PHYSICAL REVIEW D VOLUME 4, NUMBER 10 15 NOVEMBER 1971

Symmetry Breaking in a Field Theory of Currents. I

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(Received 19 May 1971)

We present a model of symmetry breaking for a field theory of currents through the divergences of vector and axial-vector currents. The assumption of partial conservation of vector current (PCVC) and partial conservation of axial-vector current (PCAC) for the nonvanishing divergences allows us to make an explicit connection between the form of the symmetry-breaking part of the energy-momentum tensor and the nonlinear realization of the internal symmetry. It is shown that for SU(2) and SU(3) the vanishing of exotic commutators fixes uniquely the symmetry breaking as well as the nonlinear realization for the divergences of the currents.

INTRODUCTION

Assuming that strong-interaction physics can be described by a set of m vector currents $V_\mu^i(x)$ and m axial-vector currents $A_\mu^i(x)$, where m is the number of generators of a special unitary group, one can construct a "field theory of currents" which is consistent with Lorentz covariance and from which the Heisenberg equations of motion emerge.[1] Many of the currents defined above are identified with the weak and electromagnetic currents and are therefore directly observable.[2,3] The basic quantities defining the theory are the equal-time commutation relations (ETCR) between the components of the currents with the extra assumption that the time-time component of the energy-momentum tensor obeys the "Schwinger condition."[4-7] Then, the complete energy-momentum tensor can be expressed as a bilinear function in those currents and is shown to be automatically invariant under the underlying group $SU(n)\otimes SU(n)$. This means that an energy-momentum tensor that is a bilinear polynomial in the currents and not invariant under the underlying group is inconsistent either with the Schwinger conditions or with the fact that the currents transform as vectors (axial vectors) under the Lorentz group.[8]

Therefore, in an attempt to obtain an energy-momentum tensor which breaks the $SU(n)\otimes SU(n)$ symmetry while keeping the original assumption of Sugawara[1] (i.e., equal-time commutation relations among the currents, and the Schwinger condition) one has to add new operators to the set of the original vector and axial-vector currents. Since in an invariant theory all the currents are conserved, the most natural way to break the symmetry is to introduce the divergences of the currents $[S^i(x)=\partial_\mu V_\mu^i(x),\ P^i(x)=\partial_\mu A_\mu^i(x)]$ as new members of the original set $\{V_\mu^i, A_\mu^i\}$.

In Sec. I we review briefly the main features of Sugawara's theory of currents. Then, in Sec. II, along with a few algebraic assumptions, we modify Sugawara's energy-momentum tensor by a term $\delta_{\mu\nu}\,\phi(x)$ where $\phi(x)$ is a scalar function only of the divergences of the currents.

Then we establish a general formalism giving a link between the form of the symmetry-breaking part of the energy-momentum tensor and the transformation properties of the divergences of the currents. It turns out that the divergences of the currents generally form a nonlinear realization of the $SU(n)\otimes SU(n)$ symmetry. It is worth mentioning that such nonlinear realizations are obtained without going through Lagrangian formalism. One finds also that in the cases considered, the vanishing of the "exotic commutators" uniquely specifies the symmetry breaking. Finally, we notice that the existence of a canonical conjugate for the fields $S^i(x)$ and $P^i(x)$ is not incompatible with their equations of motion.

In Sec. III, we apply the formalism of Sec. II to the cases where the underlying group structures are SU(2) and SU(3), respectively. In a separate publication, we will investigate the more realistic cases of $SU(2)\otimes SU(2)$ and $SU(3)\otimes SU(3)$.

I. REVIEW OF THE SUGAWARA THEORY OF CURRENTS

As stated in the Introduction, a field theory of currents of the type proposed by Sugawara[1] assumes the existence of a set of m vector currents $V_\mu^i(x)$ and m axial-vector currents $A_\mu^i(x)$. The underlying group structure of the theory is supposed to be that of the group $SU(n)\otimes SU(n)$. (We will consider in the following only the cases $n=2$, $n=3$.) The equal-time commutation relations of the components of the currents are postulated to be

$$[V_\mu^i(x), V_\nu^j(y)] = [A_\mu^i(x), A_\nu^j(y)] = t_{ijk}\{\delta_{\mu 4}V_\nu^k(x) + \tfrac{1}{2}\delta_{\nu 4}[V_\mu^k(x) + g_{\mu\sigma}V_\sigma^k(x)]\}\delta(\vec{x}-\vec{y}) + c\delta^{ij}(\delta_{\mu 4}\partial_\nu^x + \delta_{\nu 4}\partial_\mu^x)\delta(\vec{x}-\vec{y}) \tag{1.1}$$

$$[V_\mu^i(x), A_\nu^j(y)] = t_{ijk}\{\delta_{\mu 4}A_\nu^k(x) + \tfrac{1}{2}\delta_{\nu 4}[A_\mu^k(x) + g_{\mu\sigma}A_\sigma^k(x)]\}\delta(\vec{x}-\vec{y}),$$

where the t_{ijk} are the structure constants of the group SU(n),[2] c is a c-number constant, the metric used is $\delta_{\mu\nu}$ ($\mu = 1, 2, 3, 4$), $g_{\mu\nu} = (1, 1, 1, -1)$.

The Schwinger condition[4,5] must also be satisfied, i.e.,

$$[\Theta_{44}(x), \Theta_{44}(y)] = [\Theta_{4a}(x) + \Theta_{4a}(y)]\partial_a^x\delta(\vec{x}-\vec{y}) \quad (a = 1, 2, 3). \tag{1.2}$$

Here and in the following all the commutators considered are understood to be equal-time commutators. Then, the energy-momentum tensor $\Theta_{\mu\nu}(x)$ is found to be[8]

$$\Theta_{\mu\nu}(x) = -\frac{1}{2c}(\{V_\mu^i(x), V_\nu^i(x)\} + \{A_\mu^i(x), A_\nu^i(x)\} - \delta_{\mu\nu}[V_\rho^i(x)V_\rho^i(x) + A_\rho^i(x)A_\rho^i(x)]). \tag{1.3}$$

The currents obey the following equations of motion:

$$\partial_\mu V_\nu^i(x) - \partial_\nu V_\mu^i(x) = \frac{1}{2c}t_{ijk}[\{V_\mu^j(x), V_\nu^k(x)\} + \{A_\mu^j(x), A_\nu^k(x)\}], \tag{1.4a}$$

$$\partial_\mu A_\nu^i(x) - \partial_\nu A_\mu^i(x) = \frac{1}{2c}t_{ijk}[\{A_\mu^j(x), V_\nu^k(x)\} + \{V_\mu^j(x), A_\nu^k(x)\}]. \tag{1.4b}$$

From the form of $\Theta_{\mu\nu}(x)$, it follows that all the currents are conserved:

$$\partial_\mu V_\mu^i(x) = 0, \quad \partial_\mu A_\mu^i(x) = 0. \tag{1.5}$$

This theory has been shown to be consistent with Lorentz covariance and the Heisenberg equations of motion.[1]

The generators of the Poincaré group are indeed

$$P_\mu = i\int \Theta_{4\mu}(x)d^3x,$$

$$M_{\mu\nu} = i\int d^3x[x_\mu\Theta_{4\nu}(x) - x_\nu\Theta_{4\mu}(x)].$$

The Heisenberg equations of motion are expressed by

$$[P_\mu, B(x)] = i\partial_\mu B(x), \tag{1.6}$$

where $B(x)$ is any local operator.

