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Bootstrap Conditions for Photon Interactions*†

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A set of "bootstrap" self-consistency conditions, obtained previously for hadron-hadron scattering amplitudes, are generalized to amplitudes involving one external photon (defined here as any weakly interacting boson). The hadron-hadron bootstrap conditions are known to require that the mesons and baryons correspond to simple representations of one of the groups SU(n). The photon conditions require that the photon-meson-meson and photon-baryon-baryon interactions are a sum of two parts, one part transforming as a state of the regular representation of the group, the other transforming as a singlet. If the spin and parity of the photon are 1^-, this result is equivalent to vector dominance for these vertices. The experimental implications of the results are discussed briefly.

I. INTRODUCTION

Recently, the author investigated the consequences of a set of bootstrap conditions involving the trilinear interaction constants of a hypothetical set of hadrons.[1] (The symbol R1 will be used to refer to Ref. 1.) The hadron set considered consisted of four separately degenerate subsets, even- and odd-parity mesons and even- and odd-parity baryons. The number of particles in each subset was left arbitrary, *a priori*. It was shown that the bootstrap conditions imply that an SU(n)-type symmetry must be present, and that the mesons and baryons must correspond to certain simple representations of the group.

The purpose of the present paper is to apply suitable modifications of these bootstrap conditions to interactions involving hypothetical weakly interacting bosons, called "photons." The motivation is to see if bootstrap considerations can help explain the success of the vector-dominance model, and predict some observable ratios of photon-hadron-hadron interaction constants.

In order to explain the method used, we describe first the consistency relations of R1.[1] A scattering amplitude of the type hadron + hadron → hadron + hadron is considered. A bootstrap-type hypothesis, based on duality or some other appropriate dynamical principle, is used to relate the sums of the residues of the pole contributions to the amplitude in two different Mandelstam channels. Each such residue is bilinear in the coupling constants involving the virtual hadron associated with the pole. It is required that the set of virtual hadrons be identical with the set of real hadrons.

In order to generalize to weak interactions, we consider all amplitudes involving exactly one external "photon" and three external hadrons, and follow the procedure outlined above in writing the consistency equations. It is assumed that the photon interaction vertices are of the trilinear type. The weakness of the photon interactions is accounted for by omitting all contributions from virtual photons. Because of this omission, two photons with different properties would not be related to each other by the dynamical equations. Therefore, we may assume that only one photon state exists when solving the equations.

The consistency equations involving the photons are written and discussed in Sec. II. The solutions are found in Secs. III A to III E, and summarized in Sec. III F. The physical meaning of the results is discussed in Sec. IV.

II. THE CONSISTENCY EQUATIONS

We use the consistency conditions of R1 with simple modifications. The four-particle scattering amplitude in the s, t, and u Mandelstam channels is given by

$$s:\ \bar{a}+\bar{b}\to c+d,$$
$$t:\ \bar{a}+\bar{c}\to b+d, \qquad (1)$$
$$u:\ \bar{c}+\bar{b}\to a+d,$$

where the symbols a, b, etc. refer to the internal quantum numbers and spin quantum numbers along the z axis of hadrons or photons. Only collinear amplitudes are considered. The symbol $\bar{i}$ denotes the antiparticle state of i. The convention used here, with all initial particles barred, is slightly different from that of R1.

We define the hadron-hadron interaction constant $G_{ij\bar{k}}$ to correspond to the vertex $k\to i+j$, where the states i and j are emitted in the $+z$ and $-z$ directions, respectively, in the k rest system. Our convention is that the subscripts of G correspond to the quantum numbers of final-state particles in the "crossed" vertex, vacuum $\to i+j+\bar{k}$. In the case of meson-baryon-baryon (MBB) vertices, another convention used is that the third subscript corresponds to one of the baryons.

Reversal of the directions of i and j leads to the symmetry relation

$$G_{ij\bar{k}}=\eta^{ijk}G_{ji\bar{k}}, \qquad (2)$$

where η^{ijk} is the orbital parity factor of the vertex. The factors $\eta^{ij\cdots}$, with three or four superscripts, are subject to the restriction that the number of superscripts referring to baryons is even, and the number referring to antibaryons is even. The value of η is the product of the intrinsic parities of all indexed meson states, and the relative parity of any pair of baryon or antibaryon states. The phases of the G may be defined so that they have the "Hermiticity property" expressed by the relation

$$G_{ij\bar{k}}=G^{*}_{\bar{i}k\bar{j}}, \qquad (3)$$

where i is a meson state. We take the meson-meson-meson (MMM) couplings to be invariant to particle-antiparticle conjugation of all the mesons, i.e.,

$$G_{ijk}=G_{\bar{i}\bar{j}\bar{k}} \quad \text{(for } MMM \text{ couplings)}. \qquad (4)$$

The properties of photon vertices are similar to those of meson vertices. However, we omit the label distinguishing different photon states, since these states are to be treated one at a time. The small interaction constant $E_{j\bar{k}}$ is associated with the vertex $k\to\gamma+j$, where the γ is emitted in the $+z$ direction in the k rest system.

It is convenient to define coupling-constant matrices $\tilde{G}_a$ and $\tilde{E}$, denoted with a tilde. The elements of these matrices are related to coupling constants by the prescriptions

$$(\tilde{G}_a)_{ij}=\tilde{G}_{aij}=G_{ai\bar{j}}, \quad \tilde{E}_{ij}=E_{i\bar{j}}. \qquad (5)$$

The transformation from coupling constant to matrix element involves barring the last index, because the matrix notation is most appropriate when the right-hand index refers to an initial state, while the indices of the coupling constants all refer to final-state quantum numbers.

