_2+p_b)^2-\mu_b^2+i\epsilon}$$
$$+(s\to -s)\,. \tag{2.1}$$

The second term, which is obtained from the first by the crossing substitution $s\to -s$, corresponds to the diagram Fig. 2(b). The meson mass is designated by μ. For the moment we assume the masses of the external particles a and b, including the first and the last rung, to be different from μ.

$$p_a^2=\mu_a^2,\quad p_b^2=\mu_b^2\,. \tag{2.2}$$

The initial momenta p_a and p_b are directed along the third axis. We will always work in the center-of-mass system, with p_a along the positive third axis. For any four-momentum p_μ we use the "+" and "−" combinations[13]

$$p_\pm = p^0 \pm p^3 \tag{2.3}$$

and the transverse components

$$\vec{p} = (p^1, p^2) \tag{2.4}$$

as our variables. These variables are particularly useful for hadron scattering at high energies. A Lorentz transformation along the third axis changes $p_\pm$ simply by an appropriate scale factor. Furthermore, all empirical data suggest that final particles have only very limited momenta transverse to the collision axis. A decomposition of a momentum vector into transverse and longitudinal components is therefore meaningful.

B. One-Rung Amplitude

We may simplify T_1, for instance, by integrating over k_{1-} and k_{2+} by closing the contour either from above or below. We find that there are three regions of integration which contribute to the asymptotic limit of the amplitude T_1. Let ϵ, ϵ' be two small s-independent positive numbers, $1 \gg \epsilon$, $\epsilon' > 0$. These three regions are specified as follows:

(a) $|k_{1+}| > \epsilon p_{a+} = \epsilon\sqrt{s}$, $|k_{2-}| < \epsilon' p_{b-} = \epsilon'\sqrt{s}$. In this region, the virtual particle 1 will be referred to as a fragment of particle a, and is denoted graphically in Fig. 3(a). The scattering amplitude in this region is denoted by $T_1^{(a)}$,

$$T_1^{(a)} = i(ig)^6 \int \frac{dk_{1+}dk_{1-}d^2k_1}{2(2\pi)^4} \frac{dk_{2+}dk_{2-}d^2k_2}{2(2\pi)^4} \frac{i^2}{(k_{1+}k_{1-} - \vec{k}_1^{\,2} - \mu^2 + i\epsilon)^2} \frac{i^2}{(k_{2+}k_{2-} - \vec{k}_2^{\,2} - \mu^2 + i\epsilon)^2}$$
$$\times \frac{i}{(p_a - k_1)_+(p_a - k_1)_- - \vec{k}_1^{\,2} - \mu_a^{\,2} + i\epsilon} \frac{i}{(k_1 - k_2)_+(k_1 - k_2)_- - (\vec{k}_1 - \vec{k}_2)^2 - \mu^2 + i\epsilon}$$
$$\times \left[\frac{i}{(k_2 + p_b)_+(k_2 + p_b)_- - k_2^{\,2} - \mu_b^{\,2} + i\epsilon} + \frac{i}{(-k_2 + p_b)_+(-k_2 + p_b)_- - \vec{k}_2^{\,2} - \mu_b^{\,2} + i\epsilon}\right]\Bigg|_{|k_{1+}|>\epsilon\sqrt{s},\ |k_{2-}|<\epsilon'\sqrt{s}} . \tag{2.5}$$

The expression in the square brackets under the given restriction on k_{1+} and k_{2-} reduces to

$$\frac{i}{(k_2 + p_b)_+ p_{b-} - \vec{k}_2^{\,2} - \mu_b^{\,2} + i\epsilon} + \frac{i}{(-k_2 + p_b)_+ p_{b-} - \vec{k}_2^{\,2} - \mu_b^{\,2} + i\epsilon} = \frac{1}{\sqrt{s}}\left(\frac{i}{k_{2+} - \vec{k}_2^{\,2}/\sqrt{s} + i\epsilon} + \frac{i}{-k_{2+} - \vec{k}_2^{\,2}/\sqrt{s} + i\epsilon}\right) \tag{2.6}$$

$$\underset{\text{large } s}{\sim} \frac{\pi}{\sqrt{s}}[\delta(k_{2+} - \vec{k}_2^{\,2}/\sqrt{s}) + \delta(k_{2+} + \vec{k}_2^{\,2}/\sqrt{s})]. \tag{2.7}$$

Thus, at large s, the real parts in Eq. (2.6) cancel and only the δ-function parts contribute. Equation (2.7) implies that $k_{2+} = \pm\vec{k}_2^{\,2}/\sqrt{s}$. Consequently, k_{2+} is small in comparison with k_{1+} and $(\vec{k}_2^{\,2} + \mu^2)/k_{2-}$, and hence can be ignored in the remaining part of Eq. (2.5). This greatly simplifies our calculation. Note that the reduction of the last two propagators in (2.5) to simple δ functions of k_{2+} depends only on the kinematical restriction $|k_{2-}| < \epsilon'\sqrt{s}$ and is independent of the remaining part of the amplitude. This property is extremely useful in general discussion.

We now integrate Eq. (2.5) over k_{2+}, k_{1-}, and k_{2-} explicitly. The k_{2+} integral is trivial to perform. In order that the k_{1-} integration does not vanish identically, the poles of k_{1-} in the three denominators should not lie on the same side of the real axis. This implies that $p_{a+} - k_{1+}$ and k_{1+} should be of the same sign. Thus, the k_{1-} integral contributes only for $p_{a+} > k_{1+} > \epsilon p_{a+}$. Under this restriction, the following poles of k_{1-} and k_{2-} contribute to the integrations:

$$k_{1-} = p_{a-} - \frac{\vec{k}_1^{\,2} + \mu_a^{\,2}}{p_{a+} - k_{1+}} + i\epsilon, \tag{2.8}$$

$$k_{2-} = k_{1-} - \frac{(\vec{k}_1 - \vec{k}_2)^2 + \mu^2}{k_{1+} - k_{2+}} + i\epsilon, \tag{2.9}$$

corresponding to the vanishing of $(p_a - k_1)^2 - \mu_a^{\,2}$ and $(k_1 - k_2)^2 - \mu^2$, where

$$p_{a-} = \mu_a^{\,2}/p_{a+}. \tag{2.10}$$

The amplitude, after k_- integrations, becomes

$$T_1^{(a)} = -\frac{ig^6}{16\pi\sqrt{s}} \int_{\epsilon p_{a+}}^{p_{a+}} \frac{dk_{1+}}{(p_a - k_1)_+ k_{1+}^{\,3}} \frac{d^2k_1}{(2\pi)^2} \frac{d^2k_2}{(2\pi)^2} \left(\frac{\mu_a^{\,2}}{p_{a+}} - \frac{\vec{k}_1^{\,2} + \mu_a^{\,2}}{p_{a+} - k_{1+}} - \frac{\vec{k}_1^{\,2} + \mu^2}{k_{1+}}\right)^{-2} (\vec{k}_2^{\,2} + \mu^2)^{-2}. \tag{2.11}$$

In terms of scaled longitudinal momentum, $x_1 \equiv k_{1+}/p_{a+} = k_{1+}/\sqrt{s}$, we have

$$T_1^{(a)} = -i\frac{g^4}{4s}\frac{g^2}{4\pi}\int_\epsilon^1 \frac{dx_1}{x_1}\int\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{1}{(\vec{k}_2^{\,2}+\mu^2)^2}\frac{1-x_1}{[\vec{k}_1^{\,2}+(1-x_1)\mu^2+x_1^2\mu_a^{\,2}]^2}$$

$$= -\frac{ig^4}{16\pi\mu^2 s}\left(\frac{g^2}{16\pi^2\mu^2}\right)\left\{\ln\frac{1}{\epsilon}+4\pi\mu^2\int\frac{d^2k_1}{(2\pi)^2}\int_0^1\frac{dx_1}{x_1}\left[\frac{1-x_1}{(\vec{k}_1^{\,2}+(1-x_1)\mu^2+x_1^2\mu_a^{\,2})^2}-\frac{1}{(\vec{k}_1^{\,2}+\mu^2)^2}\right]\right\} \tag{2.12a}$$

$$= -\frac{ig^4}{16\pi\mu^2 s}\left(\frac{g^2}{16\pi^2\mu^2}\right)\left(\ln\frac{1}{\epsilon}-\frac{\pi}{3\sqrt{3}}\right) \quad \text{if} \quad \mu_a=\mu\,. \tag{2.12b}$$

(b) $p_{b-} > |k_{2-}| > \epsilon' p_{b-}$, $\epsilon p_{a+} > |k_{1+}|$. This region is the counterpart of (a) for particle b. [See Fig. 3(b).] After k_{i+} integrations, we find that the minus-momenta are ordered automatically $p_{b-} > -k_{2-} > \epsilon' p_{b-}$. The virtual particle 1 will now be referred to as a fragment of particle b. The scattering amplitude in this region is simply

$$T_1^{(b)} = -\frac{ig^4}{16\pi\mu^2 s}\left(\frac{g^2}{16\pi^2\mu^2}\right)\left[\ln\frac{1}{\epsilon'}+4\pi\mu^2\int\frac{d^2k_2}{(2\pi)^2}\int_0^1\frac{dy_1}{y_1}\left(\frac{1-y_1}{[\vec{k}_2^{\,2}+(1-y_1)\mu^2+y_1^2\mu_b^{\,2}]^2}-\frac{1}{(\vec{k}_2^{\,2}+\mu^2)^2}\right)\right], \tag{2.13}$$

where $y_1 = -k_{2-}/\sqrt{s}$ is the scaled longitudinal momentum of k_2 relative to p_b.

(c) $|k_{1+}| < \epsilon p_{a+}$, $|k_{2-}| < \epsilon' p_{b-}$. This region will be referred to as the pionization region [Fig. 3(c)]. The scattering amplitude in this region can be expressed as

$$T_1^{(0)} = \frac{i(ig)^6}{2}\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{i^2}{(k_1^{\,2}-\mu^2+i\epsilon)^2}\frac{i^2}{(k_2^{\,2}-\mu^2+i\epsilon)^2}\frac{i}{(k_1-k_2)^2-\mu^2+i\epsilon}$$

$$\times\left[\frac{i}{(p_a-k_1)^2-\mu_a^{\,2}+i\epsilon}+\frac{i}{(p_a+k_1)^2-\mu_a^{\,2}+i\epsilon}\right]\left[\frac{i}{(p_b+k_2)^2-\mu_b^{\,2}+i\epsilon}+\frac{i}{(p_b-k_2)^2-\mu_b^{\,2}+i\epsilon}\right]\Bigg|_{|k_{1+}|<\epsilon p_{a+},\ |k_{2-}|<\epsilon' p_{b-}}, \tag{2.14}$$

where we have symmetrized the ladder and the crossed ladder at both ends and hence included a counting factor $\frac{1}{2}$. Based on the discussion given in (a), we find that the expressions in the square brackets reduce at large s to

$$\frac{2\pi}{\sqrt{s}}\delta(k_{1-}) \quad \text{and} \quad \frac{2\pi}{\sqrt{s}}\delta(k_{2+}), \tag{2.15}$$

respectively.[14] With this simplification, we have

$$T_1^{(0)} = \frac{-ig^6}{16\pi^2 s}\int\frac{d^2k_1}{(2\pi)^2}\frac{1}{(\vec{k}_1^{\,2}+\mu^2)^2}\int\frac{d^2k_2}{(2\pi)^2}\frac{1}{(\vec{k}_2^{\,2}+\mu^2)^2}\int_0^{\epsilon\sqrt{s}}dk_{1+}\int_{-\epsilon'\sqrt{s}}^{\epsilon'\sqrt{s}}dk_{2-}\frac{i}{k_{1+}k_{2-}-(\vec{k}_1-\vec{k}_2)^2-\mu^2+i\epsilon}, \tag{2.16}$$

where the extra factor $\frac{1}{2}$ is canceled after we restrict the k_{1+} integration to $k_{1+} > 0$. In terms of scaled variables

$$x_1 = k_{1+}/p_{a+} = k_{1+}/\sqrt{s}, \qquad y_1 = k_{2-}/p_{b-} = k_{2-}/\sqrt{s},$$

Eq. (2.16) reduces to

$$T_1^{(0)} = -\frac{ig^6}{16\pi^2}\int\frac{d^2k_1}{(2\pi)^2}\frac{1}{(\vec{k}_1^{\,2}+\mu^2)^2}\int\frac{d^2k_2}{(2\pi)^2}\frac{1}{(\vec{k}_2^{\,2}+\mu^2)^2}\int_0^{\epsilon}dx_1\int_{-\epsilon'}^{\epsilon'}dy_1\frac{i}{x_1y_1s-(\vec{k}_1-\vec{k}_2)^2-\mu^2+i\epsilon}. \tag{2.17}$$

Several features of the above expression are worth mentioning. First, the dependence of the amplitude on the external particle masses μ_a and μ_b drops out completely in this region. This is actually a general result for a pionization amplitude. Namely, the distribution properties described in a pionization amplitude are independent of the nature of the target and projectile which produce the scattering. Second, the x_1, y_1 integrals are of the general form

$$I_n(s) = \lim_{s\gg 1}\int_0^1 dx_1\cdots dx_n\frac{1}{x_1x_2\cdots x_n s+1}. \tag{2.18}$$

The asymptotic behavior of $I_n(s)$ is worked out explicitly in Appendix A to the leading order in s and to all orders in $\ln s$. In particular, we have

$$I_2(s) = \frac{1}{s}\left(\frac{1}{2!}\ln^2 s+\frac{1}{3!}\pi^2\right). \tag{2.19}$$

Making use of (2.19), we obtain

$$T_1^{(0)} = -\frac{g^6}{16\pi^2 s}\int\frac{d^2k_1}{(2\pi)^2}\frac{1}{(\vec{k}_1{}^2+\mu^2)^2}\int\frac{d^2k_2}{(2\pi)^2}\frac{1}{(\vec{k}_2{}^2+\mu^2)^2}\left[\frac{1}{2}\left(\ln\frac{\epsilon\epsilon' s}{(\vec{k}_1-\vec{k}_2)^2+\mu^2}\right)^2+\frac{\pi^2}{6}\right]+(s\to -s)$$

$$= -i\frac{g^4}{4s}\left(\frac{g^2}{4\pi}\right)\int\frac{d^2k_1}{(2\pi)^2}\frac{1}{(\vec{k}_1{}^2+\mu^2)^2}\int\frac{d^2k_2}{(2\pi)^2}\frac{1}{(\vec{k}_2{}^2+\mu^2)^2}\left(\ln\frac{\epsilon\epsilon' s}{(\vec{k}_1-\vec{k}_2)^2+\mu^2}-\frac{i\pi}{2}\right), \tag{2.20}$$

where we have used the fact that

$$\ln(-s) = \ln s - i\pi . \tag{2.21}$$

A fourth region given by $|k_{1+}| > \epsilon p_{a+}$, $|k_{2-}| > \epsilon' p_{b-}$ does not contribute at large s. This is because of the extra damping due to the propagator of particle 1,

$$\frac{1}{(k_1-k_2)^2-\mu^2+i\epsilon}\approx\frac{1}{-k_{1+}k_{2-}}<\frac{1}{\epsilon\epsilon' s}.$$

The contribution from this region is at least an order of s smaller than the contribution from the previous three regions, and can be ignored in our calculation. Physically, this implies that the fragmentation regions defined in (a) and (b) do not overlap.