Lorentz covariance is expressed by[9]:

$$[M_{\mu\nu}, J_\sigma^i(x)] = i[l_{\mu\nu}(x)]_{\sigma\lambda}J_\lambda^i(x), \tag{1.7}$$

where $J_\lambda^i(x)$ is either $V_\lambda^i(x)$ or $A_\lambda^i(x)$ and

$$[l_{\mu\nu}(x)]_{\sigma\lambda} = [x_\mu\partial_\nu - x_\nu\partial_\mu]\delta_{\sigma\lambda} + [\delta_{\mu\sigma}\delta_{\nu\lambda} - \delta_{\mu\lambda}\delta_{\nu\sigma}]. \tag{1.8}$$

P_μ and $M_{\mu\nu}$ satisfy the well-known relations

$$\begin{aligned}[P_\mu, P_\nu] &= 0,\\ [P_\lambda, M_{\mu\nu}] &= i(\delta_{\lambda\nu}P_\mu - \delta_{\lambda\mu}P_\nu),\\ [M_{\mu\nu}, M_{\lambda\sigma}] &= i(\delta_{\mu\lambda}M_{\nu\sigma} - \delta_{\mu\sigma}M_{\nu\lambda} + \delta_{\nu\lambda}M_{\mu\sigma} - \delta_{\nu\sigma}M_{\mu\lambda}).\end{aligned} \tag{1.9}$$

II. SYMMETRY BREAKING IN THE SUGAWARA THEORY OF CURRENTS

A. Assumptions of the Model

As explained earlier, we intend to break the symmetry of the original theory by keeping (1.1) and (1.2) and by adding to the set of vector and axial-vector currents, a new set of operators

$$S^i(x) = \partial_\mu V_\mu^i(x), \quad P^i(x) = \partial_\mu A_\mu^i(x), \tag{2.1}$$

which are, respectively, scalar and pseudoscalar quantities. It should be noted that some of the S^i's are identically zero for those i's which correspond to conserved currents. Furthermore, accepting the common view in quantum field theory about local operators, we regard $S^i(x)$ and $P^i(x)$ as interpolating fields of physical particles. In that sense, the relations (2.1) can be called PCVC (partially conserved vector current) and PCAC (partially conserved axial-vector current) relations. At first glance, one would think that the simplest way to break the SU(n)$\otimes$SU(n) symmetry would be to write an energy-momentum tensor of the form

$$\Theta_{\mu\nu} = \alpha^{ij}[V_\mu^i(x)V_\nu^j(x) + V_\nu^i(x)V_\mu^j(x)] + \tilde{\alpha}^{ij}[A_\mu^i(x)A_\nu^j(x) + A_\nu^i(x)A_\mu^j(x)] + \delta_{\mu\nu}[\beta^{ij}V_\rho^i(x)V_\rho^j(x) + \tilde{\beta}^{ij}A_\rho^i(x)A_\rho^j(x)].$$

However, such an expression for $\Theta_{\mu\nu}$ is shown in Appendix A to be inconsistent with Lorentz covariance. Therefore, the most natural way to break the SU(n)$\otimes$SU(n) symmetry is to define a new energy-momentum tensor

$$\Theta_{\mu\nu}(x) = \Theta_{\mu\nu}^{(S)}(x) + \delta_{\mu\nu}\phi(x), \tag{2.2}$$

where $\Theta_{\mu\nu}^{(S)}(x)$ is defined by (1.3) and is the part of the energy-momentum tensor which is invariant under the group $SU(n)\otimes SU(n)$, while $\phi(x)$ is the part which breaks the $SU(n)\otimes SU(n)$ symmetry. The form (2.2) of $\Theta_{\mu\nu}(x)$ will allow us to obtain local commutation relations from integrated ones and provides a smooth extrapolation of matrix elements of the Hamiltonian to the zero four-momentum limit.[10,11] Since we add new operators to the original algebra defined by (1.1) we need to make some assumptions concerning the equal-time commutators involving $S^i(x)$, $P^i(x)$, and $\phi(x)$. It turns out that the most crucial commutators are $[O^i(x), O^j(y)]$ and $[\phi(x), J_a^i(y)]$, where $O^i(x)$ is either $S^i(x)$ or $P^i(x)$ and $J_\mu^i(x)$ is either $V_\mu^i(x)$ or $A_\mu^i(x)$. We will assume that

$$[O^i(x), O^j(y)]=0, \tag{2.3}$$

$$[\phi(x), J_a^i(y)]=0. \tag{2.4}$$

Now, we want to retain the original commutation relations (1.1) and the Schwinger condition (1.2). Therefore, we require that

$$[\Theta_{44}(x), \Theta_{44}(y)]=[\Theta_{4a}(x)+\Theta_{4a}(y)]\partial_a^x\delta(\vec{x}-\vec{y}). \tag{2.5}$$

Using (1.3) for $\Theta_{\mu\nu}^{(S)}$ it follows that

$$[\phi(x), \phi(y)]=[\phi(y), \Theta_{44}^{(S)}(x)]+[\Theta_{44}^{(S)}(y), \phi(x)]. \tag{2.6}$$

The form (1.3) of $\Theta_{44}^{(S)}(x)$ implies

$$[\phi(x), \phi(y)]=-\frac{1}{2c}(\{V_4^i(x),[V_4^i(x), \phi(y)]\}-\{V_4^i(y),[V_4^i(y), \phi(x)]\}$$
$$-\{V_a^i(x),[V_a^i(x), \phi(y)]\}+\{V_a^i(y),[V_a^i(y), \phi(x)]\}+(V\to A)). \tag{2.7}$$

But the Heisenberg equations of motion (1.6) and the relations (1.1) and (2.1) imply

$$S^i(x)=\int d^3y[\phi(y), V_4^i(x)],$$
$$P^i(x)=\int d^3y[\phi(y), A_4^i(x)]. \tag{2.8}$$

The existence of a c-number Schwinger term in the commutators of the right-hand side of (2.8) can be shown to be inconsistent with (1.1) and Jacobi identities for $[[\phi(x), V_4^i(y)], V_4^j(z)]$ and $[[\phi(x), A_4^i(y)], A_4^j(z)]$. Therefore,

$$[\phi(x), V_4^i(y)]=S^i(x)\delta(\vec{x}-\vec{y}),$$
$$[\phi(x), A_4^i(y)]=P^i(x)\delta(\vec{x}-\vec{y}). \tag{2.9}$$