We first consider the cases of MM and MB scattering; generalization to photon interactions will be trivial. The states $\bar{a}$ and c of the amplitude of Eq. (1) are taken to be meson states, while $\bar{b}$ and d are either both meson states or both baryon states. The s-u channel consistency condition of R1, which may be obtained from the application of a duality principle to backward s-channel scattering, is given in our notation by the equation

$$\sum_n G_{dc\bar{n}}G^{*}_{\bar{a}\bar{b}\bar{n}}=\eta^{a\bar{b}cd}\sum_n G_{da\bar{n}}G^{*}_{\bar{c}\bar{b}\bar{n}}. \qquad (6)$$

The sums in this equation are over intermediate states of both parities. We will not discuss the derivation of the condition in this paper. If the symmetry conditions of Eqs. (2) and (3) are used, Eq. (6) may be written in a form convenient for our purposes,

$$\sum_n G_{cd\bar{n}}G_{anb}\eta^{cdn}=\sum_n G_{ad\bar{n}}G_{cnb}\eta^{\bar{b}cn}. \qquad (7)$$

We obtain an s-t channel consistency condition for MM scattering by making the substitutions $a\rightleftarrows b$ in Eq. (7), and then using Eqs. (2), (3), and (4) to write the result in a convenient form. This leads to the equation

$$\sum_n G_{cd\bar{n}}G_{anb}=\kappa\sum_n G_{\bar{n}db}G_{acn}. \qquad (8)$$

We use this as the s-t channel condition both for MM and MB scattering. The constant κ is one for MM scattering, but is left arbitrary when $\bar{b}$ and $\bar{d}$ are baryons.

The consistency equations for the $\gamma M\to MM$ and $\gamma B\to MB$ amplitudes are obtained by replacing the external meson a by the photon. One makes the substitution

$$G_{aij}\to E_{ij} \qquad (9)$$

in Eqs. (7) and (8). The photon is excluded from the sums over virtual states.

III. SOLUTIONS TO THE BOOTSTRAP EQUATIONS

A. General Equations of the $\gamma M\to MM$ Amplitudes

When discussing specific applications, we will replace the G and E of Sec. II by more specific symbols. The MM interaction constants associated with odd and even vertex parity factors η are denoted by f_{ijk} and d_{ijk}, respectively. It follows from Eqs. (2), (3), and (4) that d_{ijk} is real and completely symmetric and f_{ijk} is imaginary and completely antisymmetric. We denote by γ_{ij} and γ'_{ij} the photon-meson-meson interaction constants corresponding to odd and even parity factors $\eta^{\gamma ij}$, respectively. These constants must have symmetry properties similar to those of *MMM* constants, i.e., γ_{ij} is imaginary and antisymmetric in i and j, while γ'_{ij} is real and symmetric.

We consider first the s-u channel consistency condition for $\gamma M \to MM$ amplitudes for which the product of the intrinsic parities of the four bosons is even. We make the photon-meson substitution of Eq. (9) in the condition of Eq. (7), and then write the condition in terms of the *MMM* interaction constants d and f and the γMM constants γ and γ'. The result is

$$\sum_n (d_{cd\bar{n}}\gamma'_{nb} - f_{cd\bar{n}}\gamma_{nb}) = \sum_n (\gamma'_{d\bar{n}} d_{cnb} - \gamma_{d\bar{n}} f_{cnb}), \quad (10)$$

where now each sum is over virtual-meson states of one parity only.

The s-t and u-t channel consistency conditions for $\gamma M \to MM$ amplitudes are equivalent to the s-u channel condition with appropriate permutations of the meson-state labels. Thus, the basic consistency equations are Eq. (10) and a corresponding equation that applies when the parity factor $\eta^{\gamma bcd}$ is negative.

B. The *MMM* Coupling Constants

We review in this subsection the results of R1 concerning the quantum numbers of the mesons and the values of the d and f of Eq. (10). The f_{ijk} must be proportional to the structure constants of one of the groups SU(n).[2] We assume that $n > 2$. The solution is parity-doubled, in that there are n^2 meson states of each parity, corresponding to the regular and singlet representations of SU(n). (In the formulation of R1, the doubling is between Regge trajectories of opposite signature.)

The interaction constants may be expressed in terms of a quark-antiquark picture. We denote the n states of the fundamental (quark) representation by Q_i, and the antiquark states by $\overline{Q}_i$. The quantum numbers of any state c of the regular or singlet representations may be expressed by picturing the state as a quark-antiquark composite, i.e.,

$$c = \sum_{ij} C_{ij}\overline{Q}_i Q_j, \quad (11)$$

where the sum is over the n^2 orthonormal $\overline{Q}Q$ states. The matrix denoted by the capital letter (in this case C) represents the meson state denoted by the corresponding small letter; the matrix is normalized by the condition $\sum_{ij} |C_{ij}|^2 = 1$. The f and d coupling constants are each invariant to changing the parities of any two mesons of the vertex. They are given in terms of traces of the matrices of Eq. (11), i.e.,

$$f_{abn} = \lambda \mathrm{Tr}[(BA - AB)N], \quad (12a)$$

$$d_{abn} = \lambda \mathrm{Tr}[(BA + AB)N], \quad (12b)$$

where λ is a real constant of proportionality.[3] The subscripts range over the n^2 states of the singlet and regular representations, and do not include the parity label. The f and d may be nonzero only when the vertex parity factor η^{abn} is odd or even, respectively.