The dependence on ϵ and ϵ' disappear when (2.12), (2.13), and (2.20) are added together to give the final answer

$$T_1 = -\frac{ig^4}{16\pi\mu^2 s}\left(\frac{g^2}{16\pi^2\mu^2}\right)\times\left[\left(a_1\ln\frac{s}{\mu^2}+b_1-\frac{i\pi}{2}\right)+F_1^{(a)}+F_1^{(b)}\right], \tag{2.22}$$

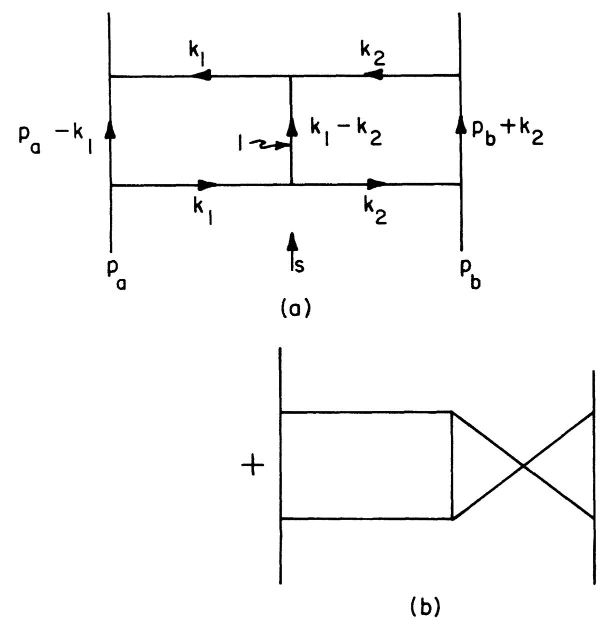

FIG. 2. (a) The straight-ladder diagram with one internal rung. (b) The crossed-ladder diagram with one internal rung.

where

$$a_1 = (4\pi\mu^2)^2\left[\int\frac{d^2k}{(2\pi)^2}\frac{1}{(\vec{k}^2+\mu^2)^2}\right]^2 = 1,$$

$$b_1 = (4\pi\mu^2)^2\int\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{1}{(\vec{k}_1{}^2+\mu^2)^2(\vec{k}_2{}^2+\mu^2)^2}\times\ln\frac{\mu^2}{(\vec{k}_1-\vec{k}_2)^2+\mu^2},$$

$$F_1^{(a)} = 4\pi\mu^2\int\frac{d^2k_1}{(2\pi)^2}\int_0^1\frac{dx_1}{x_1}\times\left(\frac{1-x_1}{[\vec{k}_1{}^2+(1-x_1)\mu^2+x_1{}^2\mu_a{}^2]^2}-\frac{1}{(\vec{k}_1{}^2+\mu^2)^2}\right), \tag{2.23}$$

$$F_1^{(b)} = 4\pi\mu^2\int\frac{d^2k_2}{(2\pi)^2}\int_0^1\frac{dy_1}{y_1}\times\left(\frac{1-y_1}{[\vec{k}_2{}^2+(1-y_1)\mu^2+y_1{}^2\mu_b{}^2]^2}-\frac{1}{(\vec{k}_2{}^2+\mu^2)^2}\right).$$

Equations (2.22) and (2.23) show explicitly the symmetry with respect to the two particles a and b.

Using the standard Feynman parameter technique, we can write b_1 as

$$b_1 = -\frac{\pi}{3\sqrt{3}}+\int_0^1 dx\frac{(1-x)^2}{(1-x+x^2)^2}\ln x(1-x) = -1.5626 . \tag{2.24}$$

If $\mu_a{}^2=\mu^2$, we have

$$F_1^{(a)} = -\frac{\pi}{3\sqrt{3}}. \tag{2.25}$$

In Appendix B, we verify the result for T_1 using the conventional Feynman parameter method.[12]

For completeness, we give the result for the fourth-order diagrams ($n=0$).

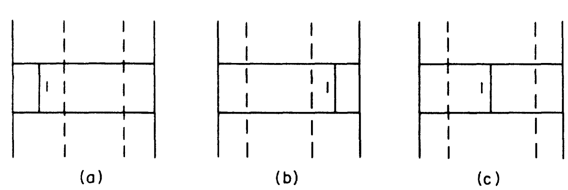

FIG. 3. The contribution of the straight-ladder diagram in Fig. 2(a) is divided according to its internal longitudinal momenta. (a) Particle 1 is in the fragmentation region of particle a; (b) particle 1 is in the fragmentation region of particle b; (c) particle 1 is in the pionization region.

$$T_0 = -\frac{ig^4}{16\pi\mu^2 s}. \qquad (2.26)$$

C. Two-Rung Amplitude

For more complicated ladder diagrams, we can divide and evaluate the amplitude analogously. In the following, we shall demonstrate explicitly how the contribution from a two-rung ladder (Fig. 4) should be divided. The important regions of integration for asymptotic behavior are classified in Figs. 5(a)–5(f) as follows:

(a) $|k_{1+}|,\ |k_{2+}| > \epsilon p_{a+},\ |k_{3-}| < \epsilon' p_{b-}$. In this region, both particles 1 and 2 are fragments of particle a. In analogy to the ordering in Fig. 3(a), the momenta $k_{1+},\ k_{2+}$ are ordered according to

$$p_{a+} > k_{1+} > k_{2+} > \epsilon p_{a+}. \qquad (2.27)$$

(b) $|k_{3-}|,\ |k_{2-}| > \epsilon' p_{b-},\ |k_{1+}| < \epsilon p_{a+}$. In this region, both particles 1 and 2 are fragments of particle b. The momentum variables are ordered as

$$p_{b-} > -k_{3-} > -k_{2-} > \epsilon' p_{b-}. \qquad (2.28)$$

(c) $|k_{1+}| < \epsilon p_{a+},\ |k_{3-}| < \epsilon' p_{b-}$. Both particles 1 and 2 are in the pionization region.

(d) $|k_{1+}| > \epsilon p_{a+},\ |k_{2+}| < \epsilon p_{a+},\ |k_{3-}| < \epsilon' p_{b-}$.

(e) $|k_{1+}| < \epsilon p_{a+},\ |k_{2-}| < \epsilon' p_{b-},\ |k_{3-}| > \epsilon' p_{b-}$. The two regions (d) and (e) describe one pionization particle and one fragmentation particle.

(f) $|k_{1+}| > \epsilon p_{a+},\ |k_{2+}| < \epsilon p_{a+},\ |k_{2-}| < \epsilon' p_{b-},\ |k_{3-}| < \epsilon' p_{b-}$. In this region, particle 1 is a fragment of a and particle 2 is a fragment of b. It is straightforward to see that no other regions of integration contribute at large s.

The simple examples studied in this section will provide the basis for discussing the notion of fragmentation and pionization in the following sections. We shall first study the contributions from pure fragmentation regions such as in Figs. 5(a) and 5(b), and from pure pionization regions such as in Fig. 5(c). These results will then be generalized to include the mixed regions such as in Figs. 5(d)–5(f).

Our calculational technique developed here relies mainly on physical interpretation rather than mathematical rigor. Extensive studies on the ladder amplitude and a more rigorous procedure for treating these diagrams exist in the literature (e.g., see Ref. 12 and references cited therein). However, our main concern in this paper is to reveal some general features of this amplitude in terms of the concepts of pionization and fragmentation. In this respect, our approach has the advantage of presenting a clear physical picture which enables us to understand the significance of various contributions from different kinematic regions leading to different high-energy behavior. We feel that such a better understanding is important. Our use of a more intuitive and physical approach at the expense of mathematical rigor is therefore justifiable. The calculation presented in Appendix B provides support for the correctness of the infinite-momentum technique.

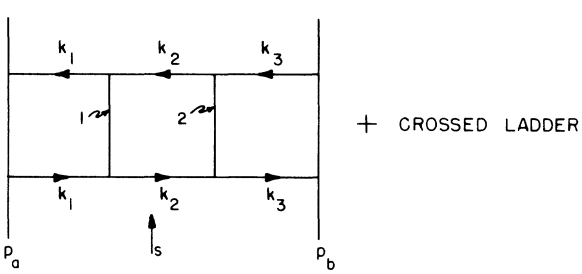

FIG. 4. The straight- and crossed-ladder diagrams with two internal rungs.

III. PIONIZATION

A. Basic Propositions

In the example of the sixth-order diagrams considered in Sec. II, the amplitude T_1 consists of three contributions of distinct nature. The contribution $T_1^{(0)}$ arises from the kinematic region where the two particles a and b retain their large momenta during the scattering; only a very small fraction ϵ (ϵ') of p_{a+} (p_{b-}) is taken away from particle a (b). The contribution $T_1^{(a)}$ ($T_1^{(b)}$) comes from the kinematic configuration in which particle a (particle b) dissociates into two constituents with comparable

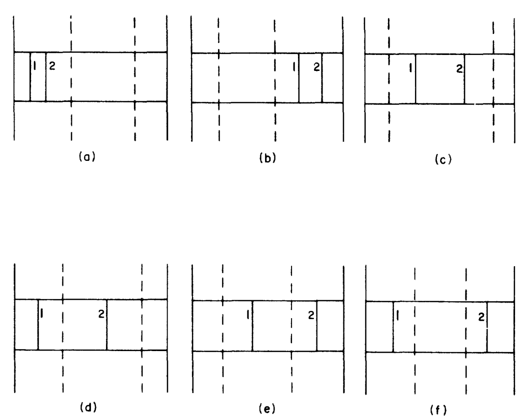

FIG. 5. The contribution of the straight-ladder diagram in Fig. 4 is divided according to its internal longitudinal momenta. (a) Both 1 and 2 are in fragmentation region a; (b) both 1 and 2 are in the fragmentation region b; (c) both 1 and 2 are in the pionization region; (d) 1 is in the fragmentation region a and 2 is in the pionization region; (e) 1 is in the pionization region and 2 is in the fragmentation region b; (f) 1 is in the fragmentation region a and 2 is in the fragmentation region b.

p_+'s (p_-'s).[15] The mechanism responsible for the first type of contributions is called pionization in which the two energetic particles a and b retain their large momenta during scattering. The particles inelastically produced by this mechanism have only very small longitudinal momenta in comparison with $(p_a)_+$ or $(p_b)_-$, these are the pionization particles.[16] The second mechanism responsible for the type of contribution $T_1^{(a)}$ ($T_1^{(b)}$) involves the dissociation of particle a (particle b) into several constituents with comparable p_+'s (p_-'s). When inelastically produced these particles have momenta proportional to $\sqrt{s}$, and they are called the fragments of particle a (particle b). This division of particles into three groups, fragments of particle a and particle b, and pionization products, is conceptually useful as we will see in this section and Sec. IV. Nevertheless, it is obvious that as the momentum of a pionization particle approaches its upper end or the momentum of a fragment approaches its lower end, the notion between pionization and fragmentation becomes a matter of definition. This point should be kept in mind as it is important for our later discussion.

In this section we shall concentrate our attention on the pure pionization contributions. Thus, we restrict k_{1+} and $k_{(n+1)-}$ to lie in the following domain:

$$\epsilon p_{a+} > |k_{1+}| > 0\,, \qquad \epsilon' p_{b-} > |k_{(n+1)-}| > 0\,. \tag{3.1}$$

What we would like to accomplish in this and Sec. IV is to use explicit calculations for diagrams up to g^{10} to demonstrate and support the following propositions for the elastic amplitude of ladder diagrams.

(a) Instead of the complete forward scattering amplitude T, we introduce a simplified amplitude $\overline{T}$. Each $\ln s$ factor in $\overline{T}$ arises from the degree of freedom in the longitudinal phase space of a subsystem, called a cluster, of one, two, ... particles. It is possible to define a basic amplitude associated with each individual cluster consisting of a number of particles with a small invariant mass. Such a basic amplitude has the form $a_i \ln(s/\mu^2) + b_i$, where a_i depends on the number and nature of the particles involved and b_i depends on the particular definition of fragmentation adopted.

(b) For a diagram involving pionization only, $\overline{T}$ can be decomposed into a sum of several terms; each term is a product of the basic amplitudes associated with individual clusters. If the same basic amplitude appears n times, a statistical factor $1/n!$ must be included.

(c) One can also define a basic amplitude for particle a or particle b to dissociate into clusters with comparable momenta. These basic amplitudes are independent of s.

(d) In a diagram involving both fragmentation and pionization, the amplitude $\overline{T}$ can be expressed as a sum of several terms. Each term is a product of three factors, corresponding to fragmentation of particle a and particle b, and pionization.

(e) Each basic amplitude associated with a subsystem of pionization particles exponentiates to a power of s. The entire s dependence of the amplitude $\overline{T}$ comes from the pionization contribution only.

(f) The complete amplitude T can be obtained from $\overline{T}$ by a simple substitution to be described below. It turns out that T and $\overline{T}$ differ only by a signature factor [Eqs. (3.29) and (3.33)], and $\overline{T}$ is the imaginary part of T. Both T and $\overline{T}$ therefore have the same s dependence.

B. General Properties of $T_n^{(0)}$

We now proceed to verify these propositions using diagrams up to tenth order in the coupling constant g. If $T_n^{(0)}$ denotes the pionization contribution to T_n, then from the result of Sec. II we have

$$T_1^{(0)} = -\frac{ig^4}{16\pi\mu^2 s}\left(\frac{g^2}{16\pi^2\mu^2}\right)\left(a_1 \ln\frac{\epsilon\epsilon' s}{\mu^2} + b_1 - i\frac{\pi}{2}\right), \tag{3.2}$$

with a_1 and b_1 given by (2.23). This defines the basic amplitude associated with a single particle to be $a_1 \ln(\epsilon\epsilon' s/\mu^2) + b_1$. Recall that T_n represents the ladder amplitude with n internal rungs. The first and the last rungs are excluded in the counting. As we shall see later, only those momentum integrations associated with internal rungs contribute to the longitudinal phase-space factor $\ln s$. Hence, T_n contributes to a maximum power of $(\ln s)^n$ to the longitudinal phase-space integrations, rather than $(\ln s)^{n+2}$.