Then it follows that (2.4) and (2.9) imply

$$[\phi(x), \phi(y)]=0. \tag{2.10}$$

Furthermore, we expect $\phi(x)$ to be identically zero in the symmetry limit [i.e., when $S^i(x)=P^i(x)=0$]. Consequently, it is natural to assume that $\phi(x)$ is a function of the P's and S's only. This assumption is further supported by the conditions (2.3) and (2.10). All the assumptions introduced up to now are consistent with the Jacobi identities. At this stage, it is interesting to notice that an important consequence of assumption (2.4) is that the equations of motion for the currents retain their original form (1.4). We further point out that sum rules not inconsistent with experimental observations have been derived[12] from equations of motion of the form (1.4). The equation of motion for $\phi(x)$ is readily obtained by the use of the condition $\partial_\mu\Theta_{\mu\nu}(x)=0$.

Indeed the condition $\partial_\mu\Theta_{\mu\nu}(x)=0$ alone implies that

$$\partial_\mu\Theta_{\mu\nu}^{(S)}=-\partial_\nu\phi(x) \tag{2.11}$$

and the form (1.3) of $\Theta_{\mu\nu}^{(S)}(x)$ gives

$$\partial_\nu\phi(x)=(1/2c)[\{S^i(x), V_\nu^i(x)\}+\{V_\rho^i(x), \partial_\rho V_\nu^i(x)-\partial_\nu V_\rho^i(x)\}+\{P^i(x), A_\nu^i(x)\}+\{A_\rho^i(x), \partial_\rho A_\nu^i(x)-\partial_\nu A_\rho^i(x)\}]. \tag{2.12}$$

Since, as noticed earlier, the equation of motion for the currents retain their original form, one finally obtains after successive use of (1.1)

$$\partial_\nu\phi(x)=(1/2c)[\{S^i(x), V_\nu^i(x)\}+\{P^i(x), A_\nu^i(x)\}]. \tag{2.13}$$

Equation (2.13) can also be obtained in a more straightforward manner from the Heisenberg equations of motion and the use of the relations (2.4), (2.9), and (2.10). However, in order to completely define our theory of currents with breaking of the symmetry, one needs to know the equations of motion for the operators $S^i(x)$ and $P^i(x)$.

One can easily see that the equations of motion we are looking for are completely governed by the knowledge of commutators of the type

$$[O^i(x), J_4^j(y)], \tag{2.14}$$

where $O^i(x)$ is either $S^i(x)$ or $P^i(x)$ and where $J_4^j(y)$ is either $V_4^j(y)$ or $A_4^j(y)$.

Indeed, the Heisenberg equations of motion for $O^i(x)$ are

$$i\partial_\mu O^i(x) = [P_\mu, O^i(x)]$$

and therefore $\partial_\mu O^i(x)$ involves only commutators of the type

$$[O^i(x), \phi(y)], \quad [O^i(x), J_a^j(y)], \quad [O^i(x), J_4^j(y)].$$

But $[O^i(x), \phi(y)] = 0$ according to our assumptions. On the other hand, if we look at Jacobi identities for quantities of the type

$$[J_a^j(z), [\phi(x), J_4^i(y)]] \tag{2.15}$$

one has from (1.1) and (2.9)

$$[J_a^j(z), [\phi(x), J_4^i(y)]] = [J_a^j(z), O^i(x)]\delta(\vec{x}-\vec{y}) = [J_a^j(z), \partial_4 J_4^i(x)]\delta(\vec{x}-\vec{y}).$$

But from the assumptions (2.4) and the Jacobi identity for (2.15),

$$[J_a^j(z), [\phi(x), J_4^i(y)]] = -[\phi(x), [J_4^i(y), J_a^j(z)]] = 0. \tag{2.16}$$

Therefore,

$$[O^i(x), J_a^j(y)] = 0. \tag{2.17}$$

We now study further the commutators of the type (2.14) by taking the time derivative of the time-time commutator in (1.1). We obtain

$$\begin{aligned}
&[S^i(x), V_4^j(y)] + [V_4^i(x), S^j(y)] = t_{ijk} S^k(x)\delta(\vec{x}-\vec{y}),\\
&[S^i(x), A_4^j(y)] + [V_4^i(x), P^j(y)] = t_{ijk} P^k(x)\delta(\vec{x}-\vec{y}),\\
&[P^i(x), A_4^j(y)] + [A_4^i(x), P^j(y)] = t_{ijk} S^k(x)\delta(\vec{x}-\vec{y}).
\end{aligned} \tag{2.18}$$

This set of relations implies the following forms for the commutators (2.14):

$$\begin{aligned}
&[S^i(x), V_4^j(y)] = \tfrac{1}{2} t_{ijk} S^k(x)\delta(\vec{x}-\vec{y}) + M^{ij}(x)\delta(\vec{x}-\vec{y}),\\
&[P^i(x), V_4^j(y)] = t_{ijk} P^k(x)\delta(\vec{x}-\vec{y}) + N^{ij}(x)\delta(\vec{x}-\vec{y}),\\
&[S^j(x), A_4^i(y)] = N^{ij}(x)\delta(\vec{x}-\vec{y}),\\
&[P^i(x), A_4^j(y)] = \tfrac{1}{2} t_{ijk} S^k(x)\delta(\vec{x}-\vec{y}) + R^{ij}(x)\delta(\vec{x}-\vec{y}),
\end{aligned} \tag{2.19}$$

where M^{ij}, N^{ij}, and R^{ij} are Hermitian with M^{ij} and R^{ij} symmetrical in i and j. Also the right-hand sides of Eqs. (2.19) are assumed to contain no gradient terms. Therefore, the quantities $M^{ij}(x)$, $N^{ij}(x)$, and $R^{ij}(x)$ must be (at least from a purely algebraic point of view) only functions of $J_\mu^i(x)$ and $O^i(x)$ with the following properties:

(1) When i corresponds to a conserved current [i.e., $S^i(x) = 0$]

$$M^{ij} = -\tfrac{1}{2} t_{ijk} S^k. \tag{2.20}$$

(2) In the symmetry limit, all these functions must vanish identically.