The matrix corresponding to the singlet states is a multiple of the unit matrix. The d interactions involving singlets are included in Eq. (12b); the f_{ijk} constants vanish if one or more indices corresponds to a singlet.

C. Antisymmetric Photon - Meson Interactions

If Eq. (10) is summed over the six permutations of the meson states b, c, and d, with a factor of -1 included with the odd permutations, the d_{ijk} terms drop out. If use is made of the antisymmetry of the γ_{ij} and the complete antisymmetry of the f_{ijk}, the result of this permutation sum may be reduced to the form

$$\sum_n (\gamma_{bn} f_{cd\bar{n}} + \gamma_{cn} f_{db\bar{n}} + \gamma_{dn} f_{bc\bar{n}}) = 0 . \quad (13)$$

This is the basic equation for γ_{ij}. The rest of this subsection consists of a proof of the following theorem.

Theorem: The condition of Eq. (13) implies that the antisymmetric γMM interaction matrix is proportional to a generator of the group. In symbols, the result is

$$\gamma_{bc} = \sum_i \mu_i f_{ibc}, \quad (14)$$

where the μ_i are constant coefficients.

The fact that Eq. (14) does satisfy the condition follows immediately from the fact that the replacement $\gamma_{ij} \to f_{aij}$ leads to one of the conditions satisfied by the f's. (This is the Jacobi identity condition, which implies that the f's are proportional to structure constants.[1]) However, we will demonstrate this fact directly, since the tools used will be needed later. The sum in Eq. (13) may be extended from the $n^2 - 1$ states of the regular representation to n^2 orthonormal states by adding a singlet state, since f_{ijk} vanishes if one of the indices refers to a singlet. The n^2 orthonormal matrices N satisfy the closure condition

$$\sum_N N_{ij}\overline{N}_{kl} = \delta_{il}\delta_{jk}, \quad (15)$$

where $\overline{N}$ is the matrix corresponding to $\bar{n}$ ($\overline{N} = N^\dagger$). It is easy to use this condition to show that Eq. (14) satisfies Eq. (13) if the f_{ijk} are given by Eq. (12a).

We have left to prove that Eq. (13) is violated if γ_{bc} has a component that does not transform as the regular representation. In the case of SU(2), all antisymmetric triplet-triplet states transform as triplets, so the result is trivial. Hence, we consider only SU(n), with $n > 2$. We construct a special state with the quantum numbers of two quark-antiquark pairs, labeled the U pair and the V pair. The state is

$$\varphi = (Q_1^U \overline{Q}_3^U Q_2^V \overline{Q}_3^V - Q_1^V \overline{Q}_3^V Q_2^U \overline{Q}_3^U)/\sqrt{2}\,, \qquad (16)$$

where the subscript denotes the state of the quark representation. The first term in this expression is represented by a pair of matrices U and V, defined by the prescription of Eq. (11) to be

$$U_{ij} = \delta_{i3}\delta_{j1}, \quad V_{ij} = \delta_{i3}\delta_{j2}\,. \qquad (17)$$

It is shown in the Appendix that φ has no component in either the singlet or regular representations, for any SU(n) (with $n > 2$).

We consider the following specific, hypothetical interaction γ_{bc},

$$\gamma_{bc} = k\,\mathrm{Tr}(UBVC - UCVB), \qquad (18)$$

where k is a constant, U and V are the matrices of Eq. (17), and B and C are the $\overline{Q}Q$ matrices of Eq. (11) representing the meson states b and c. If all matrices represent $\overline{Q}Q$ pairs, the trace of a product corresponds to a singlet. Therefore, the antisymmetric state of the mesons b and c that interact with the photon in Eq. (18) transforms under SU(n) like the conjugate of the state φ of Eq. (16).

If use is made of the closure property of Eq. (15), substitution of Eq. (18) into the bootstrap condition of Eq. (13) leads to the requirement $W = 0$ for all regular-representation meson states b, c, and d, where W is the quantity

$$W = \sum_{bcd} \mathrm{Tr}[(UBV - VBU)(DC - CD)], \qquad (19)$$

and $\sum_{bcd}$ is a sum over the three cyclic permutations of B, C, and D. A straightforward calculation shows that if U and V are given by Eq. (17), the following choice of meson $\overline{Q}Q$ matrices leads to a nonzero W:

$$B_{ij} = \delta_{i1}\delta_{j3}, \quad C_{ij} = \delta_{i2}\delta_{j3}, \quad D_{ij} = \delta_{i3}\delta_{j3}\,. \qquad (20)$$

The matrix D of Eq. (20) corresponds to a combination of singlet- and regular-representation states. However, since the matrix corresponding to a pure singlet is a multiple of the unit matrix, it is easy to see that the W in Eq. (19) vanishes if either B, C, or D corresponds to a singlet. This completes the demonstration that the consistency condition rules out the specific γMM interaction of Eq. (18).