The scattering amplitude for an n-rung ladder in Fig. 1 can be written as

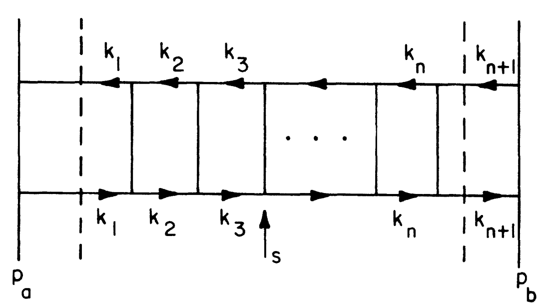

FIG. 6. The straight-ladder diagram with n rungs in the pionization region.

$$T_n = i(ig)^{2n+4}\int\frac{dk_{1+}dk_{1-}d^2k_1}{2(2\pi)^4}\cdots\int\frac{dk_{(n+1)+}dk_{(n+1)-}d^2k_{n+1}}{2(2\pi)^4}$$
$$\times\left[\frac{i}{(p_a-k_1)^2-\mu_a{}^2+i\epsilon}\,\frac{i^2}{(k_1{}^2-\mu^2+i\epsilon)^2}\,\frac{i}{(k_1-k_2)^2-\mu^2+i\epsilon}\times\cdots\right.$$
$$\left.\times\frac{i^2}{(k_n{}^2-\mu^2+i\epsilon)^2}\,\frac{i}{(k_n-k_{n+1})^2-\mu^2+i\epsilon}\,\frac{i^2}{(k_{n+1}{}^2-\mu^2+i\epsilon)^2}\,\frac{i}{(p_b-k_{n+1})^2-\mu_b{}^2+i\epsilon}+\begin{pmatrix}p_a\to -p_a\\ \text{or } p_b\to -p_b\end{pmatrix}\right]. \tag{3.3}$$

In the pionization region (Fig. 6), we have the additional restriction,

$$|k_{1+}|<\epsilon p_{a+}=\epsilon\sqrt{s}\,,\qquad |k_{n+1-}|<\epsilon' p_{b-}=\epsilon'\sqrt{s}\,. \tag{3.4}$$

As was demonstrated in Sec. II, the amplitude T_n will be greatly simplified if we average both the first and the last propagator in T_n over the straight and the crossed ladders. These propagators, after averaging, reduce to[14]

$$\left[\frac{i}{(p_a-k_1)^2-\mu_a{}^2+i\epsilon}\right]_{\text{average over}\pm p_a}=\frac{1}{2}\left[\frac{i}{(p_a-k_1)^2-\mu_a{}^2+i\epsilon}+\frac{i}{(p_a+k_1)^2-\mu_a{}^2+i\epsilon}\right]$$
$$\to\frac{\pi}{\sqrt{s}}\delta(k_{1-})\quad\text{for}\quad |k_{1+}|<\epsilon\sqrt{s} \tag{3.5}$$

and

$$\left[\frac{i}{(p_b-k_{n+1})^2-\mu_b{}^2+i\epsilon}\right]_{\text{average over}\pm p_b}=\frac{1}{2}\left[\frac{i}{(p_b-k_{n+1})^2-\mu_b{}^2+i\epsilon}+\frac{i}{(p_b+{}_{1+1})^2-\mu_b{}^2+i\epsilon}\right]$$
$$\to\frac{\pi}{\sqrt{s}}\delta(k_{(n+1)+})\quad\text{for}\quad |k_{(n+1)-}|<\epsilon'\sqrt{s}\,, \tag{3.6}$$

respectively. Expressions (3.3)–(3.6) are still manifestly symmetric with respect to particles a and b, and to plus and minus components. In an explicit calculation, however, it is convenient to carry out either all k_+ or all k_- integrations. This will lead to an expression which is no longer manifestly symmetric with respect to particles a and b. Of course, the final answer should still be symmetric. To be specific, we carry out the k_{i-} integrations for all i $(=1, 2, \ldots, n)$. As we have demonstrated in the one- and two-rung amplitudes, only the region in which all the k_+'s are positive and ordered, i.e.,

$$p_{a+}>k_{1+}>k_{2+}>\cdots>k_{n+}>0\,, \tag{3.7}$$

contributes. These k_- loop integrations put a restriction on k_+'s. For instance, there are three propagators in T_n which contain k_{n-}, namely,

$$d_{n-1,n}=\frac{i}{(k_{n-1}-k_n)^2-\mu^2+i\epsilon},\quad d_n=\frac{i}{k_n{}^2-\mu^2+i\epsilon},\quad d_{n,n+1}=\frac{i}{(k_n-k_{n+1})^2-\mu^2+i\epsilon}. \tag{3.8}$$

In order that the k_{n-} integral does not vanish, the poles of k_{n-} in $d_{n-1,n}$, d_n, and $d_{n,n-1}$ should not lie on the same side of the real axis. By the use of $k_{(n+1)+}=0$, the above restriction implies

$$|k_{(n-1)+}|>|k_{n+}|\quad\text{and}\quad \operatorname{sgn}(k_{(n-1)+})=\operatorname{sgn}(k_{n+})\,. \tag{3.9}$$

Repeating this argument on the $k_{(n-1)-}$, $k_{(n-2)-}$, ... integrations and using $p_{a+}>0$, we arrive at (3.7).

Carrying out the $k_{(n+1)+}$ integration and k_{i-} integrations for $i=1, 2, \ldots, n$, we find that the following poles of k_{i-} contribute:

$$\begin{aligned}&k_{1-}=0\,,\\ &k_{1-}-k_{2-}=\frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2-i\epsilon}{k_{1+}-k_{2+}}\,,\\ &k_{2-}-k_{3-}=\frac{(\vec{k}_2-\vec{k}_3)^2+\mu^2-i\epsilon}{k_{2+}-k_{3+}}\,,\\ &\cdots.\end{aligned} \tag{3.10}$$

Equation (3.10) corresponds to the vanishing of the propagators associated with the rungs. Substituting

(3.10) into (3.3) we have the amplitude in the pionization region,

$$T_n^{(0)} = g^2\left(\frac{g^2}{4\pi}\right)^{n+1} \prod_{i=1}^{n+1} \int \frac{d^2k_i}{(2\pi)^2} \int_0^{\epsilon} \frac{dx_1}{x_1} \int_0^{x_1} \frac{dx_2}{x_2} \cdots \int_0^{x_{n-1}} \frac{dx_n}{x_n} \int_{-\epsilon'}^{\epsilon'} dy \frac{1}{x_1(x_1-x_2)x_2(x_2-x_3)x_3\cdots(x_{n-1}-x_n)x_n}$$
$$\times\left(\frac{\vec{k}_1^{\,2}+\mu^2}{x_1}\right)^{-2}\left[\frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2}{x_1-x_2}+\frac{\vec{k}_2^{\,2}+\mu^2}{x_2}\right]^{-2}\times\cdots$$
$$\times\left[\frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2}{x_1-x_2}+\frac{(\vec{k}_2-\vec{k}_3)^2+\mu^2}{x_2-x_3}+\cdots+\frac{(\vec{k}_{n-1}-\vec{k}_n)^2+\mu^2}{x_{n-1}-x_n}+\frac{\vec{k}_n^{\,2}+\mu^2}{x_n}\right]^{-2}$$
$$\times\left(\vec{k}_{n+1}^{\,2}+\mu^2\right)^{-2}\left[x_n y s - x_n\left(\frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2}{x_1-x_2}+\cdots+\frac{(\vec{k}_n-\vec{k}_{n+1})^2+\mu^2}{x_n-0}\right)+i\epsilon\right]^{-1}, \tag{3.11}$$

where $x_i \equiv k_{i+}/p_{a+} = k_{i+}/\sqrt{s}$ and $y \equiv -k_{(n+1)-}/p_{i-} = -k_{(n+1)-}/\sqrt{s}$ are scaled longitudinal momenta. The symmetrization between straight and crossed ladders is now reflected in the y integration, which runs from 0 to ϵ' and from $-\epsilon'$ to 0.

Another useful form of $T_n^{(0)}$, which is expressed in terms of a new set of variables z_i defined by

$$z_i \equiv x_i/x_{i-1}, \quad x_i \equiv z_1 z_2 \cdots z_i, \quad i=1,2,\ldots,n \tag{3.12}$$

with all the z_i, $i \geq 2$ restricted to the range between 0 and 1, is

$$T_n^{(0)} = g^2\left(\frac{g^2}{4\pi}\right)^{n+1} \prod_{i=1}^{n+1} \int \frac{d^2k_i}{(2\pi)^2} \int_0^{\epsilon} dz_1 \int_0^1 dz_2 \cdots \int_0^1 dz_n \int_0^{\epsilon'} dy$$
$$\times \prod_{i=1}^{n}\left\{z_1^{-2}z_2^{-2}\cdots z_i^{-2}(1-z_i)^{-1}\left[\frac{\vec{k}_1^{\,2}+\mu^2}{1-z_1}+\frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2}{z_1(1-z_2)}+\frac{(\vec{k}_2-\vec{k}_3)^2+\mu^2}{z_1z_2(1-z_3)}+\cdots+\frac{\vec{k}_i^{\,2}+\mu^2}{z_1\cdots z_i}\right]^{-2}\right\}$$
$$\times\frac{1}{(\vec{k}_{n+1}^{\,2}+\mu^2)^2}\left\{z_1z_2\cdots z_n y s - z_1 z_2\cdots z_n\left[\frac{\vec{k}_1^{\,2}+\mu^2}{1-z_1}+\frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2}{z_1(1-z_2)}+\cdots\right.\right.$$
$$\left.\left.+\frac{(\vec{k}_{n-1}-\vec{k}_n)^2+\mu^2}{z_1\cdots z_{n-1}(1-z_n)}+\frac{(\vec{k}_n-\vec{k}_{n+1})^2+\mu^2}{z_1\cdots z_n}\right]^2+i\epsilon\right\}^{-1}+(s\to -s), \tag{3.13}$$

where we now restrict the y integration to $(0, \epsilon')$ only and include the other half of the integration $(-\epsilon', 0)$ by an explicit $(s\to -s)$ crossing substitution.

A simple inspection shows that each of the $z_1, \ldots, z_n$ integrations in $T_n^{(0)}$ contributes a factor of $\ln(s/\mu^2)$ arising from the integration region $(0, \epsilon_i)$, where ϵ_i is a small but finite number. Thus, it is very useful for the purpose of extracting the $\ln^n s$ factors to divide each of the $z_1, \ldots, z_n$ integrations into two regions, $(0, \epsilon_i)$ and $(\epsilon_i, 1)$. These ϵ's can be chosen to be very small so that in the region $(0, \epsilon_i)$ this particular variable z can be set to be zero everywhere except where it appears in the coefficient of s in the denominator of the last factor in (3.13). It will be clear that to evaluate these z_i integrations we encounter the integral $I_n(s)$ at large s as introduced in (2.18). This integral is worked out in Appendix A, and gives an asymptotic expression $I_n(s) \propto (1/sn!)(\ln s)^n$. It will also be clear that a z integral over the region $(\epsilon_i, 1)$ does not contribute to a $\ln s$ factor. The significance of this fact will be demonstrated in the following explicit calculations.

C. The Amplitude $T_2^{(0)}$

The amplitude $T_2^{(0)}$ can be calculated easily from (3.13), and is given by

$$T_2^{(0)} = \frac{g^4}{4\pi}\left(\frac{g^2}{4\pi}\right)^2 \int \frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{d^2k_3}{(2\pi)^2}\frac{1}{(\vec{k}_1^{\,2}+\mu^2)^2(\vec{k}_3^{\,2}+\mu^2)^2}\int_0^{\epsilon} dz_1 \int_0^1 dz_2 \int_0^{\epsilon} dy \frac{1-z_2}{[z_2(\vec{k}_1-\vec{k}_2)^2+(1-z_2)\vec{k}_2^{\,2}+\mu^2]^2}$$
$$\times\left[z_1z_2ys-\frac{z_2}{1-z_2}[(\vec{k}_1-\vec{k}_2)^2+\mu^2]-(\vec{k}_2-\vec{k}_3)^2-\mu^2+i\epsilon\right]^{-1}+(s\to -s), \tag{3.14}$$

where we have neglected terms which vanish as $\epsilon, \epsilon' \to 0$. Choosing a small parameter ϵ_2, we can rewrite $T_2^{(0)}$ as

$$T_2^{(0)}=\frac{g^4}{4\pi}\left(\frac{g^2}{4\pi}\right)^2\int\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{d^2k_3}{(2\pi)^2}\frac{1}{(\vec{k}_1^{\,2}+\mu^2)^2(\vec{k}_3^{\,2}+\mu^2)^2}$$
$$\times\left\{\int_0^\epsilon dz_1\int_0^{\epsilon_2}dz_2\int_0^{\epsilon'}dy\frac{1}{(\vec{k}_2^{\,2}+\mu^2)^2}\frac{1}{z_1z_2ys-(\vec{k}_2-\vec{k}_3)^2-\mu^2+i\epsilon}\right.$$
$$+\int_0^\epsilon dz_1\int_{\epsilon_2}^1 dz_2\int_0^{\epsilon'}dy\frac{1-z_2}{[z_2(\vec{k}_1-\vec{k}_2)^2+(1-z_2)\vec{k}_2^{\,2}+\mu^2]^2}$$
$$\left.\times\left[z_1z_2ys-\frac{z_2(\vec{k}_1-\vec{k}_2)^2+(1-z_2)(\vec{k}_2-\vec{k}_3)^2+\mu^2}{1-z_2}+i\epsilon\right]^{-1}\right\}+(s\to -s)$$
$$=\frac{g^4}{4\pi}\left(\frac{g^2}{4\pi}\right)^2\int\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{d^2k_3}{(2\pi)^2}\frac{1}{(\vec{k}_1^{\,2}+\mu^2)^2(\vec{k}_3^{\,2}+\mu^2)^2}$$
$$\times\left[\frac{1}{(\vec{k}_2^{\,2}+\mu^2)^2}\frac{1}{s}J_3\left(-\frac{\epsilon\epsilon_2\epsilon' s}{(\vec{k}_2-\vec{k}_3)^2+\mu^2}\right)+\int_{\epsilon_2}^1 dz_2\frac{1-z_2}{[z_2(\vec{k}_1-\vec{k}_2)^2+(1-z_2)\vec{k}_2^{\,2}+\mu^2]^2}\right.$$
$$\left.\times\frac{1}{z_2 s}J_2\left(-\frac{\epsilon\epsilon' s z_2(1-z_2)}{z_2(\vec{k}_1-\vec{k}_2)^2+(1-z_2)(\vec{k}_2-\vec{k}_3)^2+\mu^2}\right)\right]+(s\to -s)\,. \tag{3.15}$$