(3) By commuting both sides of (2.19) with ϕ one has

$$[\phi(x), M^{ij}(y)] = [\phi(x), N^{ij}(y)] = [\phi(x), R^{ij}(y)] = 0. \tag{2.21}$$

Therefore, it is reasonable to assume that $M^{ij}(x)$, $N^{ij}(x)$, and $R^{ij}(x)$ are functions of the P's and S's only. This assumption shows that the P's and S's form a nonlinear realization of $SU(n)\otimes SU(n)$ if they correspond to interpolating Heisenberg fields. (Nonlinear realizations of chiral symmetry in particular have been investigated by Weinberg.[13]) The equations of motion for $S^i(x)$ and $P^i(x)$ can now be obtained:

$$\begin{aligned}
\partial_\mu S^j(y) &= \frac{1}{2c}\left[\{\tfrac{1}{2} t_{jmk} S^k(y) + M^{jm}(y), V_\mu^m(y)\} + \{N^{mj}(y), A_\mu^m(y)\}\right],\\
\partial_\mu P^j(y) &= \frac{1}{2c}\left[\{\tfrac{1}{2} t_{jmk} S^k(y) + R^{jm}(y), A_\mu^m(y)\} + \{t_{jmk} P^k(y) + N^{jm}(y), V_\mu^m(y)\}\right].
\end{aligned} \tag{2.22}$$

B. General Constraints on the Functions M^{ij}, N^{ij}, R^{ij}, and ϕ

The functions M^{ij}, N^{ij}, R^{ij}, and ϕ are not independent of each other since there are numerous Jacobi identities connecting them. Then, the constraints imposed on those functions will take the form of functional differential equations. In order to have more compact notations, we will drop the arguments of the operators as well as the $\delta(\vec{x}-\vec{y})$ and we will write $J_4^i(x)$ as J^i.

Thus, from the Jacobi identities for $[[S^i, V^j], V^k]$, $[[S^i, V^j], A^k]$, $[[P^i, V^j], V^k]$, $[[P^i, V^j], A^k]$, and $[[P^i, A^j], A^k]$ one obtains, respectively,

$$\begin{aligned}
&[M^{ij}, V^k] - [M^{ik}, V^j] = \tfrac{1}{4} t_{jkl} t_{ilm} S^m + t_{jkl} M^{il} + \tfrac{1}{2} t_{ikl} M^{jl} - \tfrac{1}{2} t_{ijl} M^{kl},\\
&[M^{ij}, A^k] - [N^{ki}, V^j] = t_{jkl} N^{li} - \tfrac{1}{2} t_{ijl} N^{kl},\\
&[N^{ij}, V^k] - [N^{ik}, V^j] = t_{jkl} N^{il} + t_{ikl} N^{lj} - t_{ijl} N^{lk},\\
&[N^{ij}, A^k] - [R^{ik}, V^j] = \tfrac{1}{4} t_{ikl} t_{jlm} S^m + t_{jkl} R^{il} + \tfrac{1}{2} t_{ikl} M^{lj} - t_{ijl} R^{kl},\\
&[R^{ij}, A^k] - [R^{ik}, A^j] = t_{jkl} t_{ilm} P^m + t_{jkl} N^{il} + \tfrac{1}{2} t_{ikl} N^{jl} - \tfrac{1}{2} t_{ijl} N^{kl}.
\end{aligned} \tag{2.23}$$

The differential equations obtained from (2.23) are explicitly written in Appendix B. Now using the relation (2.9) for the symmetry-breaking part of the energy-momentum tensor, one obtains

$$S^i = \tfrac{1}{2}t_{iml}S^m\partial_l\phi + m^{li}\,\partial_l\phi + t_{iml}P^m\tilde{\partial}_l\phi + N^{li}\tilde{\partial}_l\phi,$$
$$P^i = \tfrac{1}{2}t_{iml}S^m\tilde{\partial}_l\phi + R^{li}\tilde{\partial}_l\phi + N^{il}\tilde{\partial}_l\phi, \qquad (2.24)$$

where $\partial_l = \partial/\partial S^l$ and $\tilde{\partial}_l = \partial/\partial P^l$.

In order to achieve a consistent field theory of currents with symmetry breaking, the equations of Appendix B and (2.24) are fundamental since they relate, although not necessarily in a unique way, the form of the symmetry-breaking part of the energy-momentum tensor to the transformation properties of the divergences of the currents. For example, it will be shown in Sec. III that when the underlying group structure is that of SU(2) or SU(3), the vanishing of those commutators $[S^i, V^j]$, which are exotic, determines ϕ uniquely.

C. Possible Existence of Canonical Conjugates for the Fields $S^i(x)$ and $P^i(x)$

Since, according to our PCVC-PCAC assumption, we interpreted the nonzero fields $S^i(x)$ and $P^i(x)$ as interpolating fields of physical particles, it might be helpful in order to find, eventually, a particle interpretation of a field theory of currents, to investigate whether or not such a theory is incompatible with the existence of canonical conjugates $\pi^i(x)$ and $\tilde{\pi}^i(x)$ such that

$$[\pi^i(x), S^j(y)] = -\delta^{ij}\delta(\vec{x}-\vec{y}),$$
$$[\tilde{\pi}^i(x), P^j(y)] = -\delta^{ij}\delta(\vec{x}-\vec{y}), \qquad (2.25)$$

$$[\pi^i(x), \tilde{\pi}^j(y)] = [\pi^i(x), P^j(y)] = [\tilde{\pi}^i(x), S^j(y)] = 0, \qquad (2.26)$$

where $\pi^i(x)$ and $\tilde{\pi}^i(x)$ are defined only for those i's which correspond to $S^i(x)\neq 0$ and $P^i(x)\neq 0$.

By commuting $\pi^i(x)$ and $\tilde{\pi}^i(x)$ with both sides of the space part of Eqs. (2.22) we obtain equations defining $[\pi^i(x), V_a^m(y)]$ and $[\tilde{\pi}^i(x), A_a^m(y)]$:

$$-\delta^{ij}\partial_a^y\delta(\vec{x}-\vec{y}) = \frac{1}{2c}\big([t_{jmk}S^k(y) + 2M^{jm}(y)][\pi^i(x), V_a^m(y)] - \{\partial_i[t_{jmk}S^k(y) + 2M^{jm}(y)]\}V_a^m(y)\delta(\vec{x}-\vec{y})$$
$$+ 2N^{jm}(y)[\pi^i(x), A_a^m(y)] - 2[\partial_i N^{jm}(y)]A_a^m(y)\delta(\vec{x}-\vec{y})\big), \qquad (2.27a)$$

$$-\delta^{ij}\partial_a^y\delta(\vec{x}-\vec{y}) = \frac{1}{2c}\big([t_{jmk}S^k(y) + 2R^{jm}(y)][\tilde{\pi}^i(x), A_a^m(y)] - 2[\tilde{\partial}_i R^{jm}(y)]A_a^m(y)\delta(\vec{x}-\vec{y})$$
$$+ 2[t_{jmk}P^k(y) + N^{jm}(y)][\tilde{\pi}^i(x), V_a^m(y)] - 2\{\tilde{\partial}_i[t_{jmk}P^k(y) + N^{jm}(y)]\}A_a^m(y)\delta(\vec{x}-\vec{y})\big), \qquad (2.27b)$$

$$0 = [t_{jmk}S^k(y) + 2M^{jm}(y)][\tilde{\pi}^i(x), V_a^m(y)] - 2[\tilde{\partial}_i M^{jm}(y)]V_a^m(y)\delta(\vec{x}-\vec{y})$$
$$+ 2N^{jm}(y)[\tilde{\pi}^i(x), A_a^m(y)] - 2[\tilde{\partial}_i N^{jm}(y)]A_a^m(y)\delta(\vec{x}-\vec{y}), \qquad (2.27c)$$

$$0 = [t_{jmk}S^k(y) + 2R^{jm}(y)][\pi^i(x), A_a^m(y)] - \{\partial_i[t_{jmk}S^k(y) + 2R^{jm}(y)]\}A_a^m(y)\delta(\vec{x}-\vec{y})$$
$$+ 2[t_{jmk}P^k(y) + N^{jm}(y)][\pi^i(x), V_a^m(y)] - 2[\partial_i N^{jm}(y)]A_a^m(y)\delta(\vec{x}-\vec{y}). \qquad (2.27d)$$