We now extend the argument to show that the bootstrap condition is violated if the photon interacts antisymmetrically with any MM state that has a component not in the regular representation. The antisymmetric terms in the reduction of the direct product $R \otimes R$ are

$$R \oplus T \oplus T^*, \qquad (21)$$

where R is the regular representation, T and T^* are conjugate, and T is given (when $n > 3$) by $(\lambda_1, \ldots, \lambda_{n-2}, \lambda_{n-1}) = (2, \ldots, 1, 0)$.[4,5] All the λ_i omitted here are zero. The quantity λ_i is the number of boxes in the ith row of the Young tableau of the representation, minus the number of boxes in the $(i+1)$th row. In the case of SU(3), the representation is $(\lambda_1, \lambda_2) = (3, 0)$. The argument of the Appendix shows that the state φ of Eq. (16) is completely in the representation $T \oplus T^*$.

The meaning of a nonzero W in Eq. (19) may be understood in terms of a wave function constructed in a particular way from the three meson states b, c, and d. One forms a regular-representation state antisymmetrically from c and d, and combines this state antisymmetrically with b to form a state conjugate to the photon state represented by Eq. (16). Then one adds the three cyclic permutations of the states b, c, and d. In symbols, this prescription for forming the MMM state (denoted by Ψ) is represented by

$$\Psi = \sum_{bcd} \{[B(CD)_R]_\Psi\}, \qquad (22)$$

where $(\alpha\beta)_j$ denotes the antisymmetric combination of the regular-representation states α and β to form the state j, $\sum_{bcd}$ is a sum over cyclic permutations and R is a regular-representation state. The fact that the W of Eq. (19) does not vanish for all choices of the matrices B, C, and D is equivalent to the statement that a state can be formed according to the prescription of Eq. (22) with SU(n) quantum numbers conjugate to those of the φ of Eq. (16).

One can produce any state or linear combination of states of the representation $T \oplus T^*$ by operating on the state of Eq. (22) with a suitably constructed polynomial composed of raising and lowering operators and the particle-antiparticle conjugation operator.[6] Such an operator would not change the symmetry of the state; i.e., the resulting state would still be of the type shown in Eq. (22). This implies that for any state in $T \oplus T^*$, matrices B, C, and D can be found for which Eq. (13) is violated. Therefore, any antisymmetric γMM coupling that satisfies Eq. (13) for all b, c, and d must correspond to the regular representation.

For definiteness, we now limit ourselves to the "physical" case in which the photon parity is odd. The antisymmetric γMM interactions then connect mesons of the same parity. We consider first the

$\gamma M \to MM$ amplitudes where all the mesons are of odd intrinsic parity. All the γ_{ij} of Eq. (13) then refer to couplings between odd-parity mesons. The result of the last paragraph shows that these coupling constants are given by

$$\gamma_{ij} = \mu f_{aij} , \tag{23}$$

where a is one of the states of the regular representation and μ is a constant of proportionality. We choose our basis for the odd-parity meson states of the regular representation to consist of the state a and states orthogonal to a.

We next consider the amplitudes for which the parities of the mesons b, c, and d are even, odd, and even, respectively. We use the symmetries of the f's and γ's to rewrite Eq. (13) as the $b\bar{d}$ element of the following matrix equation:

$$[\tilde{\gamma}, \tilde{f}_c] = \sum_n \gamma_{nc} \tilde{f}_{\bar{n}} . \tag{24}$$

The matrices are related to the coupling constants by the prescription of Eq. (5). We define a matrix x by the condition $\tilde{\gamma} = \mu \tilde{f}_a + x$. The result of Eq. (23) (which we have derived for the case that i and j are odd-parity states) implies that x_{ij} can be nonzero only when i and j are even-parity states. In Eq. (24), the $\tilde{\gamma}$ matrices on the left refer to couplings between states of even parity, while the γ_{nc} on the right refer to meson states of odd parity. If we substitute Eq. (23) into the right-hand side of Eq. (24), the latter equation reduces to the condition $[x, \tilde{f}_c] = 0$. Because of Schur's lemma, this implies that x transforms as a singlet. Since there is no antisymmetric singlet coupling, x must be zero. We conclude that Eq. (23) applies to the coupling of the photon with two states of even parity, as well as two states of odd parity. The condition of Eq. (23) is equivalent to Eq. (14) if the state a is chosen properly. The theorem is proven.

It should be emphasized that this solution for γ_{ij} is independent of the symmetric MMM interaction, and so does not depend on the extent that the values of d_{ijk} predicted in R1 agree with experiment.

D. Symmetric Photon - Meson Interactions

We now consider the symmetric γMM interactions γ' of Eq. (10). If the photon parity is odd, these interactions connect meson states of opposite parities.

It is convenient to start with the s-t channel $MM \to MM$ condition of Eq. (8), with κ set equal to one. One writes this equation for the two cases of even- and odd-parity factor η^{abcd}, adds and subtracts the two resulting equations, and antisymmetrizes with respect to the labels a and c. If one then replaces the a-meson couplings d_a and f_a by the photon couplings γ' and γ, respectively, the result is the two equations

$$[\tilde{p}_c, (\tilde{\gamma} + \tilde{\gamma}')] = 2 \sum_n \tilde{\gamma}_{cn} \tilde{p}_n , \tag{25a}$$

$$[\tilde{m}_c, (\tilde{\gamma} - \tilde{\gamma}')] = 2 \sum_n \tilde{\gamma}_{cn} \tilde{m}_n , \tag{25b}$$

where $\tilde{p}_i = \tilde{f}_i + \tilde{d}_i$, $\tilde{m}_i = \tilde{f}_i - \tilde{d}_i$, and the tildes denote matrices, defined in terms of the coupling constants by Eq. (5).