The first term in the above square brackets gives rise to a factor

$$\frac{1}{s}J_3\left(\frac{\epsilon\epsilon'\epsilon_2 s}{(\vec{k}_2-\vec{k}_3)^2+\mu^2}\right)\propto(1/s)(\ln\epsilon\epsilon'\epsilon_2 s)^3,$$

while the second term contributes only a $(1/s)\ln^2(\epsilon\epsilon' s)$ factor. In forming the s-u crossing-symmetric amplitude, we always encounter the following combination:

$$\frac{1}{s}\left[J_n\left(\frac{s}{\mu^2}\right)-J_n\left(-\frac{s}{\mu^2}\right)\right]=\frac{1}{s}\left[\ln^n\frac{s}{\mu^2}-\left(\ln\frac{s}{\mu^2}-i\pi\right)^n+\cdots\right]$$
$$=\frac{i\pi n}{s}\left[\ln^{n-1}\frac{s}{\mu^2}-\frac{n-1}{2}(i\pi)\ln^{n-2}\frac{s}{\mu^2}+\cdots\right], \tag{3.16}$$

where J_n is the function introduced in Eq. (A11). We note that keeping the first term is sufficient to keep track of all the remaining terms. To recover the other terms from the first term, we make the substitution

$$\frac{1}{(n-1)!}\ln^{n-1}\frac{s}{\mu^2}\to\frac{1}{i\pi}\left[J_n\left(\frac{s}{\mu^2}\right)-J_n\left(-\frac{s}{\mu^2}\right)\right]. \tag{3.17}$$

This substitution works for two reasons. The quantities $J_n(s/\mu^2)-J_n(s/\mu^2)$ appear only linearly in T, and (3.17) is independent of μ^2. Namely, one can change the scale μ^2 without affecting the final answer. To be more precise, if we change μ^2 to μ_0^2, after keeping only the first term of (3.16), we have

$$\frac{1}{(n-1)!}\ln^{n-1}\frac{s}{\mu^2}=\frac{1}{(n-1)!}\left(\ln\frac{s}{\mu_0^2}+\ln\frac{\mu_0^2}{\mu^2}\right)^{n-1}$$
$$=\sum_{m=0}^{n-1}\frac{1}{m!(n-m-1)!}\ln^{n-m-1}\left(\frac{s}{\mu_0^2}\right)\ln^m\left(\frac{\mu_0^2}{\mu^2}\right).$$

The substitution (3.17), when applied to the right-hand side of the above equation, gives

$$\frac{1}{(n-1)!}\ln^{n-1}\frac{s}{\mu^2}\to\frac{1}{i\pi}\sum_{m=0}^{n}\frac{1}{m!}\left[J_{n-m}\left(\frac{s}{\mu_0^2}\right)-J_{n-m}\left(-\frac{s}{\mu_0^2}\right)\right]\ln^m\frac{\mu_0^2}{\mu^2}$$
$$=\frac{1}{i\pi}\left[J_n\left(\frac{s}{\mu^2}\right)-J_n\left(-\frac{s}{\mu^2}\right)\right],$$

which is identical to (3.17). The last step follows from the identity

$$J_n\left(\frac{s}{\mu^2}\right)=\sum_{m=0}^{n}\frac{1}{m!}J_{n-m}\left(\frac{s}{\mu_0^2}\right)\ln^m\frac{\mu_0^2}{\mu^2}. \tag{3.18}$$

This identity (3.18) can be easily verified for $n \leq 4$ and can be established in general by comparing the coefficients of g^n of the power-series expansion on both sides of the following relation between the generating functionals for $J_n(s/\mu^2)$ and $J_n(s/\mu_0^2)$:

$$\left(\frac{s}{\mu^2}\right)^g \frac{\pi g}{\sin\pi g} = \left(\frac{\mu_0^2}{\mu^2}\right)^g \left(\frac{s}{\mu_0^2}\right)^g \frac{\pi g}{\sin\pi g}.$$

Identity (3.18) implies that after keeping only the first leading terms in the expansions of $J_n(s/\mu^2) - J_n(-s/\mu^2)$, one can simplify the resultant expressions by grouping the logarithms or changing the scales of the masses without altering the final answer when the substitution (3.17) is made to recover the full amplitude. Now we introduce a simplified amplitude $\overline{T}_n^{(0)}$ by keeping only the leading $(i\pi n/s)\ln^{n-1}(s/\mu^2)$ terms in $T_n^{(0)}$. We can recover $T_n^{(0)}$ from $\overline{T}_n^{(0)}$ in the final answer by the substitution (3.17) or (3.18).

From (3.15), we have

$$\overline{T}_2^{(0)} = -\frac{ig^4}{4s}\left(\frac{g^2}{4\pi}\right)^2 \int \frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{d^2k_3}{(2\pi)^2}\frac{1}{(\vec{k}_1^{\,2}+\mu^2)^2(\vec{k}_3^{\,2}+\mu^2)^2}$$
$$\times\left(\frac{1}{2!}\frac{1}{(\vec{k}_2^{\,2}+\mu^2)^2}\ln^2\frac{\epsilon\epsilon'\epsilon_2 s}{(\vec{k}_2-\vec{k}_3)^2+\mu^2} + \int_{\epsilon_2}^1 \frac{dz_2}{z_2}\frac{1-z_2}{[z_2(\vec{k}_1-\vec{k}_2)^2+(1-z_2)\vec{k}_2^{\,2}+\mu^2]^2}\ln\frac{\epsilon\epsilon' s z_2(1-z_2)}{(1-z_2)(\vec{k}_2-\vec{k}_3)^2+z_2(\vec{k}_1-\vec{k}_2)^2+\mu^2}\right). \tag{3.19}$$

Note that the sum is independent of ϵ_2, and can be rearranged to yield the following form:

$$\overline{T}_2^{(0)} = -\frac{ig^4}{16\pi\mu^2 s}\lambda^2\left[\frac{1}{2}\left(a_1\ln\frac{\epsilon\epsilon' s}{\mu^2}+b_1\right)^2 + a_2\ln\frac{\epsilon\epsilon' s}{\mu^2}+b_2\right], \tag{3.20}$$

where

$$\lambda = \frac{g^2}{16\pi^2\mu^2}, \tag{3.21}$$

$$a_2 = -(4\pi\mu^2)^2\int\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{1}{(\vec{k}_1^{\,2}+\mu^2)^2(\vec{k}_2^{\,2}+\mu^2)^2}\ln\frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2}{\mu^2}, \tag{3.22}$$

and

$$b_2 = -\tfrac{1}{2}b_1^{\,2} + \tfrac{1}{2}(4\pi\mu^2)^2\int\frac{d^2k_2}{(2\pi)^2}\frac{d^2k_3}{(2\pi)^2}\frac{1}{(\vec{k}_2^{\,2}+\mu^2)^2(\vec{k}_3^{\,2}+\mu^2)^2}\ln^2\frac{(\vec{k}_2-\vec{k}_3)^2+\mu^2}{\mu^2}$$
$$+(4\pi\mu^2)^3\int\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{d^2k_3}{(2\pi)^2}\frac{1}{(\vec{k}_1^{\,2}+\mu^2)^2(\vec{k}_3^{\,2}+\mu^2)^2}$$
$$\times\int_0^1\frac{dz_2}{z_2}\left[\left(\frac{1-z_2}{[z_2(\vec{k}_1-\vec{k}_2)^2+(1-z_2)\vec{k}_2^{\,2}+\mu^2]^2}-\frac{1}{(\vec{k}_2^{\,2}+\mu^2)^2}\right)\ln\frac{\mu^2 z_2}{(\vec{k}_2-\vec{k}_3)^2+\mu^2}\right.$$
$$\left.+\frac{1-z_2}{[z_2(\vec{k}_1-\vec{k}_2)^2+(1-z_2)\vec{k}_2^{\,2}+\mu^2]^2}\ln\frac{(1-z_2)[(\vec{k}_2-\vec{k}_3)^2+\mu^2]}{z_2(\vec{k}_1-\vec{k}_2)^2+(1-z_2)(\vec{k}_2-\vec{k}_3)^2+\mu^2}\right]. \tag{3.23}$$

To arrive at (3.23) we have used the results

$$\int_0^1\frac{dz_2}{z_2}\left[\frac{1-z_2}{[z_2(\vec{k}_1-\vec{k}_2)^2+(1-z_2)\vec{k}_2^{\,2}+\mu^2]^2}-\frac{1}{(\vec{k}_2^{\,2}+\mu^2)^2}\right]=\frac{1}{(\vec{k}_2^{\,2}+\mu^2)^2}\left[\ln\frac{\vec{k}_2^{\,2}+\mu^2}{\mu^2}-1+\ln\frac{\mu^2}{(\vec{k}_1-\vec{k}_2)^2+\mu^2}\right] \tag{3.24}$$

and

$$\int\frac{d^2k}{(2\pi)^2}\frac{1}{(\vec{k}^2+\mu^2)^2}\left(\ln\frac{\vec{k}^2+\mu^2}{\mu^2}-1\right)=0. \tag{3.25}$$

Thus, from (3.20), the quantity $a_2\ln(\epsilon\epsilon' s/\mu^2)+b_2$ is seen to be the basic amplitude associated with a subsystem of two particles, since this contribution originates from the region where z_2 is not small. Perhaps the result (3.20) constitutes nothing new since the leading terms are known to form an exponential series. Therefore, the amplitude $\overline{T}_2^{(0)}$ can always be written in the form as given in (3.20). The consistency of this identification can only be appreciated by exhibiting the structure for $\overline{T}_3^{(0)}$ which we now calculate.

D. The Amplitude $T_3^{(0)}$

The amplitude $T_3^{(0)}$ is obtained from (3.13) to be

$$T_3^{(0)}=\frac{g^4}{4\pi}\left(\frac{g^2}{4\pi}\right)^3\int\frac{d^2k_1}{(2\pi)^2}\cdots\frac{d^2k_4}{(2\pi)^2}\frac{1}{(\vec{k}_1{}^2+\mu^2)^2(\vec{k}_4{}^2+\mu^2)^2}\int_0^\epsilon dz_1\int_0^1 dz_2\int_0^1 dz_3\int_{-\epsilon'}^{\epsilon'}dy$$
$$\times\frac{(1-z_2)(1-z_3)}{[z_2(\vec{k}_1-\vec{k}_2)^2+(1-z_2)\vec{k}_2{}^2+\mu^2]^2}\left\{\frac{(1-z_3)z_2z_3}{1-z_2}[(\vec{k}_1-\vec{k}_2)^2+\mu^2]+z_3(\vec{k}_2-\vec{k}_3)^2+(1-z_3)\vec{k}_3{}^2+\mu^2\right\}^{-2}$$
$$\times\left\{z_1z_2z_3ys-\left[(\vec{k}_3-\vec{k}_4)^2+\mu^2+\left(z_2z_3\frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2}{1-z_2}+z_3\frac{(\vec{k}_2-\vec{k}_3)^2+\mu^2}{1-z_3}\right)\right]+i\epsilon\right\}^{-1}. \tag{3.26}$$

A similar calculation gives

$$\overline{T}_3^{(0)}=-\frac{ig^4}{4s}\frac{\lambda^3}{4\pi\mu^2}\left[\frac{1}{3!}\left(a_1\ln\frac{\epsilon\epsilon's}{\mu^2}+b_1\right)^3+\left(a_1\ln\frac{\epsilon\epsilon's}{\mu^2}+b_1\right)\left(a_2\ln\frac{\epsilon\epsilon's}{\mu^2}+b_2\right)+\left(a_3\ln\frac{\epsilon\epsilon's}{\mu^2}+b_3\right)\right], \tag{3.27}$$

where

$$\lambda=g^2/16\pi^2\mu^2,$$
$$a_3=(4\pi\mu^2)^3\left(\frac{1}{4\pi\mu^2}\int\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{1}{(\vec{k}_1{}^2+\mu^2)^2}\int_0^1\frac{dz_2}{z_2}\ln z_2\left(\frac{1-z_2}{[z_2(\vec{k}_1-\vec{k}_2)^2+(1-z_2)\vec{k}_2{}^2+\mu^2]^2}-\frac{1}{(\vec{k}_2{}^2+\mu^2)^2}\right)\right.$$
$$+\int\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{d^2k_3}{(2\pi)^2}\frac{1}{(\vec{k}_1{}^2+\mu^2)^2}\int_0^1\frac{dz_2}{z_2}\frac{dz_3}{z_3}\left\{\frac{(1-z_2)(1-z_3)}{[z_2(\vec{k}_1-\vec{k}_2)^2+(1-z_2)\vec{k}_2{}^2+\mu^2]^2}\right.$$
$$\times\left[\left(\frac{(1-z_3)z_2z_3}{1-z_2}[(\vec{k}_1-\vec{k}_2)^2+\mu^2]+z_3(\vec{k}_2-\vec{k}_3)^2+(1-z_3)\vec{k}_3{}^2+\mu^2\right)^{-2}-[z_3(\vec{k}_2-\vec{k}_3)^2+(1-z_3)\vec{k}_3{}^2+\mu^2]^{-2}\right]$$
$$\left.\left.+\left(\frac{1-z_2}{[z_2(\vec{k}_1-\vec{k}_2)^2+(1-z_2)\vec{k}_2{}^2+\mu^2]^2}-\frac{1}{(\vec{k}_2{}^2+\mu^2)^2}\right)\left(\frac{1-z_3}{(z_3(\vec{k}_2-\vec{k}_3)^2+(1-z_3)\vec{k}_3{}^2+\mu^2)^2}-\frac{1}{(\vec{k}_3{}^2+\mu^2)^2}\right)\right\}\right), \tag{3.28}$$

and b_3 is a very complicated expression which is of no special interest to us. The basic amplitude associated with a three-particle system is identified from (3.27) to be $a_3\ln(\epsilon\epsilon's/\mu^2)+b_3$.