Thus we see from the above relations that there is no obvious incompatibility between the equations of motion and the existence of the canonical conjugates $\pi^i(x)$ and $\tilde{\pi}^i(x)$. In fact, in some simple cases it is possible to find an explicit form for the $\pi^i(x)$ and $\tilde{\pi}^i(x)$. We refer the reader to Sec. III for a brief study of a particular form of canonical conjugates in the case of SU(2) and SU(3), respectively.

III. APPLICATIONS OF THE GENERAL FORMALISM TO THE CASES OF SU(2) AND SU(3)

A. The case of SU(2)

Let us assume that we deal with a world where we have no axial-vector currents and no strange currents. Then, the energy-momentum tensor describing strong and electromagnetic interactions is such that the third component of the isospin current is conserved,

$$S^3 = \partial_\mu V_\mu^3 = 0.$$

Since we have no axial-vector currents, Eqs. (2.19) reduce to the form

$$[S^i, V^j] = \tfrac{1}{2}\epsilon_{ijk}S^k + M^{ij}, \qquad (3.1)$$

where ϵ_{ijk} are the SU(2) structure constants.

At this point, it is more convenient to express the equations (3.1) in terms of lowering and raising operators. Therefore, we define in a usual manner

$$V^+ = V^1 + iV^2, \quad V^- = V^1 - iV^2.$$

Since $S^3 = 0$, it follows that the S's transform linearly with respect to V^3.

Indeed $S^3 = 0$ implies

$$M^{3j} = -\tfrac{1}{2}\epsilon_{3jk}S^k = M^{j3}, \tag{3.2}$$

which leads to

$$[S^j, V^3] = \epsilon_{j3k}S^k. \tag{3.3}$$

Then, when expressed in terms of $S^\pm$, S^3, $V^\pm$, V^3, the Eqs. (3.1) become

$$\begin{aligned} &[S^+, V^3] = iS^+, \\ &[S^+, V^+] = M^{11} - M^{22} + 2iM^{12}, \\ &[S^+, V^-] = M^{11} + M^{22}. \end{aligned} \tag{3.4}$$

The remaining commutators are obtained by taking the Hermitian conjugate of (3.4).

Now, one notices that each commutator in (3.4) belongs to the class of operators $\{T\}$ of infinite polynomials in S^+, S^- such that

$$\begin{aligned} &[T, V^3] = iqT, \\ &[T^\dagger, V^3] = -iqT \quad (q = 0, 1, 2, \ldots, n). \end{aligned} \tag{3.5}$$

We show in Appendix C that

$$T = g(X)(S^+)^q, \tag{3.6}$$

where

$$X = S^+S^-$$

and $g(X)$ is any infinite polynomial function of X which commutes with V^3.

Therefore, we define two functions f and h, which commute with V^3, such that

$$\begin{aligned} &[S^+, V^+] = 4f(S^+)^2, \\ &[S^+, V^-] = 2h. \end{aligned} \tag{3.7}$$

It is easy to see that one obtains a first fundamental differential equation relating ϕ to f and h through the relations

$$[\phi, V^+] = S^+, \tag{3.8a}$$

$$[\phi, V^3] = 0. \tag{3.8b}$$

But, from (3.5) and (3.8b) ϕ is only a function of X so that according to (2.24) one obtains

$$2(\partial_X\phi)(h + 2fX) = 1. \tag{3.9}$$

Since ϕ is Hermitian as well as h, f *must also be Hermitian*. The other fundamental differential equations relating f and h are obtained by using Jacobi identities of the type (2.23). For example, the Jacobi identity for $[[S^+, V^+], V^-]$ gives the relation

$$S^+ = 2(S^+)^2[f, V^-] + 8fhS^+ - [h, V^+]. \tag{3.10}$$

Since, for any function $g(X)$

$$[g(X), V^+] = \frac{\partial g}{\partial x}[X, V^+] = 2S^+(h + 2fX)\partial_X g, \tag{3.11}$$

Eq. (3.10) becomes

$$S^+ = 2S^+[2(\partial_X f)(h + 2fX)X + 4fh - (\partial_X h)(h + 2fX)],$$

or

$$1 = 2[2(\partial_X f)(h + 2fX)X + 4fh - (\partial_X h)(h + 2fX)]. \tag{3.12}$$

It turns out that all the Jacobi identities of the type (2.23) reduce to the same Eq. (3.12) which therefore becomes one fundamental differential equation. Thus Eqs. (3.9) and (3.12) tell us how the form of the symmetry breaking is related to the transformation properties of the S's. In particular, Eq. (3.9) shows that if the functions h and f are known and obey (3.12), then ϕ is uniquely determined. For example, let us assume that the so-called "exotic commutator" $[S^+, V^+]$ is identically zero (i.e., $f = 0$). Then

$$\begin{aligned} &h\partial_X h = -\tfrac{1}{2}, \\ &h = (\alpha - X)^{1/2}, \end{aligned} \tag{3.13}$$

where $\alpha \neq 0$ and $(\alpha - X)^{1/2}$ is understood to be the corresponding polynomial expansion in X.

From (3.9) one obtains

$$\begin{aligned} &\partial_X\phi = \frac{1}{2(\alpha - X)^{1/2}}, \\ &\phi = -(\alpha - X)^{1/2} + \beta. \end{aligned} \tag{3.14}$$

It is easy to see that in this case ϕ transforms like the third component of an SU(2) triplet. It is interesting to notice here that ϕ can always be made positive definite by choosing $\alpha > 0$ and $\beta = \sqrt{\alpha}$. Then ϕ takes the form

$$\phi = \sqrt{\alpha}\left[\frac{1}{2}\frac{X}{\alpha} + \frac{1}{8}\left(\frac{X}{\alpha}\right)^2 + \frac{1}{24}\left(\frac{X}{\alpha}\right)^3 + \cdots\right], \tag{3.15}$$

which ensures the positive definiteness of the energy spectrum. Conversely, if one fixes the form of the symmetry-breaking $\phi(X)$, one can obtain the transformation properties of the S's in a unique way.