We define the matrix y by the equation

$$\tilde{\gamma}'_{ij} = \mu \tilde{d}_{aij} + y_{ij} , \tag{26}$$

where μ is the proportionality constant of Eq. (23). If Eqs. (23) and (26) and the expressions for d and f of Eqs. (12a) and (12b) are used, the conditions of Eqs. (25a) and (25b) may be reduced to the form

$$[\tilde{p}_c, y] = [\tilde{m}_c, y] = 0 . \tag{27}$$

We have made use of the fact that the equations for MM scattering that are analogous to Eqs. (25a) and (25b) are satisfied. [These are Eqs. (3.8) and (3.10) of R1.]

It is shown in R1 that $\tilde{p}_i$ and $\tilde{m}_i$ correspond to two conjugate fundamental representations of SU(n), which we call the quark and antiquark representations.[2] The meson states interact like direct products of these two representations, i.e., like quark-antiquark states. Because of Schur's lemma, Eq. (27) implies that y transforms as a singlet. Furthermore, this singlet must be coupled equally to all the meson states. In other words, Eq. (26) may be replaced by the equation

$$\tilde{\gamma}'_{ij} = \mu \tilde{d}_{aij} + \nu \delta_{ij} , \tag{28}$$

where ν is a constant.

There is no consistency condition for $\gamma M \to MM$ scattering that relates μ and ν, the proportionality constants of the regular-representation part and singlet part of the γMM interaction. This contrasts with the consistency conditions for the MMM couplings, where the occurrence of the M as both real and virtual states leads to the prediction of all coupling ratios.[1]

E. Coupling of the Photon to Baryons

We next consider the $\gamma B \to MB$ processes, using the results of R1 for the MBB and MMM interaction constants. As mentioned in Sec. III B, the meson states of each parity correspond to the n^2 states of the regular and singlet representations of SU(n). Furthermore, it is shown in R1 that the constants of interaction with any pair of baryons of the corresponding mesons of opposite parities are equal. On the other hand, the representations of the baryons of opposite parities are not the same. We use the symbol D_c to denote the coupling matrix between baryons of the same parity of the meson c (of either parity) and F_c to denote the corresponding coupling matrix between states of opposite parities. We define "plus" and "minus" coupling matrices by the equations

$$\tilde{P}_c = \tilde{F}_c + \tilde{D}_c, \quad \tilde{M}_c = \tilde{F}_c - \tilde{D}_c .$$

The vector space includes the baryon states of both parities, ordered so that even-parity states have the lowest indices. The $\tilde{D}$ are nonzero only in the diagonal quadrants of this vector space, and the $\tilde{F}$ are nonzero only in the off-diagonal quadrants. We use the capital letter $\tilde{\Gamma}$ to denote the matrix of the photon coupling between baryon states of the same parity, and $\tilde{\Gamma}'$ to denote the corresponding matrix between states of opposite parity.

Since the baryonic couplings of corresponding mesons of opposite parity are the same, the s-u conditions are independent of the meson parity. In the s-t channel conditions, the roles of odd- and even-parity mesons in the t channel are reversed if one reverses the parity of the external meson. However, when one sums over all virtual particles, these conditions also are independent of the parity of the external meson. There are only two different parity cases; the two baryons have the same parity in one case and opposite parities in the other. The s-u channel conditions are found by writing Eq. (7) for these two parity cases, adding and subtracting the results, and replacing the MBB coupling matrices $\tilde{D}_a$ and $\tilde{F}_a$ by $\tilde{\Gamma}$ and $\tilde{\Gamma}'$. The results are

$$[\tilde{P}_c, \tilde{\Gamma}' - \tilde{\Gamma}] = 0, \tag{29a}$$

$$[\tilde{M}_c, \tilde{\Gamma}' + \tilde{\Gamma}] = 0 . \tag{29b}$$

The s-t and u-t channel conditions are obtained by writing Eq. (8) for the two parity cases, and adding and subtracting. One then symmetrizes and antisymmetrizes with respect to the indices a and c, and makes the replacements $\tilde{D}_a \to \tilde{\Gamma}$, $\tilde{F}_a \to \tilde{\Gamma}'$, $\tilde{f}_a \to \tilde{\gamma}$, and $\tilde{d}_a \to \tilde{\gamma}'$. [An alternate method of deriving all the conditions is to make these hadron → photon replacements in Eqs. (4.9)–(4.13) of R1.] The results are

$$[\tilde{P}_c, \tilde{\Gamma}' + \tilde{\Gamma}] = 2\kappa \sum_n \tilde{\gamma}_{cn} \tilde{P}_n, \tag{30a}$$

$$[\tilde{M}_c, \tilde{\Gamma}' - \tilde{\Gamma}] = -2\kappa \sum_n \tilde{\gamma}_{cn} \tilde{M}_n, \tag{30b}$$

$$\{\tilde{P}_c, \tilde{\Gamma}' + \tilde{\Gamma}\}_+ = 2\kappa \sum_n \tilde{\gamma}'_{cn} \tilde{P}_n, \tag{30c}$$

$$\{\tilde{M}_c, \tilde{\Gamma}' - \tilde{\Gamma}\}_+ = -2\kappa \sum_n \tilde{\gamma}'_{cn} \tilde{M}_n . \tag{30d}$$