The result (3.27) provides a crucial test of our previous assignment of $a_2\ln(\epsilon\epsilon's/\mu^2)+b_2$ as the basic amplitude associated with a two-particle system. A different choice for a_2 will change the coefficient of $\ln^2(\epsilon\epsilon's/\mu^2)$ and the decomposition of $T_2^{(0)}$ ($\overline{T}_2^{(0)}$) into the form (3.27) will be impossible. The fact that $T_3^{(0)}$ ($\overline{T}_3^{(0)}$) comes out automatically in a structure as expected from our propositions is not accidental. It is a nontrivial support to our suggestions. Our identification of a_1, a_2, and a_3 as the coefficients of $\ln(\epsilon\epsilon's/\mu^2)$ associated with a one-, two-, and three-particle cluster acquires more credence when we take into account the fragmentation events. As we shall see in Sec. IV, not only these parameters consistently reappear in the right places, but there is a striking resemblance between these quantities and the fragmentation amplitudes. In fact there is a one-to-one correspondence, and we can infer one from the others. We therefore content ourselves to stop here and conclude that the pionization contributions sum up to a general result for the amplitude $T^{(0)}$ ($\overline{T}^{(0)}$)[17]:

$$\overline{T}^{(0)}=\sum_n\overline{T}_n^{(0)}=-\frac{ig^4}{16\pi\mu^2s}\exp\sum_n\lambda^n(a_n\ln(\epsilon\epsilon's/\mu^2)+b_n)$$
$$=-\frac{ig^4}{16\pi\mu^2s}\left(\frac{\epsilon\epsilon's}{s_0(\lambda)}\right)^{a(\lambda)}, \tag{3.29}$$

where

$$\frac{1}{s_0(\lambda)}=\frac{1}{\mu^2}e^{b(\lambda)/a(\lambda)}, \tag{3.30}$$

$$a(\lambda)=\sum\lambda^na_n,\quad b(\lambda)=\sum\lambda^nb_n. \tag{3.31}$$

As we shall see later, $a(\lambda)$ is related to the conventional Regge trajectory function $\alpha(\lambda,t)$ at $t=0$ through

$$\alpha(\lambda,0)=a(\lambda)-1. \tag{3.32}$$

Thus, the trajectory function is expressed as a sum of contributions due to one-, two-, . . . , -particle clusters. It should be emphasized again that $a(\lambda)$, or the trajectory function $\alpha(\lambda, 0)$, is independent of the masses of the external particles μ_a and μ_b. Now

$$\overline{T}^{(0)} = -\frac{ig^4}{16\pi\mu^2 s}\sum \frac{a(\lambda)^{n-1}}{(n-1)!}\ln^{n-1}\frac{\epsilon\epsilon' s}{s_0}$$

implies

$$\begin{aligned} T^{(0)} &= -\frac{ig^4}{16\pi\mu^2 s}\sum \frac{a(\lambda)^{n-1}}{i\pi}\left[J_n\left(\frac{\epsilon\epsilon' s}{s_0}\right) - J_n\left(-\frac{\epsilon\epsilon' s}{s_0}\right)\right] \\ &= -\frac{ig^4}{16\pi\mu^2 s}\frac{1}{i\pi a(\lambda)}\left[\left(\frac{\epsilon\epsilon' s}{s_0}\right)^{a(\lambda)} - \left(-\frac{\epsilon\epsilon' s}{s_0}\right)^{a(\lambda)}\right]\frac{\pi a(\lambda)}{\sin\pi a(\lambda)} \\ &= -\frac{ig^4}{16\pi\mu^2 s}\left(\frac{\epsilon\epsilon' s}{s_0}\right)^{a(\lambda)}\frac{1-e^{-i\pi a(\lambda)}}{i\sin\pi a(\lambda)}. \end{aligned} \tag{3.33}$$

Thus, the substitution does not affect the power dependence in s. It only supplies us with the required signature factor. Important physics is all contained in the simplified amplitude $\overline{T}$ which has the intuitive interpretation as suggested in our basic propositions.

The combination $\epsilon\epsilon' s$ which appears in all these results simply reflects the fact that at most only the fraction $\epsilon\epsilon'$ of the total s is available for pionization particles since k_{1+} and $k_{(n+1)-}$ are restricted to be less than ϵp_{a+} and $\epsilon' p_{b-}$, respectively.

IV. FRAGMENTATION

A. Fragmentation of Particle a

To describe the fragmentation of particle a, we restrict k_{i+} $(i=1, 2, \ldots, n)$ to vary between $(\epsilon p_{a+}, p_{a+})$. We shall use the notation $T_n^{(a)}$ to denote the amplitude with $(n+1)$ loops which includes the fragments of particle a only (see Fig. 7). From Eq. (2.12), we have for $n=1$,

$$T_1^{(a)} = -\frac{ig^4}{16\pi\mu^2 s}\left(\frac{g^2}{16\pi^2\mu^2}\right)\left(a_1\ln\frac{1}{\epsilon} + F_1^{(a)}\right), \tag{4.1}$$

where

$$\begin{aligned} &a_1 = 1, \\ &F_1^{(a)} = 4\pi\mu^2\int\frac{d^2k_1}{(2\pi)^2}\int_0^1\frac{dx_1}{x_1}\left(\frac{1-x_1}{[\vec{k}_1^{\,2} + (1-x_1)\mu^2 + x_1^{\,2}\mu_a^{\,2}]^2} - \frac{1}{(\vec{k}_1^{\,2}+\mu^2)^2}\right). \end{aligned} \tag{4.2}$$

The fragmentation amplitude $T_n^{(a)}$ for an n-rung ladder can be obtained in an analogous way as introduced in the pionization region. The general n-rung amplitude was given in Eq. (3.3). After summing the straight and crossed ladders, we obtain an important simplification on the last propagator in $T_n^{(a)}$,

$$\frac{i}{(p_b - k_{n+1})^2 - \mu_b^{\,2} + i\epsilon} + \frac{i}{(p_b + k_{n+1})^2 - \mu_b^{\,2} + i\epsilon} \to \frac{2\pi}{\sqrt{s}}\delta(k_{(n+1)+}). \tag{4.3}$$

We then carry out the (k_{i-})-integrals for $i=1, 2, \ldots, n+1$ and find as in the pionization region that all k_+'s are ordered automatically:

$$p_{a+} > k_{1+} > \cdots > k_{n+} > \epsilon p_{a+}. \tag{4.4}$$

The amplitude $T_n^{(a)}$ then reduces to

$$T_n^{(a)} = -\frac{ig^4}{16\pi\mu^2 s}\left(\frac{g^2}{4\pi}\right)^n\prod_i\int\frac{d^2k_1}{(2\pi)^2}\int_\epsilon^1\frac{dx_1}{x_1}\int_\epsilon^{x_1}\frac{dx_2}{x_2}\cdots\int_\epsilon^{x_{n-1}}\frac{dx_n}{x_n}f_n(1, 2, \ldots, n), \tag{4.5}$$

with

$$\begin{aligned} f_n(1, 2, \ldots, n) &\equiv f_n(x_1, \vec{k}_1; x_2, \vec{k}_2; \cdots; x_n, \vec{k}_n) \\ &\equiv \frac{1/x_n}{(1-x_1)x_1(x_1-x_2)x_2\cdots(x_{n-1}-x_n)x_n}\left(-\mu_a^{\,2} + \frac{\vec{k}_1^{\,2}+\mu_a^{\,2}}{1-x_1} + \frac{\vec{k}_1^{\,2}+\mu^2}{x_1}\right)^{-2} \\ &\quad\times\left(-\mu_a^{\,2} + \frac{\vec{k}_1^{\,2}+\mu_a^{\,2}}{1-x_1} + \frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2}{x_1-x_2} + \frac{\vec{k}_2^{\,2}+\mu^2}{x_2}\right)^{-2}\times\cdots \\ &\quad\times\left(-\mu_a^{\,2} + \frac{\vec{k}_1^{\,2}+\mu_a^{\,2}}{1-x_1} + \frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2}{x_1-x_2} + \cdots + \frac{\vec{k}_n^{\,2}+\mu^2}{x_n}\right)^{-2}, \end{aligned} \tag{4.6}$$

where $x_i \equiv k_{i+}/\sqrt{s}$ are scaled longitudinal momenta. There are two important differences between (4.5) and its counterpart (3.11) in the pionization region. First, the x integrations are restricted to $(\epsilon, 1)$ rather than $(0, \epsilon)$ which differentiate the fragmentation from the pionization. Second, the y integration, which plays an important role in understanding the asymptotic behavior of $T_n^{(0)}$, is carried out trivially here. Therefore, we shall use a somewhat different method to determine the $\ln^n\epsilon$ behavior of $T_n^{(a)}$.

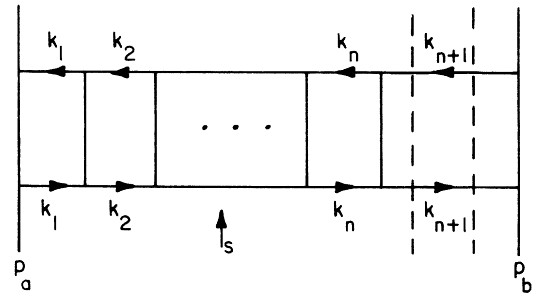

FIG. 7. The straight-ladder diagram with n (internal) rungs in the fragmentation region a.

B. The Amplitude $T_2^{(a)}$

The amplitude $T_2^{(a)}$ is given by (4.5) as

$$T_2^{(a)} = -\frac{ig^4}{16\pi\mu^2 s}\left(\frac{g^2}{4\pi}\right)^2 \int \frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\int_\epsilon^1 \frac{dx_1}{x_1}\int_\epsilon^{x_1}\frac{dx_2}{x_2} f_2(1,2), \tag{4.7}$$

with

$$f_2(1,2) = \frac{1}{(1-x_1)x_1(x_1-x_2)x_2^2}\left(-\mu_a^2 + \frac{\vec{k}_1^2+\mu_a^2}{1-x_1} + \frac{\vec{k}_1^2+\mu^2}{x_1}\right)^{-2}\left(-\mu_a^2 + \frac{\vec{k}_1^2+\mu_a^2}{1-x_1} + \frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2}{x_1-x_2} + \frac{\vec{k}_2^2+\mu^2}{x_2}\right)^{-2}. \tag{4.8}$$

The amplitude $f_2(1, 2)$ has the important factorization property that, for $x_1 \gg x_2$,

$$f_2(1,2) = \frac{1-x_1}{[\vec{k}_1^2 + (1-x_1)\mu^2 + x_1^2\mu_a^2]^2}\frac{1}{(\vec{k}_2^2+\mu^2)^2} + O(x_2/x_1). \tag{4.9}$$

It is easy to see that the approximate relation $f_2(1,2) \approx f_1(1)f_1(2)$ leads to

$$T_2^{(a)} \approx -\frac{ig^4}{16\pi\mu^2 s}\frac{\lambda^2}{2}\left(a_1 \ln\frac{1}{\epsilon} + F_1^{(a)}\right)^2,$$

where

$$f_1(1) = \frac{1-x_1}{[\vec{k}_1^2 + (1-x_1)\mu^2 + x_1^2\mu_a^2]^2} \tag{4.10}$$

is the corresponding one-rung amplitude. Thus, the deviation of $f_2(1, 2)$ from $f_1(1)f_1(2)$, i.e.,

$$\begin{aligned} g_2(1,2) &\equiv f_2(1,2) - f_1(1)f_1(2) \\ &= \frac{1-x_1}{[\vec{k}_1^2+(1-x_1)\mu^2+x_1^2\mu_a^2]^2}\left\{\frac{x_1}{x_1-x_2}\left[\vec{k}_2^2 + \frac{x_2}{x_1-x_2}(\vec{k}_1-\vec{k}_2)^2 + \frac{x_2}{1-x_1}\vec{k}_1^2 + \frac{x_1\mu^2}{x_1-x_2} + \frac{x_1x_2\mu_a^2}{1-x_1}\right]^{-2}\right. \\ &\qquad \left. - \frac{1-x_2}{[\vec{k}_2^2+(1-x_2)\mu^2+x_2^2\mu_a^2]^2}\right\}, \end{aligned} \tag{4.11}$$

measures the correlated part of the two-particle amplitude. Note that $g_2(1, 2)$ is the Mayers' cluster function introduced in statistical mechanics[18] and has the property that, for $x_1 \gg x_2$,

$$g_2(1,2) = O(x_2/x_1). \tag{4.12}$$

Equation (4.12) implies that the integration over $g_2(1, 2)$ contributes to a single $\ln\epsilon$ factor,

$$\int \frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\int_\epsilon^1\frac{dx_1}{x_1}\int_\epsilon^{x_1}\frac{dx_2}{x_2}g_2(1,2) = \frac{1}{(4\pi\mu^2)^2}\left(a_2\ln\frac{1}{\epsilon} + F_2^{(a)} - R_2\right), \tag{4.13}$$

where

$$a_2 \ln\frac{1}{\epsilon} + F_2^{(a)} = (4\pi\mu^2)^2\int\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\int_\epsilon^1\frac{dx_1}{x_1}\int_0^{x_1}\frac{dx_2}{x_2}g_2(1,2) \tag{4.14}$$

and

$$R_2 = (4\pi\mu^2)^2\int\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\int_\epsilon^1\frac{dx_1}{x_1}\int_0^\epsilon\frac{dx_2}{x_2}g_2(1,2). \tag{4.15}$$

Note that the dx_2/x_2 integration over $g_2(1, 2)$ is well defined and has a finite limit at $x_1=0$. Thus, the dx_1/x_1 integral gives rise to a single $\ln\epsilon$ factor. Since the cluster function $g_2(1, 2)$ is nonvanishing when $x_1>\epsilon>x_2$, Eq. (4.14) includes the contribution from the region where the cluster function $g_2(1, 2)$ lies in the fragmentation region of one variable and in the pionization region of the other. R_2 is a correction due to this phenomenon. This term R_2 will be canceled by contributions from other mixed regions when they are added together. This will be demonstrated later.

We present a physical argument to support our using (4.14) as the appropriate definition for a two-particle cluster in the fragmentation region. The two-particle cluster function $g_2(1, 2)$ measures only the correlated part of the two-particle amplitude. The effective invariant mass of a two-particle cluster is therefore small, and its value is independent of the total longitudinal momentum of the two particles. Thus, it is natural to assign a correlated system to the fragmentation region whenever its total longitudinal momentum exceeds a fraction ϵ of the incident momentum. To be consistent, when that fraction is less than ϵ the cluster should be called pionization. This is exactly what our treatment of pionization amplitude in Sec. III implicitly implies. There all the k_{i+} integrations, like (4.14), extend from zero to the maximum values allowed by momentum conservation and our definition of pionization. This explains why we separate out the constant R_2 in (4.13).

The coefficient a_2 can be determined easily from (4.14) by setting $x_1=0$ in the integrand of dx_1/x_1 integral, giving

$$\begin{aligned}
a_2 &= \lim_{x_1\to 0}(4\pi\mu^2)^2\int\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\int_0^{x_1}\frac{dx_2}{x_2}g_2(1,2)\\
&=(4\pi\mu^2)^2\int\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{1}{(\vec{k}_1^{\,2}+\mu^2)^2}\int_0^1\frac{dz_2}{z_2}\left(\frac{1-z_2}{[z_2(\vec{k}_1-\vec{k}_2)^2+(1-z_2)\vec{k}_2^{\,2}+\mu^2]^2}-\frac{1}{(\vec{k}_2^{\,2}+\mu^2)^2}\right)\\
&=-(4\pi\mu^2)^2\int\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{1}{(\vec{k}_1^{\,2}+\mu^2)^2(\vec{k}_2^{\,2}+\mu^2)^2}\ln\frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2}{\mu^2},
\end{aligned}\tag{4.16}$$

where $z_2\equiv x_2/x_1$. To arrive at (4.16), we have used (3.24) and (3.25). Note that we have encountered the identical expression a_2 as appeared in Eq. (3.22) in the pionization region. This implies that the clusters introduced in the fragmentation region are the continuation of the clusters introduced in the pionization region.