For example, let us assume that $\phi(X) = \alpha X$. Then Eq. (3.9) gives

$$h + 2fX = 1/2\alpha. \tag{3.16}$$

From the Eq. (3.12) one obtains the relation

$$1 = 2\left(\frac{2}{\alpha}X\partial_X f - 8f^2X + \frac{3}{\alpha}f\right). \tag{3.17}$$

The first terms of the polynomial expansion in X give

$$f = \frac{\alpha}{6} + \frac{2\alpha^3}{45}X + \frac{16}{(7)(135)}\alpha^5X^2 + \cdots. \tag{3.18}$$

B. Conditions for the Existence of a Particular Canonical Conjugate

Let us now investigate the possible existence of a canonical conjugate $\pi^i(x)$ $(i=1, 2)$ for the fields $S^i(x)$. We look for an operator $\pi^i(x)$ such that

$$[\pi^i(x), S^j(y)] = -\delta^{ij}\delta(\vec{x}-\vec{y}).$$

From (2.22), the equations of motion for $S^j(y)$ are

$$\partial_\mu S^j(y) = (1/2c)[\{\tfrac{1}{2}\epsilon_{jmk}S^k(y) + M^{jm}(Y), V_\mu^m(y)\}]. \tag{3.19}$$

Commuting $\pi^i(x)$ with both sides of (3.19), Eq. (2.28a) takes the form

$$\delta^{ij}\partial_a^x\delta(\vec{x}-\vec{y}) = \frac{1}{c}\{[\tfrac{1}{2}\epsilon_{jmi} + \partial_i M^{jm}(y)]V_a^m(y)\delta(\vec{x}-\vec{y}) + [\tfrac{1}{2}\epsilon_{jmk}S^k(y) + M^{jm}(y)][\pi^i(x), V_a^m(y)]\}, \tag{3.20}$$

where use has been made of $[[\pi^i(x), V_a^m(y)], S^j(z)] = 0$. Now Eq. (3.20) suggests that

$$[\pi^i(x), V_a^m(y)] = F_n^{im}(x)V_a^n(y)\delta(\vec{x}-\vec{y}) + G^{im}(x)\partial_a^x\delta(\vec{x}-\vec{y}), \tag{3.21}$$

where F_n^{im} and G^{im} are functions of S's only. The above equation suggests in turn that $\pi^i(x)$ might be of the form

$$\pi^i(x) = \tfrac{1}{2}\{H^{ik}(x), V_4^k(x)\}, \tag{3.22}$$

where $H^{im}(x)$ is a function of S's only. Thus we look for the constraints imposed upon the function $H^{ik}(x)$ in order for $\pi^i(x)$ to be a solution of (3.20). Putting

$$A^{jm} = \tfrac{1}{2}\epsilon_{jmk}S^k + M^{jm}, \tag{3.23}$$

one finds the following conditions:

$$G^{im} = cH^{im}, \qquad F_n^{im} = H^{ik}\epsilon_{kmn}, \tag{3.24}$$

which imply that H^{im} must obey the constraints

$$\delta^{ij} = H^{im}A^{jm}, \qquad 0 = \partial_i A^{jn} + H^{ik}[\epsilon_{kmn}A^{jm} + (\partial_l A^{jk})A^{ln}], \tag{3.25}$$

where we recall that $i=1, 2$ and the other indices run from 1 to 3.

Furthermore, the constraints (3.25) have to be consistent with the differential equation (3.12). For example, if one assumes that $\pi^i(x) = \partial_4 S^i(x)$, one finds that such a canonical conjugate is not compatible with (3.12). It also turns out that when the exotic commutators vanish, Eqs. (3.25) are incompatible.

Therefore, the form (3.22) for the canonical conjugate imposes additional constraints on the nonlinear realizations of the SU(2) symmetry.

C. The Case of SU(3)

The case of SU(2) gave us some ideas about the manipulations involved in the applications of the general formalism developed in the previous sections. The results obtained for SU(2) look indeed very simple and we are encouraged to apply the same methods to the case of SU(3).

In a theory where the underlying group structure is SU(3) we will assume, in order to describe strong interactions, that the total isospin and hypercharge are conserved, i.e.,

$$S^1 = S^2 = S^3 = S^8 = 0.$$

Because of this fact, some of the operators M^{ij} can be determined. Indeed [with f_{ijk} being the structure constants of SU(3)]

$$[S^i, V^j] = \tfrac{1}{2}f_{ijk}S^k + M^{ij} = 0 \quad (i=1, 2, 3, 8) \tag{3.26}$$

implies

$$M^{ji} = M^{ij} = -\tfrac{1}{2}f_{ijk}S^i \quad (i=1, 2, 3, 8), \tag{3.27}$$

so that

$$[S^i, V^j] = f_{ijk}S^k \quad (j=1, 2, 3, 8). \tag{3.28}$$

This equation shows that the S's transform linearly under the isospin and hypercharge generators. Now, as we did in the case of SU(2), we express the commutators $[S^i, V^j]$ in terms of raising and lowering operators:

$$V^\pm = V^1 \pm iV^2, \qquad S_1^\pm = S^4 \pm iS^5, \quad S_2^\pm = S^6 \pm iS^7, \qquad V_1^\pm = V^4 \pm iV^5, \quad V_2^\pm = V^6 \pm iV^7. \tag{3.29}$$

Then, when expressed in terms of (3.29), the commutators containing the unknown M^{ij} are easily seen to belong to the class of operators T for which

$$[T, V^3] = n\tfrac{1}{2}iT, \quad n = 0, \pm1, \pm2, \ldots \qquad [T, V^8] = mi\tfrac{1}{2}\sqrt{3}T, \quad m = 0, \pm1, \pm2, \ldots. \tag{3.30}$$

Then we show in Appendix C that such an operator can be written as

$$T = g(X, Y)(S_1^{\epsilon(n+m)})^{\epsilon(n+m)(n+m)/2}(S_2^{\epsilon(m-n)})^{\epsilon(m-n)(m-n)/2}, \tag{3.31}$$

$$\epsilon(n+m) = \begin{cases} +, & n+m>0 \\ -, & n+m<0, \end{cases}$$

$$\epsilon(m-n) = \begin{cases} +, & m-n>0 \\ -, & m-n<0, \end{cases}$$

where $g(X, Y)$ is some function of the variables

$$X = S_1^+ S_1^- \quad \text{and} \quad Y = S_2^+ S_2^- .$$

Then, using (3.31) we can express the commutators (3.26) in the following form:

$$[S_1^+, V_1^+] = 4f(S_1^+)^2 , \qquad [S_1^+, V_1^-] = 2h , \tag{3.32}$$

where f and h are functions of X and Y only. Then the rest of the commutators (3.26) can be generated by some Jacobi identities of the type $[[S^i, V^j], V^\pm]$. One obtains

$$[S_2^+, V_1^+] = 4f S_2^+ S_1^+ , \quad [S_2^+, V_1^-] = 2F S_2^+ S_1^- , \quad [S_2^+, V_2^+] = 4f(S_2^+)^2 , \quad [S_2^+, V_1^-] = 2h - 2F(X - Y) , \tag{3.33}$$

where $F = (\partial_X - \partial_Y)h$. The use of Jacobi identities of the type $[[S_1^+, V_1^+], V^+] = 0$ and $[[S_2^+, V_1^-], V^-] = 0$ imposes further restrictions on f and h:

$$(\partial_X - \partial_Y) f = 0 , \qquad (\partial_X - \partial_Y)^2 h = (\partial_X - \partial_Y)F = 0 . \tag{3.34}$$