We now define the matrices X and Y by the equations

$$\tilde{\Gamma}' + \tilde{\Gamma} = \mu(\tilde{F}_a + \tilde{D}_a) + X, \tag{31a}$$

$$\tilde{\Gamma}' - \tilde{\Gamma} = \mu(\tilde{F}_a - \tilde{D}_a) + Y, \tag{31b}$$

where μ is the γMM proportionality constant of Secs. III C and III D. If Eqs. (31a) and (31b) and the expressions for γ and γ' of Eqs. (23) and (28) are substituted into Eqs. (29a), (29b), and (30a)–(30d), and the MB scattering conditions [Eqs. (4.9)–(4.13) of R1] are used, the results are

$$[X, \tilde{P}_c] = [X, \tilde{M}_c] = [Y, \tilde{P}_c] = [Y, \tilde{M}_c] = 0, \tag{32}$$

$$\{\tilde{P}_c, X\}_+ = 2\kappa\nu \tilde{P}_c, \tag{33a}$$

$$\{\tilde{M}_c, Y\}_+ = -2\kappa\nu \tilde{M}_c . \tag{33b}$$

It is shown in R1 that if c is the SU(n) singlet state, the matrices $\tilde{P}_c$ and $\tilde{M}_c$ are nonzero multiples of the unit matrix. (We are assuming a solution of the type that is not parity-doubled for the baryons.) It follows from this fact and Eqs. (33a) and (33b) that X and Y are given by

$$X_{ij} = \kappa\nu\delta_{ij}, \quad Y_{ij} = -\kappa\nu\delta_{ij} . \tag{34}$$

If one adds and subtracts Eqs. (31a) and (31b) and makes use of Eq. (34), the result is

$$\tilde{\Gamma}' = \mu \tilde{F}_a, \tag{35}$$

$$\tilde{\Gamma} = \mu \tilde{D}_a + \kappa\nu \tilde{1}, \tag{36}$$

where $\tilde{1}$ is the unit matrix. It is seen that the singlet part of the γBB interaction contributes only when the parities of the two baryons are the same.

In order to clarify this result, we review briefly the results of R1 concerning the MBB interaction constants, obtained from the MB-scattering consistency equations.[7] In the solution that corresponds most closely with experiment, the baryons have the symmetry of three-quark composites, and the parity of the baryon corresponds to the symmetry under interchange of two of the quarks. In this solution, the predicted even-parity and odd-parity baryons correspond to the SU(6) representations $\underline{56} \oplus \underline{70}$ and $\underline{70} \oplus \underline{20}$, respectively. Only the even-parity $\underline{56}$ and the odd-parity $\underline{70}$ are observed in nature. The $\tilde{D}_a$ and $\tilde{F}_a$ of Eqs. (35) and (36) are the SU(6)$_W$-symmetric interactions of the meson state a between baryon states of like and opposite parity, respectively.[8] The physical significance of our results for the γBB interactions is discussed further in Sec. IV.

F. Summary of Predicted Form of Photon Interactions

The implications of the bootstrap conditions on the weak couplings of a boson state to the various mesons and baryons are given in Eqs. (23), (28), (35), and (36). The ratios of all photon couplings to hadron-hadron-hadron couplings are given in terms of the regular-representation state denoted by a and the two proportionality constants μ and ν, which refer to the regular-representation part and singlet part of the photon interactions, respectively.

IV. PHYSICAL INTERPRETATION

It is pointed out in Sec. I that if more than one boson state is weakly coupled to the hadrons, the consistency equations apply separately to each such state. If one such state is not its own antiparticle state, *CPT* invariance implies that the antiparticle state must exist and be coupled also. In our formalism, where the indices refer both to spin components and internal quantum numbers, the physical photon is a pair of conjugate states, corresponding to positive and negative helicities.

The consistency equations do not predict particle-mass ratios, and certainly do not predict the masslessness of the photon. However, the photon masslessness causes no difficulty in the derivation of the consistency conditions.[9] It is assumed in the derivation that the Regge residues for similar types of couplings are proportional as functions of the various energy and momentum-transfer variables. Our use of these conditions in this paper involves the assumption that this proportionality condition applies also to photon-hadron vertices of hadronic Regge residues. This assumption enables us not only to use the consistency equations, but also to apply the results directly to ratios of observed γMM and γBB interaction constants.

Physically, the symmetry group that involves both spin and internal quantum numbers of the hadrons is SU(6), with the spin treated by the W-spin prescription.[8] The parity-doubling condition for mesons implies that each Regge trajectory through an odd-parity meson state should be exchange-degenerate with a trajectory of opposite signature. The lowest states on these latter trajectories are meson resonances of even parity. Similarly, the odd-parity baryons are resonances with masses larger than those of the nucleon octet and Δ decuplet. Many of the predictions of this paper concern the photon couplings of these resonances; these predictions are difficult to check. The predictions most accessible to experimental check concern the photon couplings of various states of the vector- and pseudoscalar-meson nonets [which correspond to the $\underline{35}\oplus\underline{1}$ representation of $\mathrm{SU}(6)_W$], and the couplings between the 56 states of the nucleon octet and Δ decuplet. These are couplings of the types denoted by γ and Γ in Sec. III.