The constant $F_2^{(a)}$ can be calculated from (4.14) by subtracting our $a_2\ln(1/\epsilon)$, giving

$$\begin{aligned}
F_2^{(a)} &= (4\pi\mu^2)\int\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\int_0^1\frac{dx_1}{x_1}\int_0^1\frac{dz_2}{z_2}[g_2(1,2)-g_2(1,2)|_{x_1=0}]\\
&=(4\pi\mu^2)^2\int\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\int_0^1\frac{dx_1}{x_1}\int_0^1\frac{dz_2}{z_2}\\
&\quad\times\left\{\frac{1-x_1}{[\vec{k}_1^{\,2}+(1-x_1)\mu^2+x_1^{\,2}\mu_a^{\,2}]^2}\left[\frac{1}{1-z_2}\left(\vec{k}_2^{\,2}+\frac{z_2}{1-z_2}(\vec{k}_1-\vec{k}_2)^2+\frac{x_1z_2}{1-x_1}\vec{k}_1^{\,2}+\frac{\mu^2}{1-z_2}+\frac{x_1^{\,2}z_2}{1-x_1}\mu_a^{\,2}\right)^{-2}\right.\right.\\
&\qquad\left.\left.-\frac{1-x_1z_2}{[\vec{k}_2^{\,2}+(1-x_1z_2)\mu^2+x_1^{\,2}z_2^{\,2}\mu_a^{\,2}]^2}\right]-\frac{1}{(\vec{k}_1^{\,2}+\mu^2)^2}\left(\frac{1-z_2}{[z_2(\vec{k}_1-\vec{k}_2)^2+(1-z_2)\vec{k}_2^{\,2}+\mu^2]^2}-\frac{1}{(\vec{k}_2^{\,2}+\mu^2)^2}\right)\right\},
\end{aligned}\tag{4.17}$$

where we have ignored terms of $O(\epsilon)$.

Putting (4.7)–(4.11) and (4.13) together, we have

$$T_2^{(a)}=-\frac{ig^4}{16\pi\mu^2 s}\lambda^2\left[\frac{1}{2}\left(a_1\ln\frac{1}{\epsilon}+F_1^{(a)}\right)^2+a_2\ln\frac{1}{\epsilon}+F_2^{(a)}-R_2\right].\tag{4.18}$$

As we have pointed out earlier, R_2 will be canceled out by other contributions and does not play any significant role. This will be shown later on.

C. High-Order Amplitudes

The idea of cluster decomposition can be generalized to higher-order amplitudes.[19,20] Let us demonstrate the technique with $T_3^{(a)}$. We have for $n=3$,

$$f_3(1,2,3)=\frac{1}{(1-x_1)x_1(x_1-x_2)x_2(x_2-x_3)x_3^2}\left(-\mu_a^2+\frac{\vec{k}_1^2+\mu_a^2}{1-x_1}+\frac{\vec{k}_1^2+\mu^2}{x_1}\right)^{-2}$$
$$\times\left(-\mu_a^2+\frac{\vec{k}_1^2+\mu_a^2}{1-x_1}+\frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2}{x_1-x_2}+\frac{\vec{k}_2^2+\mu^2}{x_2}\right)^{-2}$$
$$\times\left(-\mu_a^2+\frac{\vec{k}_1^2+\mu_a^2}{1-x_1}+\frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2}{x_1-x_2}+\frac{(\vec{k}_2-\vec{k}_3)^2+\mu^2}{x_2-x_3}+\frac{\vec{k}_3^2+\mu^2}{x_3}\right)^{-2}. \quad (4.19)$$

The amplitude $f_3(1, 2, 3)$ satisfies the following factorization properties. For $x_1 \gg (x_2, x_3)$:

$$f_3(1,2,3)=f_1(1)f_2(2,3)+O(x_2/x_1 \text{ or } x_3/x_1).$$

For $(x_1, x_2) \gg x_3$: (4.20)

$$f_3(1,2,3)=f_2(1,2)f_1(3)+O(x_3/x_1 \text{ or } x_3/x_2).$$

Equations (4.9) and (4.20) indicate that we should introduce a three-particle cluster function through

$$g_3(1,2,3)\equiv f_3(1,2,3)-f_1(1)f_1(2)f_1(3)-f_1(1)g_2(2,3)-f_1(2)g_2(1,3)-f_1(3)g_2(1,2). \quad (4.21)$$

The cluster function g_3 has the property that, for either $x_1 \gg (x_2, x_3)$ or $(x_1, x_2) \gg x_3$,

$$g_3(1,2,3)\to 0 \quad \text{as} \quad O(x_3/x_1). \quad (4.22)$$

Just as in the $n=2$ case, Eq. (4.22) implies that the integral over $g_3(1, 2, 3)$ contributes to a single $\ln(1/\epsilon)$ factor.

$$\int\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{d^2k_3}{(2\pi)^2}\int_\epsilon^1\frac{dx_1}{x_1}\int_\epsilon^{x_1}\frac{dx_2}{x_2}\int_\epsilon^{x_2}\frac{dx_3}{x_3}g_3(1,2,3)=\frac{1}{(4\pi\mu^2)^3}\left(a_3\ln\frac{1}{\epsilon}+F_3^{(a)}-R_3\right), \quad (4.23)$$

where

$$a_3\ln\frac{1}{\epsilon}+F_3^{(a)}=(4\pi\mu^2)^3\int\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{d^2k_3}{(2\pi)^2}\int_\epsilon^1\frac{dx_1}{x_1}\int_0^{x_1}\frac{dx_2}{x_2}\int_0^{x_2}\frac{dx_3}{x_3}g_3(1,2,3) \quad (4.24)$$

and R_3 is a correction term due to the possible overlapping of the clusters in the pionization region. The coefficient a_3 can be computed easily from (4.24) as

$$a_3=\lim_{x_1\to 0}(4\pi\mu^2)^3\int\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{d^2k_3}{(2\pi)^2}\int_0^{x_1}\frac{dx_2}{x_2}\int_0^{x_2}\frac{dx_3}{x_3}g_3(1,2,3). \quad (4.25)$$

In terms of $z_2\equiv x_2/x_1$ and $z_3\equiv x_3/x_2$, Eq. (4.25) can be brought into the same form (3.28) as obtained in the pionization region. $F_3^{(a)}$ can be calculated in an analogous way as in Eq. (4.17).

By the use of Eqs. (4.21) and (4.23), it is now straightforward to show that

$$T_3^{(a)}=-\frac{ig^4}{16\pi\mu^2 s}\left(\frac{g^2}{4\pi}\right)^3\int\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{d^2k_3}{(2\pi)^2}\int_\epsilon^1\frac{dx_1}{x_1}\int_\epsilon^{x_1}\frac{dx_2}{x_2}\int_\epsilon^{x_2}\frac{dx_3}{x_3}f_3(1,2,3)$$
$$=-\frac{ig^4}{16\pi\mu^2 s}\lambda^3\left[\frac{1}{3!}\left(a_1\ln\frac{1}{\epsilon}+F_1^{(a)}\right)^3+\left(a_1\ln\frac{1}{\epsilon}+F_1^{(a)}\right)\left(a_2\ln\frac{1}{\epsilon}+F_2^{(a)}\right)+a_3\ln\frac{1}{\epsilon}+F_3^{(a)}-\left(a_1\ln\frac{1}{\epsilon}+F_1^{(a)}\right)R_2-R_3\right]. \quad (4.26)$$

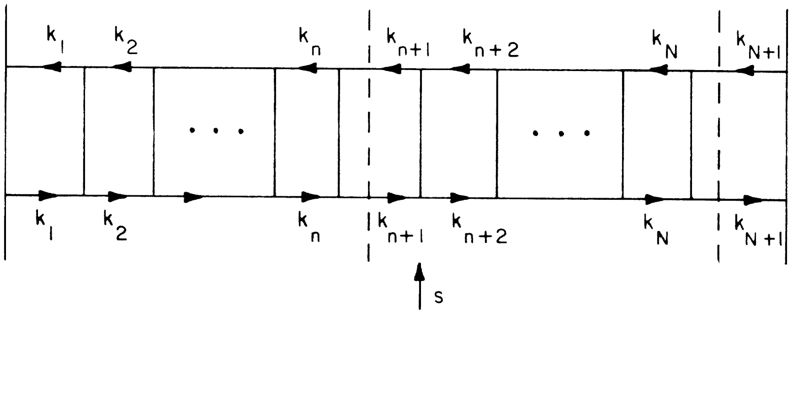

FIG. 8. The straight-ladder diagram with n (internal) rungs in the fragmentation region a and $m(=N-n)$ rungs in the pionization region.

These correction terms involving R's will be canceled out in the full amplitude after contributions from other regions are included. Equations (4.1), (4.17), and (4.26) and the generality of our method demonstrate that the pure fragmentation amplitude also exponentiates,

$$T^{(a)} = \sum T_n^{(a)} = -\frac{ig^4}{16\pi\mu^2 s}\exp\sum \lambda^n\left(a_n \ln\frac{1}{\epsilon} + F_n^{(a)}\right) + \text{terms due to } R\text{'s} = -\frac{ig^4}{16\pi\mu^2 s}e^{F^{(a)}(\lambda)}\left(\frac{1}{\epsilon}\right)^{a(\lambda)} + \text{terms due to } R\text{'s}, \tag{4.27}$$

with

$$F^{(a)}(\lambda) \equiv \sum \lambda^n F_n^{(a)}. \tag{4.28}$$

D. Fragmentation Amplitude $T^{(b)}$

The method developed so far can be applied equally well to the pure fragmentation amplitude of particle b since the two particles a and b are treated symmetrically. We can introduce various cluster functions and their corresponding amplitudes

$$a_n \ln\frac{1}{\epsilon'} + F_n^{(b)}, \tag{4.29}$$

just as in the fragmentation region of particle a. Thus, we shall not repeat the calculation here. The pure fragmentation amplitude for particle b can be written down immediately as

$$T^{(b)} = \sum T_n^{(b)} = -\frac{ig^4}{16\pi\mu^2 s}e^{F^{(b)}(\lambda)}\left(\frac{1}{\epsilon'}\right)^{a(\lambda)} + \text{terms due to overlapping effects}, \tag{4.30}$$

with

$$F^{(b)}(\lambda) \equiv \sum \lambda^n F_n^{(b)}. \tag{4.31}$$

V. GENERAL AMPLITUDE

A. Amplitude from Mixed Regions

A general n-rung amplitude T_n should include integration regions consisting of both the fragmentation regions and the pionization region. To gain some insights into the amplitude covering these mixed regions, we first restrict ourselves to integration regions associated with fragmentation of particle a and pionization. The generalization to cover all three regions is straightforward. The N-rung amplitude in this combined region is denoted by $T_N^{(a+0)}$, while the partial amplitude for n particles in the fragmentation region of particle a and m particles in the pionization region is denoted by $T_{n,m}^{(a,0)}$ (see Fig. 8). Obviously,

$$T_N^{(a+0)} = \sum_{n+m=N} T_{n,m}^{(a,0)}. \tag{5.1}$$

A general amplitude, $T_{n,m}^{(a,0)}$, can be worked out analogously just as in the pure pionization region, giving

$$T_{n,m}^{(a,0)} = g^2\left(\frac{g^2}{4\pi}\right)^{N+1}\prod_{i=1}^{N+1}\int\frac{d^2k_i}{(2\pi)^2}\int_\epsilon^1\frac{dx_1}{x_1}\int_\epsilon^{x_1}\frac{dx_2}{x_2}\cdots\int_\epsilon^{x_{n-1}}\frac{dx_n}{x_n}\int_0^\epsilon\frac{dx_{n+1}}{x_{n+1}}\cdots\int_0^{x_{N-1}}\frac{dx_N}{x_N}\int_{-\epsilon'}^{\epsilon'}dy\, a_N(1,2,\ldots,N;\vec{k}_{N+1},y), \tag{5.2}$$

with

$$\begin{aligned} a_N(1,2,\ldots,N;\vec{k}_{N+1},y) &\equiv a_N(x_1,\vec{k}_1;x_2,\vec{k}_2;\ldots;\vec{k}_{N+1},y) \\ &\equiv \frac{1}{(1-x_1)x_1(x_1-x_2)\cdots(x_{N-1}-x_N)x_N}\left(-\mu_a^2+\frac{\vec{k}_1^2+\mu_a^2}{1-x_1}+\frac{\vec{k}_1^2+\mu^2}{x_1}\right)^{-2} \\ &\times\left(-\mu_a^2+\frac{\vec{k}_1^2+\mu_a^2}{1-x_1}+\frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2}{x_1-x_2}+\frac{\vec{k}_2^2+\mu^2}{x_2}\right)^{-2}\times\cdots \\ &\times\left(-\mu_a^2+\frac{\vec{k}_1^2+\mu_a^2}{1-x_1}+\frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2}{x_1-x_2}+\cdots+\frac{\vec{k}_N^2+\mu^2}{x_N}\right)^{-2}(\vec{k}_{N+1}^2+\mu^2)^{-2} \\ &\times\left[x_N y s - x_N\left(-\mu_a^2+\frac{\vec{k}_1^2+\mu_a^2}{1-x_1}+\frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2}{x_1-x_2}+\cdots+\frac{(\vec{k}_N-\vec{k}_{N+1})^2+\mu^2}{x_N}\right)+i\epsilon\right]^{-1}, \end{aligned} \tag{5.3}$$

where $N=n+m$, and $x_i = k_{i+}/\sqrt{s}$ and $y = -k_{(N+1)-}/\sqrt{s}$ are scaled longitudinal momenta. $T_N^{(a+0)}$ is given by the same equation (5.2) except that the limits of x integrations are replaced by

$$1 > x_1 > \cdots > x_N > 0. \tag{5.4}$$

If none of the x's are near the dividing point, i.e., if $x_n \gg \epsilon \gg x_{n+1}$, we find that

$$a_N(1, 2, \ldots, N; \vec{k}_{N+1}, y) = f_n(1, 2, \ldots, n) a_m^{(0)}(n+1, n+2, \ldots, N; \vec{k}_{N+1}, y), \tag{5.5}$$

where $a_m^{(0)}(n+1, n+2, \ldots, N; \vec{k}_{N+1}, y)$ is the analog of $a_N(1, 2, \ldots, N; \vec{k}_{N+1}, y)$ appropriate for a pure pionization amplitude. Under this condition, $T_{n,m}^{(a,0)}$ reduces to the product of $T_n^{(a)}$ and $T_m^{(0)}$,

$$T^{(a,0)} \to \left(-\frac{ig^4}{16\pi\mu^2 s}\right)^{-1} T_n^{(a)} T_m^{(0)}. \tag{5.6}$$

The possible correction to Eq. (5.6) is solely due to the contribution of those x integrations which are very close to the dividing point. In terms of cluster functions, this correction originates from the possibilities that clusters may overlap with the dividing point. As we shall see, this type of correction term will cancel the corresponding terms $-R_N$ appearing in the fragmentation region. We shall demonstrate this cancellation for $N=2$ in the following explicit calculation.