Similarly, from the fact that the symmetry-breaking part ϕ commutes with V^+ and V^-, one deduces

$$(\partial_X - \partial_Y)\phi = 0 . \tag{3.35}$$

Then the conditions

$$[\phi, V_1^+] = S_1^+ , \qquad [\phi, V_2^+] = S_2^+ \tag{3.36}$$

lead to two fundamental differential equations relating ϕ, h, f, and F:

$$1 = 2\partial_X\phi[(h + 2fX) + (F + 2f)Y] , \qquad 1 = 2\partial_X\phi[2f(X+Y) + FY + h] . \tag{3.37}$$

It follows from (3.37) that f is Hermitian. Then the remaining of the Jacobi identities of the type (2.24) will provide differential equations relating F, h, and f. Those differential equations are

$$\begin{aligned} \tfrac{1}{2} &= (2X\partial_X f - \partial_X h)[h + FY + 2f(X+Y)] + 4fh + F(F+2f)Y , \\ \tfrac{1}{4} &= 2fh - F(h - FX + 2fX) , \\ 0 &= (2\partial_X f - \partial_X F)[h + FY + 2f(X+Y)] + F(2f - F) . \end{aligned} \tag{3.38}$$

We notice that the conditions (3.34) and (3.35) imply that F, f, and ϕ are only functions of the single variable $X+Y$ and that h can be written in the form

$$h = K + XF = \tilde{K} - YF , \tag{3.39}$$

where K and $\tilde{K}$ are also functions of $X+Y$ only. Again, we see from (3.37) that if F, f, and h are known, then ϕ is completely determined.

As a particularly interesting solution to our system of differential equations (3.38) let us consider the case when the exotic commutators[14] are zero. Then, $f = 0$ and h obeys the differential equations

$$\tfrac{1}{2} = -(\partial_X h)(h + FY) + F^2 Y , \qquad \tfrac{1}{4} = -F(h - FX) , \qquad 0 = +(\partial_X F)(h + FY) + F^2 . \tag{3.40}$$

The most general solution to (3.40) is

$$F = \frac{1}{2(X+Y)}\{[\alpha - (X+Y)]^{1/2} - \sqrt{\alpha}\} , \quad \text{with } \alpha > 0$$
$$h = [\alpha - (X+Y)]^{1/2} - \frac{Y}{2(X+Y)}\{[\alpha - (X+Y)]^{1/2} - \sqrt{\alpha}\} , \tag{3.41}$$

where use has been made of (3.39).

Eqs. (3.37) give

$$\partial_X \phi = \frac{1}{2(h + YF)} ,$$

the solution of which is

$$\phi = -[\alpha - (X+Y)]^{1/2} + \beta , \tag{3.42}$$

where β is an integration constant of no physical significance. The positive definiteness of the energy spectrum is again obtained by choosing $\beta = \sqrt{\alpha}$ ($\alpha > 0$). Let us investigate further the consequences coming from the conditions that the exotic commutators are zero.

From (3.41) and (3.42) one notices that

$$\phi(X, Y) = -\tfrac{2}{3}[2h - (X - Y)F] + \tfrac{1}{3}\sqrt{\alpha} + \beta .$$

Defining

$$U_8 = -[2h - F(X - Y)] ,$$

it is possible to show, by using successively the commutators of U_8 with V^i, that one generates seven other quantities U^i such that $[U^i, V^j] = f_{ijk}U^k$.

Those U^i's are

$$\begin{aligned} U^1 &= -F(S_1^+ S_2^- + S_1^- S_2^+)\sqrt{3} , \\ U^2 &= iF(S_1^+ S_2^- - S_1^- S_2^+)\sqrt{3} , \\ U^3 &= -F(X - Y)\sqrt{3} , \\ U^4 &= -\sqrt{3}\, S^5 , \\ U^5 &= \sqrt{3}\, S^4 , \\ U^6 &= -\sqrt{3}\, S^7 , \\ U^7 &= \sqrt{3}\, S^6 . \end{aligned}$$

Therefore, the fact that the exotic commutators are zero implies that $\phi(x)$ transforms like the eighth component of an octet and fixes completely the nonlinear realization of SU(3). Conversely, it is well known that if $\phi(x)$ is proportional to the eighth component of an octet, then the exotic commutator vanishes.

Finally, what we have said about the existence of canonical conjugates in the case of SU(2) applies also in the case of SU(3), i.e., there exists a class of nonlinear realizations of the SU(3) symmetry which is not compatible with the existence of canonical conjugates of the form (3.22).

CONCLUSION

In this paper we presented a model of symmetry breaking in a field theory of currents. It is very interesting to notice that starting with a few simple assumptions we were able to develop a complete framework of symmetry breaking which has many common features with nonlinear Lagrangian theories.

In fact, once we fix the scalar and pseudoscalar interpolating fields by the PCVC-PCAC assumption, we find [at least for SU(2) and SU(3)] that the vanishing of the exotic commutators fixes the symmetry breaking as well as the nonlinear realization in a unique way. This uniqueness property is beyond the restrictions of group theory and shows that the above PCVC-PCAC assumption is a very strong one. Although the examples considered in this paper are not physically very interesting, many of the features discovered appear also in more realistic cases. Indeed, in a subsequent paper[15] we investigate in detail the case of SU(2) $\otimes$ SU(2) [with SU(2) conserved] and find that the formalism proposed above leads uniquely to the σ-model type of nonlinear realization of symmetry breaking. There are many problems to be investigated in this model. The basic question is that of the realization of the algebra and we believe that the existence of a canonical conjugate for some of the variables will help in this respect.

One can also investigate the breaking of scale invariance in this model. In fact, within our assumptions, scale invariance is obviously broken. Since ϕ is only a function of S's and P's and $\Theta_{\mu\nu}^{(S)}$ is only a function of V's and A's, $\Theta_{\mu\mu}$ cannot be zero.