Strictly speaking, the photons considered here are real, and hence are in states of helicity ± 1. The γMM and γBB vertices are of the magnetic type. Of course, electrical charge may be measured by the Compton scattering of electrically coupled, real photons, but this process involves vertices of a type different from that considered here. (For example, the Thomson limit of photon scattering from mesons involves the A^2 term in the interaction Lagrangian.) The $\mathrm{SU}(6)_W$ singlet term [ν term of Eqs. (28) and (36)] cannot contribute to the coupling of helicity states ± 1. The prediction of Sec. III is that the magnetic photon coupling is proportional to the coupling of the helicity states of a linear combination of the ρ_0, ω, and φ vector mesons. The observed $-\frac{2}{3}$ neutron/proton magnetic-moment ratio, predicted originally from SU(6) symmetry, is a confirmation of a combination of our photon predictions and the predictions of R1 concerning vector-meson interactions.[10,11]

Our predictions involve real photons. However, if one can extend the treatment to longitudinally polarized virtual photons, a similar prediction may be made for the electric coupling. Again, the prediction is that the coupling is proportional to the coupling of a linear combination of the ρ_0, ω, and φ.

The vector-dominance hypothesis usually is applied to observed amplitudes involving photons.[12] Our predictions apply only to photon-hadron-hadron vertices. However, if amplitudes are dominated by poles in various channels, it is clear that vertex vector dominance can lead to amplitude vector dominance. The success of the vector-dominance hypothesis may be the result of consistency conditions of the bootstrap type.

The present approach is complementary to the "octet-dominance" approach to electromagnetic interactions, used by Dashen and Frautschi and others several years ago.[13] In both approaches it is assumed that the strong interactions are SU(n)-symmetric. Dashen and Frautschi show that if basic EM (electromagnetic) interactions corresponding to different representations of the group are present, the effect of the strong interactions may be such as to enhance the observed effect of the regular-representation type, EM interactions. On the other hand, our result is that self-consistency conditions may force the basic EM interactions to correspond to the regular and singlet representations.

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APPENDIX

We prove here that the $Q_1\overline{Q}_3Q_2\overline{Q}_3$ state of Eq. (16) has no component in the singlet or regular representation for any SU(n), $n \geq 3$. The algebra of SU(n) contains $n-1$ independent, commuting, Hermitian, n-by-n matrices. We diagonalize these matrices simultaneously, denoting by $H_{i\alpha}$ the αth eigenvalue of the ith matrix. We choose the $n-1$ matrices H_i to satisfy the following orthogonality and normality relation,

$$\sum_{\alpha=1}^{n} H_{i\alpha} H_{j\alpha} = \delta_{ij} . \tag{A1}$$

We define an nth matrix H_n to be a multiple of the unit matrix, with diagonal elements $H_{n\alpha}$ equal to $n^{-1/2}$. The requirement that the generators be traceless then allows us to extend the range of i and j in Eq. (A1) from $n-1$ to n. If $\overline{H}$ is defined to be the n-by-n matrix whose elements are $H_{i\alpha}$, Eq. (A1) is equivalent to $\overline{H}\overline{H}^T = 1$, where T denotes the transpose. Since the dimension is finite, this implies that $\overline{H}^T H = 1$, or

$$\sum_{i=1}^{n} H_{i\alpha} H_{i\beta} = \delta_{\alpha\beta} . \tag{A2}$$

We consider only states that are eigenvectors of the diagonal generators H_1 to H_{n-1}. For every such state of every representation there exists a "weight" vector in the $(n-1)$-dimensional space defined by the generators. The ith component of the weight vector is the eigenvalue of H_i. We use the norm, or length squared, of the weight vectors to classify states. The quark states are those whose eigenvalues are the elements $H_{i\alpha}$ of the matrices H_1 to H_{n-1}. The norm N_Q of any state Q_α of the quark representation is given by $\sum_{i=1}^{n-1} H_{i\alpha}{}^2$, or

$$N_Q = 1 - H_{n\alpha}{}^2 = (n-1)/n . \tag{A3}$$

The cosine of the angle $\theta_{\alpha\beta}$ between any two different quark states Q_α and Q_β, is $\sum_{i=1}^{n-1} H_{i\alpha} H_{i\beta}/N_Q$, or

$$\cos\theta_{\alpha\beta} = 0 - H_{n\alpha} H_{n\beta}/N_Q$$
$$= -(n-1)^{-1} . \tag{A4}$$

The weight vector of an antiquark state is the negative of that of the corresponding quark state. Thus, the cosine of the angle $\overline{\theta}_{\alpha\beta}$ between the vectors corresponding to Q_α and $\overline{Q}_\beta$ is

$$\cos\theta_{\alpha\beta} = (n-1)^{-1} \quad (\text{if } \alpha \neq \beta) . \tag{A5}$$

The weight vector of a composite state is the sum of the weight vectors of the constituents. The norm of the vector of a singlet state is zero. The norm for the regular-representation state $Q_\alpha \overline{Q}_\beta$(for $\alpha \neq \beta$), computed by using Eqs. (A3) and (A5) and the law of cosines, is two. Thus all states of the singlet and regular representations have norms of 0 or 2. On the other hand, a quick calculation shows that the norm for the state $Q_1\overline{Q}_3 Q_2 \overline{Q}_3$ is 6. This concludes the proof that the state φ of Eq. (16) has no component in the singlet or regular representation.

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[1]R. H. Capps, Phys. Rev. D 3, 3059 (1971).