B. The Amplitude $T_{1,1}^{(a,0)}$

Equations (5.2) and (5.3) imply that

$$T_{1,1}^{(a,0)} = g^2\left(\frac{g^2}{4\pi}\right)^3 \int \frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{d^2k_3}{(2\pi)^2}\int_\epsilon^1 \frac{dx_1}{x_1}\int_0^\epsilon \frac{dx_2}{x_2}\int_{-\epsilon'}^{\epsilon'} dy\, a_2(1, 2; \vec{k}_3, y), \tag{5.7}$$

where

$$a_2(1, 2; \vec{k}_3, y) = \frac{1}{(1-x_1)x_1(x_1-x_2)x_2}\left(-\mu_a^2 + \frac{\vec{k}_1^2+\mu_a^2}{1-x_1} + \frac{\vec{k}_1^2+\mu^2}{x_1}\right)^{-2}\left(-\mu_a^2 + \frac{\vec{k}_1^2+\mu_a^2}{1-x_1} + \frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2}{x_1-x_2} + \frac{\vec{k}_2^2+\mu^2}{x_2}\right)^{-2}$$
$$\times (k_3^2+\mu^2)^{-2}\left[x_2 y s - x_2\left(-\mu_a^2 + \frac{\vec{k}_1^2+\mu_a^2}{1-x_1} + \frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2}{x_1-x_2} + \frac{(\vec{k}_2-\vec{k}_3)^2+\mu^2}{x_2}\right) + i\epsilon\right]^{-1}. \tag{5.8}$$

The difference between $a_2(1, 2; \vec{k}_3, y)$ and $f_1(1)a_1^{(0)}(2; \vec{k}_3, y)$ in the region $1 > x_1 > \epsilon > x_2 > 0$ can be written as

$$c_2(1, 2; \vec{k}_3, y) \equiv a_2(1, 2; \vec{k}_3, y) - f_1(1)a_1^{(0)}(2; \vec{k}_3, y)$$
$$= (\vec{k}_3^2+\mu^2)^{-2} f_1(1)\left\{\frac{x_1}{(x_1-x_2)x_2}\left[\frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2}{x_1-x_2} + \frac{\vec{k}_2^2+\mu^2}{x_2}\right]^{-2}\right.$$
$$\times\left[x_2 y s - x_2\left(\frac{(\vec{k}_1-\vec{k}_2)^2+\mu^2}{x_1-x_2} + \frac{(\vec{k}_2-\vec{k}_3)^2+\mu^2}{x_2}\right) + i\epsilon\right]^{-1}$$
$$\left. - \frac{1}{x_2}\left(\frac{\vec{k}_2^2+\mu^2}{x_2}\right)^{-2}[x_2 y_2 s - (\vec{k}_2-\vec{k}_3)^2 - \mu^2 + i\epsilon]^{-1}\right\}, \tag{5.9}$$

where we have ignored terms of $O(\epsilon)$ or smaller. Note that $c_2(1, 2; \vec{k}_3, y) = O(x_2)$ at $x_2 \to 0$, as expected from (5.5). Thus, the contribution from the small-x_2 integration region ($x_2 \ll \epsilon$) is damped out, and we can carry out the y integration of $c_2(1, 2; \vec{k}_3, y)$ by assuming that x_2 is always $\gg 1/\epsilon' s$. Then we can extend the limits of y integration to $\pm\infty$, giving

$$\int_{-\infty}^{\infty} dy\, c_2(1, 2; \vec{k}_3, y) = -\frac{i\pi}{s}(\vec{k}_3^2+\mu^2)^{-2} f_1(1)\left[\frac{x_1}{x_1-x_2}\left(\frac{x_2[(\vec{k}_1-\vec{k}_2)^2+\mu^2]}{x_1-x_2} + \vec{k}_2^2+\mu^2\right)^{-2} - (\vec{k}_2^2+\mu^2)^{-2}\right]. \tag{5.10}$$

Thus, the contribution due to $c_2(1, 2; \vec{k}_3, y)$ leads to

$$\int \frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{d^2k_3}{(2\pi)^2}\int_\epsilon^1\frac{dx_1}{x_1}\int_0^\epsilon\frac{dx_2}{x_2}\int_{-\infty}^{\infty} dy\, c_2(1, 2; \vec{k}_3, y) = -\frac{i\pi}{s}\frac{1}{(4\pi\mu^2)^3}R_2, \tag{5.11}$$

where R_2 is defined in (4.13) and (4.15). Putting (5.7), (5.9), and (5.11) together, we have

$$T_{1,1}^{(a,0)} = g^2\left(\frac{g^2}{4\pi}\right)^3\int \frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{d^2k_3}{(2\pi)^2}\int_\epsilon^1\frac{dx_1}{x_1}\int_0^\epsilon\frac{dx_2}{x_2}\int_{-\epsilon'}^{\epsilon'} dy f_1(1)a_1^{(0)}(2; \vec{k}_3, y) + g^2\left(\frac{g^2}{4\pi}\right)^3\left(-\frac{i\pi}{s}\right)\frac{1}{(4\pi\mu^2)^3}R_2,$$

and consequently

$$\overline{T}_{1,1}^{(a,0)} = -\frac{ig^4}{16\pi\mu^2 s}\lambda^2\left[\left(a_1\ln\frac{1}{\epsilon} + F_1^{(a)}\right)\left(a_1\ln\frac{\epsilon\epsilon' s}{\mu^2} + b_1\right) + R_2\right], \tag{5.12}$$

where we have introduced a simplified amplitude $\overline{T}$ through the substitution (3.17). Indeed, R_2 in (5.12) cancels out a similar term in (4.18). Once we understand the origin of these terms, the cancellation among these correction terms here and R_n (and R_n') appearing in the fragmentation amplitudes emerges naturally.

C. Exponentiation of a General Amplitude

Now we know how to calculate the amplitude for a general ladder once we know the pure pionization and the pure fragmentation amplitudes. Let us summarize the amplitudes for $N=2$ associated with various kinematical regions in Figs. 5(a) – (f):

$$T_2^{(a)} = -\frac{ig^4}{16\pi\mu^2 s}\lambda^2\left[\frac{1}{2}\left(a_1 \ln\frac{1}{\epsilon}+F_1^{(a)}\right)^2 + a_2\ln\frac{1}{\epsilon}+F_2^{(a)} - R_2\right], \tag{5.13a}$$

$$T_2^{(b)} = -\frac{ig^4}{16\pi\mu^2 s}\lambda^2\left[\frac{1}{2}\left(a_1 \ln\frac{1}{\epsilon'}+F_1^{(b)}\right)^2 + a_2\ln\frac{1}{\epsilon'}+F_2^{(b)} - R_2'\right], \tag{5.13b}$$

$$\overline{T}_2^{(0)} = -\frac{ig^4}{16\pi\mu^2 s}\lambda^2\left[\frac{1}{2}\left(a_1 \ln\frac{\epsilon\epsilon' s}{\mu^2}+b_1\right)^2 + a_2\ln\frac{\epsilon\epsilon' s}{\mu^2}+b_2\right], \tag{5.13c}$$

$$\overline{T}_{1,1}^{(a,0)} = -\frac{ig^4}{16\pi\mu^2 s}\lambda^2\left[\left(a_1 \ln\frac{1}{\epsilon}+F_1^{(a)}\right)\left(a_1\ln\frac{\epsilon\epsilon' s}{\mu^2}+b_1\right)+R_2\right], \tag{5.13d}$$

$$\overline{T}_{1,1}^{(0,b)} = -\frac{ig^4}{16\pi\mu^2 s}\lambda^2\left[\left(a_1 \ln\frac{\epsilon\epsilon' s}{\mu^2}+b_1\right)\left(a_1\ln\frac{1}{\epsilon'}+F_1^{(b)}\right)+R_2'\right], \tag{5.13e}$$

$$T_{1,1}^{(a,b)} = -\frac{ig^4}{16\pi\mu^2 s}\lambda^2\left(a_1 \ln\frac{1}{\epsilon}+F_1^{(a)}\right)\left(a_1\ln\frac{1}{\epsilon'}+F_1^{(b)}\right). \tag{5.13f}$$

The only contribution which has not been obtained before is (5.13f), corresponding to Fig. 5(f). However, the contribution from Fig. 5(f) is simple to obtain because the amplitude f_2 factors in this region. The over-all amplitude $\overline{T}_2$ is the sum of all six subamplitudes, giving

$$\begin{aligned}\overline{T}_2 &= -\frac{ig^4}{16\pi\mu^2 s}\lambda^2\left[\frac{1}{2}\left(a_1\ln\frac{1}{\epsilon}+F_1^{(a)}+a_1\ln\frac{1}{\epsilon'}+F_1^{(b)}+a_1\ln\frac{\epsilon\epsilon' s}{\mu^2}+b_1\right)^2 + a_2\ln\frac{1}{\epsilon}+F_2^{(a)}+a_2\ln\frac{1}{\epsilon'}+F_2^{(b)}+a_2\ln\frac{\epsilon\epsilon' s}{\mu^2}+b_2\right]\\ &= -\frac{ig^4}{16\pi\mu^2 s}\lambda^2\left[\frac{1}{2}\left(a_1\ln\frac{s}{\mu^2}+F_1^{(a)}+b_1+F_1^{(b)}\right)^2 + a_2\ln\frac{s}{\mu^2}+F_2^{(a)}+b_2+F_2^{(b)}\right].\end{aligned} \tag{5.14}$$

Note that all the ϵ and ϵ' dependences drop out. The only modifications of $\overline{T}_2$ in comparison with the corresponding amplitude $\overline{T}_2^{(0)}$ in the pionization region are

$$\epsilon\epsilon' s \to s \tag{5.15a}$$

and

$$b_n \to F_n^{(a)} + b_n + F_n^{(b)}. \tag{5.15b}$$

The coefficient a_n of $\ln s$ is not affected. We have verified this point explicitly for $n \leq 3$. One can conclude from the general property of the amplitude that Eq. (5.15) should be valid in general.

Once we establish the relation (5.15), the sum over the general amplitude is straightforward. We have, in analogy to (3.29),

$$\begin{aligned}\overline{T} = \sum_n \overline{T}_n &= -\frac{ig^4}{16\pi\mu^2 s}\exp\sum_n \lambda^n\left(a_n\ln\frac{s}{\mu^2}+F_n^{(a)}+b_n+F_n^{(b)}\right)\\ &= -\frac{ig^4}{16\pi\mu^2 s}G^{(a)}(\lambda)\left(\frac{s}{s_0(\lambda)}\right)^{a(\lambda)}G^{(b)}(\lambda),\end{aligned} \tag{5.16}$$

where

$$\begin{aligned}G^{(a)}(\lambda) &= \exp\sum \lambda^n F_n^{(a)},\\ G^{(b)}(\lambda) &= \exp\sum \lambda^n F_n^{(b)},\end{aligned} \tag{5.17}$$

and $s_0(\lambda)$ and $a(\lambda)$ are defined in (3.30)–(3.31).

After making the substitution (3.17), we obtain the final expression for the sum of straight and crossed ladders:

$$T=-\frac{ig^4}{16\pi\mu^2 s}G^{(a)}(\lambda)\left(\frac{s}{s_0(\lambda)}\right)^{\alpha(\lambda)}G^{(b)}(\lambda)\frac{1-e^{-i\pi\alpha(\lambda)}}{i\sin\pi\alpha(\lambda)}$$

$$=-\frac{ig^4}{16\pi\mu^2 s}G^{(a)}(\lambda)\left(\frac{s}{s_0(\lambda)}\right)^{\alpha(\lambda)}G^{(b)}(\lambda)\left(1-i\tan\frac{\pi\alpha(\lambda)}{2}\right). \tag{5.18}$$

Equation (5.18) implies that $\mathrm{Im}(-T)$ is positive definite, as required by the optical theorem.

Equation (5.16) makes it clear that the constants $F_n^{(a)}+b_n+F_n^{(b)}$ can be written many different ways as the sum of three terms without affecting the final answer. This is a manifestation of the arbitrariness, mentioned in Sec. III, of calling particle a a pionization or fragmentation product when its momentum is close to the dividing point. To give an example, let us return to (3.11). Inspection of (3.11) shows that all the x_i's have a natural cutoff at the lower end of order $1/\epsilon's$ with an unknown finite squared-mass scale. An alternative definition for the pionization amplitude can be made by restricting all the x_i to be $\epsilon > x_i > \mu^2/\epsilon's$, with μ^2 being the squared mass of the internal particles. This definition does not affect the values of a_n, but alters the values of $F_n^{(a)}$, b_n, and $F_n^{(b)}$. This definition is used in Ref. 19.

VI. DISCUSSION

Based on the results up to tenth order in the coupling constant obtained in previous sections, we conclude that the forward elastic amplitude for the ladder diagrams has the structure

$$T=-i\frac{g^4}{16\pi\mu^2 s}G^{(a)}(\lambda)\exp\sum_n\lambda^n\left(a_n\ln\frac{s}{\mu^2}+b_n\right)G^{(b)}(\lambda)\frac{1-e^{-i\pi\alpha(\lambda)}}{i\sin\pi\alpha(\lambda)}, \tag{6.1}$$

where $a_n\ln(s/\mu^2)+b_n$ is the pionization contribution due to a cluster of n particles with a small invariant mass. The factors $G^{(a)}$ and $G^{(b)}$ are given in (5.17) and describe the (s-independent) amplitudes for particle a and particle b to dissociate into one, two, ... particles. It should be appreciated that the simplicity of the structure for the amplitude (6.1) is discovered by a very unconventional approach. We do not group together as a single term all the contributions of the same order in g^2 (or λ) and of the same power in $\ln(s/\mu^2)$. Instead, we treat pionization and fragmentation separately. In this way it is discovered that the pionization contributions sum up to a simple structure; so do the fragmentation contributions. It is clearly very difficult to recognize the simple structure as given in (6.1) if it were expanded in a power series of $(g^2)^n\ln^m(s/\mu^2)$.

A similar situation may also exist in the case of the amplitude for multiladder exchanges. This was the attitude adopted in Ref. 7. In that paper only the kinematic region where particles a and b retain their large momenta is considered. Perhaps the basic structure obtained in this restricted kinematic region will appear repeatedly in an improved approach that includes the fragmentation events. However, multiladder exchange is much more complicated than the simple ladder considered in this paper. It is beyond the scope of the present paper.

To conclude, we make two remarks:

(1) Since the sum of nonleading logarithms corresponding to clusters of two or more pionization particles also exponentiates to a power of s, these contributions are comparable to the sum of the leading terms if the coupling is not weak. We learn again that summing the leading terms only is not quantitatively meaningful. This suggests that a nonperturbative approach is essential to a quantitative understanding of high-energy scattering of hadrons.

(2) The remark just made above also implies that, in general, correlation among pionization particles may be important since the probability for several pionization particles to form a cluster of small invariant mass is not negligible. Thus, the number distribution is no longer a simple Poisson distribution. For example, inelastic two-particle events now involve processes in which the two uncorrelated particles are produced and those in which the two particles are strongly correlated. In a separate note, we shall apply the considerations here to the prong distribution in p-p interactions to extract the information about the two-particle cluster.[21]

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APPENDIX A

In this Appendix, we evaluate the integral of the general form

$$I_n(s) = \lim_{s\gg 1}\int_0^1 dx_1\cdots dx_n \frac{1}{x_1x_2\cdots x_n s+1} \tag{A1}$$

for large s up to the accuracy of $1/s$. To simplify (A1), we introduce the new variables

$$x_1\cdots x_i = e^{-y_i}, \quad i=1,2,\ldots,n \tag{A2}$$

then

$$I_n(s) = \int_0^\infty dy_1 \int_{y_1}^\infty dy_2 \cdots \int_{y_{n-1}}^\infty dy_n \frac{e^{-y_n}}{e^{-y_n}s+1}\,. \tag{A3}$$

Interchange the order of integrations and we get

$$I_n(s) = \int_0^\infty dy_n \frac{e^{-y_n}}{e^{-y_n}s+1}\int_0^{y_n} dy_1 \int_{y_1}^{y_n} dy_2 \cdots \int_{y_{n-2}}^{y_n} dy_{n-1}\,. \tag{A4}$$

This gives

$$\begin{aligned} I_n(s) &= \frac{1}{(n-1)!}\int_0^\infty dy\, y^{n-1}\frac{e^{-y}}{e^{-y}s+1} \\ &= \frac{1}{(n-1)!}\frac{1}{s}\int_0^s dx \frac{\ln^{n-1}s/x}{x+1}\,. \end{aligned} \tag{A5}$$

To evaluate (A5), it is convenient to consider the generating function

$$I(s,g) = \sum_{n=0}^\infty g^n I_n(s), \quad I_0(s) \equiv \frac{1}{s+1}\,, \quad |g|<1\,. \tag{A6}$$

Then

$$\begin{aligned} I(s,g) &= \frac{1}{s+1} + \frac{g}{s}\int_0^s dx \left(\frac{s}{x}\right)^g \frac{1}{x+1} \\ &= \frac{1}{s+1} + \frac{g}{s}\, s^g \left[\int_0^s dx \frac{(x+1)^{-g}}{x+1} + \int_0^s dx \frac{x^{-g}-(1+x)^{-g}}{1+x}\right]. \end{aligned} \tag{A7}$$

Since we only keep terms up to $1/s$ or larger, we can replace the upper limit in the last term by infinity. The resulting integral then becomes a standard one. We get finally

$$I(s,g) = \frac{1}{s}\left(s^g \frac{\pi g}{\sin \pi g}\right). \tag{A8}$$

The original integral $I_n(s)$ can now be expressed as

$$I_n(s) = \frac{1}{n!}\frac{d^n}{dg^n} I(s,g)\Big|_{g=0}\,. \tag{A9}$$

The results for $n=1,2,3,4$ will be needed for our discussion. Explicit expressions for these cases are

$$\begin{aligned} I_1(s) &= \frac{1}{s}\ln s\,, \\ I_2(s) &= \frac{1}{s}\left(\frac{1}{2!}\ln^2 s + \frac{1}{3!}\pi^2\right), \\ I_3(s) &= \frac{1}{s}\left(\frac{1}{3!}\ln^3 s + \frac{1}{3!}\pi^2 \ln s\right), \\ I_4(s) &= \frac{1}{s}\left(\frac{1}{4!}\ln^4 s + \frac{1}{2!3!}\pi^2\ln^2 s + \frac{7}{360}\pi^4\right). \end{aligned} \tag{A10}$$

We shall denote the expression in the large parentheses of (A10) by $J_n(s)$, i.e.,

$$I_n(s)=\frac{1}{s}J_n(s), \quad J_n(s)=s\,I_n(s)\,. \tag{A11}$$

Clearly, $J_n(s)$ is an nth-order polynomial in $\ln s$.

A slight variation of (A1), which appears repeated in this paper, is

$$\lim_{s\gg 1}\int_0^{\epsilon_1}dx_1\cdots\int_0^{\epsilon_{n-1}}dx_{n-1}\int_{-\epsilon_n}^{\epsilon_n}dx_n\frac{1}{x_1x_2\cdots x_n s+c}\,. \tag{A12}$$

In terms of $z_i=x_i/\epsilon_i$, Eq. (A12) reduces to

$$\frac{\epsilon_1\epsilon_2\cdots\epsilon_n}{c}\lim_{s\gg 1}\int_0^1 dz_1\cdots\int_0^1 dz_{n-1}\int_{-1}^1 dz_n\frac{1}{z_1z_2\cdots z_n\left(\frac{\epsilon_1\epsilon_2\cdots\epsilon_n s}{c}\right)+1}=\frac{\epsilon_1\epsilon_2\cdots\epsilon_n}{c}\left[I_n\left(\frac{\epsilon_1\epsilon_2\cdots\epsilon_n s}{c}\right)+(s\to -s)\right]$$

$$=\frac{1}{s}\left[J_n\left(\frac{\epsilon_1\epsilon_2\cdots\epsilon_n s}{c}\right)-J_n\left(-\frac{\epsilon_1\epsilon_2\cdots\epsilon_n s}{c}\right)\right]. \tag{A13}$$

By the use of

$$\ln(-s)=\ln s-i\pi,$$

the expression in the square brackets of (A13) can be reduced into an $(n-1)$th-order polynomial in $\ln s$.

APPENDIX B

In this Appendix we shall work out the scattering amplitude associated with Fig. 2 by means of the conventional Feynman parameter technique. In our notation, this amplitude is denoted by T_1 and is

$$T_1=-g^6\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{1}{(p_a-k_1)^2-\mu^2+i\epsilon}\,\frac{1}{(k_1-k_2)^2-\mu^2+i\epsilon}\,\frac{1}{(p_b+k_2)^2-\mu^2+i\epsilon}\,\frac{1}{(k_1^2-\mu^2+i\epsilon)^2}\,\frac{1}{(k_2^2-\mu^2+i\epsilon)^2}+(s\to -s)\,, \tag{B1}$$

where for simplicity we choose $\mu_a=\mu_b=\mu$. The seven propagators in (B1) can be combined through a set of Feynman parameters $\alpha_1,\alpha_2,\ldots,\alpha_7$:

$$T_1=-g^6 6!\prod_i\int d\alpha_i\delta\left(\sum\alpha_i-1\right)\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}$$

$$\times[\alpha_1(p_a-k_1)^2+\alpha_2(k_1-k_2)^2+\alpha_3(p_b+k_2)^2+(\alpha_4+\alpha_5)k_1^2+(\alpha_6+\alpha_7)k_2^2-\mu^2+i\epsilon]^{-7}+(s\to -s). \tag{B2}$$

It is now straightforward to carry out the d^4k_1 and d^4k_2 integrations, giving

$$T_1=-\frac{2g^6}{(16\pi^2)^2}\prod_i\int d\alpha_i\delta\left(\sum\alpha_i-1\right)\left(\frac{\Delta}{(D-i\epsilon)^3}+(s\to -s)\right), \tag{B3}$$

where

$$\Delta\equiv\begin{vmatrix}\alpha_1+\alpha_2+\alpha_4+\alpha_5 & -\alpha_2\\ -\alpha_2 & \alpha_2+\alpha_3+\alpha_6+\alpha_7\end{vmatrix}=(\alpha_1+\alpha_2+\alpha_4+\alpha_5)(\alpha_2+\alpha_3+\alpha_6+\alpha_7)-\alpha_2^2 \tag{B4}$$

and

$$D(s)\equiv-\begin{vmatrix}\alpha_1+\alpha_2+\alpha_4+\alpha_5 & -\alpha_2 & -\alpha_1p_a\\ -\alpha_2 & \alpha_2+\alpha_3+\alpha_6+\alpha_7 & \alpha_3p_b\\ -\alpha_1p_a & \alpha_3p_b & -\mu^2(1-\alpha_1-\alpha_3)\end{vmatrix}$$

$$=-\alpha_1\alpha_2\alpha_3(s-2\mu^2)+\mu^2[(1-\alpha_1-\alpha_3)\Delta+\alpha_1^2(\alpha_2+\alpha_3+\alpha_6+\alpha_7)+\alpha_3^2(\alpha_1+\alpha_2+\alpha_4+\alpha_5)]\,. \tag{B5}$$

It is easier to carry out the α integrations for $D(-s)$ first because the latter is positive definite. We can obtain the final expression T_1 by adding the contribution from $D(s)$ through the crossing substitution $(s\to -s)$ as before. For terms of order $(\ln s/s)$ and $1/s$, it is enough to integrate over the regions

(1) $0\leq\alpha_1\leq\epsilon_1,\quad 0\leq\alpha_2\leq\epsilon_2,\quad 0\leq\alpha_3\leq\epsilon_3\,,$

(2) $\epsilon_1\leq\alpha_1\leq 1\,,\quad 0\leq\alpha_2<\epsilon_2,\quad 0\leq\alpha_3<\epsilon_3\,,$

(3) $0\leq\alpha_1<\epsilon_1,\quad 0\leq\alpha_2<\epsilon_2,\quad \epsilon_3\leq\alpha_3\leq 1\,,$

and

(4) $0 \leq \alpha_1 < \epsilon_1, \quad \epsilon_2 \leq \alpha_2 \leq 1, \quad 0 \leq \alpha_3 < \epsilon_3$.

The integrations are quite simple, and they yield

$$T_{(1)} = -\frac{g^6}{(16\pi^2\mu^2)^2 s}\left[\frac{1}{2}\left(\ln\frac{s}{\mu^2}\right)^2 + \ln\frac{s}{\mu^2}(\ln\epsilon_1\epsilon_2\epsilon_3 + 3)\right] + (s \to -s), \tag{B6}$$

$$T_{(2)} = -\frac{g^6}{(16\pi^2\mu^2)^2 s}\left[-\ln\frac{s}{\mu^2}\ln\epsilon_1 - \ln\frac{s}{\mu^2}\left(1 + \frac{\pi}{3\sqrt{3}}\right)\right] + (s \to -s), \tag{B7}$$

$$T_{(3)} = -\frac{g^6}{(16\pi^2\mu^2)^2 s}\left[-\ln\frac{s}{\mu^2}\ln\epsilon_3 - \ln\frac{s}{\mu^2}\left(1 + \frac{\pi}{3\sqrt{3}}\right)\right] + (s \to -s), \tag{B8}$$

and

$$T_{(4)} = -\frac{g^6}{(16\pi^2\mu^2)^2 s}\left[-\ln\frac{s}{\mu^2}(\ln\epsilon_2 + 1) + \ln\frac{s}{\mu^2}A\right], \tag{B9}$$

where

$$\begin{aligned} A &= -\int_0^1 x\,dx \int_0^1 \frac{dy}{1 - x(1 - y + y^2)} \\ &= \int_0^1 dy\left(\frac{1}{(1 - y + y^2)^2}\ln y(1 - y) + \frac{1}{1 - y + y^2}\right) \\ &= -1.5626\,. \end{aligned} \tag{B10}$$

We have ignored in (B6)–(B9) terms of order ϵ or smaller. Finally, the resultant amplitude is

$$\begin{aligned} T_1 &= T_{(1)} + T_{(2)} + T_{(3)} + T_{(4)} \\ &= -\frac{g^6}{(16\pi^2\mu^2)^2 s}\left[\frac{1}{2}\left(\ln\frac{s}{\mu^2}\right)^2 - \ln\frac{s}{\mu^2}\left(\frac{2\pi}{3\sqrt{3}} + 1.5626\right)\right] + (s \to -s) \\ &= -\frac{ig^4}{16\pi\mu^2 s}\left(\frac{g^2}{16\pi^2\mu^2}\right)\left(\ln\frac{s}{\mu^2} - \frac{2\pi}{3\sqrt{3}} - 1.5626 - \frac{i\pi}{2}\right) \\ &= -\frac{ig^4}{16\pi\mu^2 s}\left(\frac{g^2}{16\pi^2\mu^2}\right)\left(a_1\ln\frac{s}{\mu^2} + b_1 - i\frac{\pi}{2} + F_1^{(a)} + F_1^{(b)}\right), \end{aligned} \tag{B11}$$

which is exactly (2.22) with[22]

$$a_1 = 1, \quad b_1 + F_1^{(a)} + F_1^{(b)} = -\frac{2\pi}{3\sqrt{3}} - 1.5626\,.$$

We have compared the infinite-momentum technique with the α-space method up to $(\ln s)^2/s$ and $(\ln s)/s$ for the two-rung ladder and crossed ladder [i.e. to $O(g^8)$]. They also agree with each other. This gives us confidence that the two approaches are equally valid.

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[13]We follow the notations used in Ref. 6.

[14]The approximation of ignoring k_{1-} (k_{2+}) in this kinematical region neglects correction terms of order $O(1/s)$ or $O(\epsilon)$ smaller.

[15]Observe that there exists a natural cutoff for the maximum value that k_{1+} and k_{2-} can take due to the momentum conservation. The cutoff at the lower end is of course provided by ϵ (or ϵ').

[16]Pionization particles defined in this paper contain all possible final or intermediate particles of c.m. energy $E \propto s^a$, $a < \frac{1}{2}$. This definition differs from that introduced in Ref. 4.

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Subtracted Spectral Representations and Singular Equal-Time Commutators*

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We consider the question of subtractions in a spectral representation which, we argue, should be regarded as a dispersion relation in energy (the Low equation) for the true propagator function, rather than as a dispersion relation in the square of the momentum for some suitably covariantized version. We demonstrate how a generalization of the theorem of Bjorken, Johnson, and Low to subtracted spectral representations yields a vacuum expectation value of the equal-time commutator which, together with the propagator function, satisfies a sequence of partial differential equations in momentum space. We find that the commutator corresponding to a subtracted representation contains singular terms which are not completely specified by the spectral functions. As an application we consider any model in which currents are represented by bilinear forms of free spinor fields. There such singular terms can invalidate, for example, the usual statement of universality of weak interactions and the simple $SU_3 \times SU_3$ algebra of quark charges.

In this note we would like to discuss an important topic, which has not received sufficient attention before, concerning the question of subtractions in spectral representations. Subtractions might be required, for example, if, defining the vacuum expectation value of an equal-time commutator by means of the Bjorken-Johnson-Low (BJL) theorem,[1,2] we were led to a *divergent* spectral integration. This would mean that the (possibly frame-dependent) propagator $\Delta(p)$ with which we started converged more slowly than p_0^{-1} in the limit $p_0 \to \infty$ with $\vec{p}$ fixed. To illustrate our conclusions, for a once-subtracted $\Delta(p)$, we find the corresponding equal-time commutator to have a