[2]In R1 it was shown that the matrices $\tilde{p}_i = \tilde{f}_i + \tilde{d}_i$ have the property that all products $\tilde{p}_j \tilde{p}_k$ may be expressed as a linear combination of the $\tilde{p}_i$ and the identity matrix, in any irreducible subspace of the $\tilde{p}_i$. It was stated without proof that this requires the $\tilde{p}_i$ to be the fundamental representation of one of the groups SU(n). It has since been pointed out to the author by Dr. T. K. Kuo (private communication) that this follows from Burnside's theorem for group representations. See Hermann Boerner, *Representations of Groups* (North-Holland, Amsterdam, 1963), p. 69. The theorem was pointed out to Dr. Kuo by Dr. S. P. Wang.

[3]A similar expression is derived in R1 for a representation in which all meson states are self-conjugate. We emphasize that the meson states in the present treatment need not be self-conjugate.

[4]Rules for decomposing the direct product of two representations of SU(n) are given by C. Itzykson and M. Nauenberg, Rev. Mod. Phys. 38, 95 (1966).

[5]The separation into symmetric and antisymmetric states may be performed by the following procedure. One takes the direct product of the symmetric two-quark state and the antisymmetric two-antiquark state, and adds to this the direct product of the antisymmetric two-quark state and the symmetric two-antiquark state. This leads to the representation $R \oplus R \oplus T \oplus T^*$, as can be seen from the rules of Ref. 4. These states are antisymmetric in exchange of the two quark-antiquark pairs. The R state resulting from the antisymmetric singlet-R combination must be subtracted, yielding Eq. (21).

[6]The raising and lowering operators of SU(n) groups are discussed in Ref. 4, and also by R. E. Behrends, J. Dreitlein, C. Fronsdal, and B. W. Lee, Rev. Mod. Phys. 34, 1 (1962).

[7]It is shown in R1 that if one applies the bootstrap conditions to $B\overline{B}$ scattering, as well as to MM and MB scattering, the baryon representations are required to be two-quark representations. This result does not agree with experiment, so we neglect the implications of the $B\overline{B}$ conditions here. The fact that $B\overline{B}$ consistency conditions of this type are not satisfied by three-quark representations has been emphasized by J. L. Rosner, Phys. Rev. Letters 21, 950 (1968); 21, 1468(E) (1968).

[8]W spin is defined and discussed by H. J. Lipkin and S. Meshkov, Phys. Rev. 143, 1269 (1966).

[9]This bootstrap condition, used in R1, is derived by R. H. Capps, Phys. Rev. D 2, 2640 (1970).

[10]It is well known that the first derivations based on

SU(6) of the neutron/proton magnetic-moment ratio were wrong, essentially because the moment operator did not behave properly under particle-antiparticle conjugation. However, the discovery of W spin made possible a correct derivation, given by K. M. Bitar and F. Gürsey, Phys. Rev. 164, 1805 (1967).

[11]The prediction of the moment ratio does not follow from our results alone. One must use the requirement that the SU(3) structure of the state that corresponds to the magnetic interactions is the U-spin scalar state of the octet. Since the electric charge is proportional to the U-spin scalar of the octet, this requirement may be obtained from the requirement that the charge and magnetic-moment operators [within the regular representation of SU(6)] have the same SU(3) structure.

[12]The vector-dominance hypothesis has been discussed in many papers. Some references to early papers are contained in the review talk of N. M. Kroll, in *Proceedings of the Fourteenth International Conference on High Energy Physics, Vienna, 1968,* edited by J. Prentki and J. Steinberger (CERN, Geveva, 1968), pp. 75–86. A more recent paper on the subject with some references, is A. P. Contogouris, R. Gaskell, and M. Svec, Phys. Rev. D 3, 145 (1971). The extent of the validity of the vector-dominance hypothesis is not yet clear.

[13]R. Dashen and S. Frautschi, Phys. Rev. Letters 13, 497 (1964); Phys. Rev. 143, 1171 (1966); R. E. Cutkosky and Pekka Tarjanne, *ibid.* 132, 1354 (1963); R. H. Capps, *ibid.* 134, B649 (1964).

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Unitarized Veneziano $\pi\pi$ Scattering Amplitude Consistent with Positivity and Crossing-Symmetry Constraints

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A complete K-matrix unitarization of the Veneziano amplitudes for pion-pion scattering is carried out. In contrast to previous works on this subject, the correct behavior of the partial waves at the left-hand cut threshold is taken into account. Crossing symmetry is enforced by determining the parameters from the Roskies sum rules, and the Martin inequalities are used to limit further the range of solutions. The resulting s and p waves are in good agreement with the most recent experimental information.

I. INTRODUCTION

The problem of pion-pion scattering has been a very fundamental subject in hadron physics and has served as a model example in the S-matrix approach to strong interactions.[1] The development of current algebra coupled with the hypothesis of the partially conserved axial-vector currents (PCAC) on the one hand and the discovery of the Veneziano model on the other have stimulated many authors[2] to have a renewed interest in the quest for pion-pion interactions.

The Veneziano model is based on Regge asymptotic behavior, crossing symmetry, and real linear trajectories. However, the resulting amplitudes lack unitarity and cut-plane analyticity. Furthermore, while the K-matrix method, suggested for example by Lovelace,[3] can remedy the unitarity to some extent, the resulting partial-wave amplitudes do not possess the crossing symmetry of the original Veneziano amplitudes. Though some attempts[4] have been made to retain the crossing symmetry in the course of unitarization, no complete K-matrix unitarization, which is consistent with cut-plane analyticity and crossing symmetry, has been available till now.

The purpose of this paper is to carry out a complete K-matrix unitarization of the Veneziano am-