Finally, it should be pointed out that there have been many attempts to introduce symmetry breaking in Sugawara's model.[16-19] Those attempts, among which Refs. 18 and 19 display great similarity with our work, deal with linear realizations for the divergences of the vector currents while we allow a more general transformation. A common feature of all these models including the present one is that the equations of motion for the currents are unaffected by the symmetry breaking.

APPENDIX A: COVARIANCE AND THE FORM OF THE ENERGY-MOMENTUM TENSOR

We show in the following that an energy-momentum tensor bilinear in the currents is inconsistent with Lorentz covariance when the internal symmetry is broken.

Consider the most general form for a symmetrical energy-momentum tensor bilinear in the vector and axial-vector currents:

$$\Theta_{\mu\nu}=\alpha^{ij}[V_\mu^i(x)V_\nu^j(x)+V_\nu^i(x)V_\mu^j(x)]+\tilde{\alpha}^{ij}[A_\mu^i(x)A_\nu^j(x)+A_\nu^i(x)A_\mu^j(x)]+\delta_{\mu\nu}[\beta^{ij}V_\rho^i(x)V_\rho^j(x)+\tilde{\beta}^{ij}A_\rho^i(x)A_\rho^j(x)]\,. \tag{A1}$$

Then, using the Heisenberg equations of motion,

$$\partial_4 V_4^k(x)=\int d^3y[\Theta_{44}(y),\,V_4^k(x)] \qquad \text{and} \qquad \partial_4 A_4^k(x)=\int d^3y[\Theta_{44}(y),A_4^k(x)]\,,$$

one obtains

$$\begin{aligned}\partial_4 V_4^k(y)=&[(2\alpha^{ij}+\beta^{ij})f_{jkl}+(2\alpha^{jl}+\beta^{jl})f_{jki}]V_4^i(y)V_4^l(y)+[(2\tilde{\alpha}^{ij}+\tilde{\beta}^{ij})f_{jkl}+(2\tilde{\alpha}^{jl}+\tilde{\beta}^{jl})f_{jki}]V_4^i(x)A_4^l(y)\\&+(\beta^{ij}f_{jkl}+\beta^{jl}f_{jki})V_a^i(y)V_a^l(y)+(\tilde{\beta}^{ij}f_{jkl}+\tilde{\beta}^{jl}f_{jki})A_a^i(y)A_a^l(y)-c(\beta^{kj}+\beta^{jk})\partial_a V_a^j(y)\,,\end{aligned} \tag{A2}$$

$$\begin{aligned}\partial_4 A_4^k(y)=&[(2\tilde{\alpha}^{ij}+\tilde{\beta}^{ij})f_{jkl}+(2\alpha^{jl}+\beta^{jl})f_{jki}]A_4^i(y)V_4^l(y)+[(2\alpha^{ij}+\beta^{ij})f_{jkl}+(2\tilde{\alpha}^{jl}+\tilde{\beta}^{jl})f_{jki}]V_4^i(y)A_4^l(y)\\&+[\beta^{ij}f_{jkl}+\tilde{\beta}^{jl}f_{jki}]V_a^i(y)A_a^l(y)+[\tilde{\beta}^{ij}f_{jkl}+\beta^{jl}f_{jki}]A_a^i(y)V_a^l(y)-c[\tilde{\beta}^{kj}+\tilde{\beta}^{jk}]\partial_a A_a^j(y)\,.\end{aligned} \tag{A3}$$

Therefore, Lorentz covariance implies

$$\alpha^{ij}f_{jkl}+\alpha^{jl}f_{jki}=0\,, \tag{A4a}$$

$$\tilde{\alpha}^{ij}f_{jkl}+\tilde{\alpha}^{jl}f_{jki}=0\,, \tag{A4b}$$

$$\alpha^{ij}f_{jkl}+\tilde{\alpha}^{jl}f_{jki}=0\,, \tag{A4c}$$

$$\tilde{\alpha}^{ij}f_{jkl}+\alpha^{jl}f_{jki}=0\,, \tag{A4d}$$

$$\beta^{kj}+\beta^{jk}=(1/c)\delta^{jk}\,, \tag{A4e}$$

$$\tilde{\beta}^{kj}+\tilde{\beta}^{jk}=(1/c)\delta^{jk}\,. \tag{A4f}$$

But the conditions (A4a) and (A4c) imply

$$\alpha^{ij}=\tilde{\alpha}^{ij}\,. \tag{A5}$$

Now the Heisenberg equation of motion

$$\partial_a V_4^k(y)=\int d^3y[\Theta_{4a}(y),\,V_4^k(x)]$$

gives

$$\alpha^{jk}+\alpha^{kj}=-(1/c)\delta^{jk}\,, \tag{A6}$$

where use has been made of the relations (A1). Now assuming that

$$V_\mu^i(x)V_\nu^j(x)=V_\nu^j(x)V_\mu^i(x)+\lim_{x\to y}[V_\mu^i(x),\,V_\nu^j(y)]\,, \tag{A7}$$

with

$$\lim_{x\to y}[V_\mu^i(x),\,V_\nu^j(y)] =f_{ijk}\lim_{x\to y}\{\delta_{\mu 4}V_\nu^k(x)+\tfrac{1}{2}\delta_{\nu 4}[V_\mu^k(x)+V_\mu^{\dagger k}(x)]\}\delta(\vec{x}-\vec{y})\,, \tag{A8}$$

one has

$$\alpha^{ij}V_\mu^i(x)V_\nu^j(x)=-\frac{1}{2c}V_\mu^i(x)V_\nu^i(x) +\tfrac{1}{2}(\alpha^{ij}-\alpha^{ji})\lim_{x\to y}[V_\mu^i(x),\,V_\nu^j(y)]\,. \tag{A9}$$

But from (A4a) it is easy to see that $(\alpha^{ij}-\alpha^{ji})f_{ijk}=0$. Similarly, from (A2) one deduces that

$$\beta^{ij}f_{jkl}+\beta^{jl}f_{jki}=\beta^{lj}f_{jki}+\beta^{ji}f_{jkl} \tag{A10}$$

so that

$$(\beta^{ij}-\beta^{ji})f_{jil}=0\,.$$

Therefore using (A4e) one has

$$\beta^{ij}V_\mu^i(x)V_\nu^j(x)=\frac{1}{2c}V_\mu^i(x)V_\nu^i(x)\,. \tag{A11a}$$

Similarly,

$$\tilde{\beta}^{ij}A_\mu^i(x)A_\nu^j(x)=\frac{1}{2c}A_\mu^i(x)A_\nu^i(x)\,. \tag{A11b}$$

Consequently,

$$\Theta_{\mu\nu}^{(x)}=-\frac{1}{2c}(\{V_\mu^i(x),\,V_\nu^i(x)\}+\{A_\mu^i(x),A_\nu^i(x)\} -\delta_{\mu\nu}[V_\rho^i(x)V_\rho^i(x)+A_\rho^i(x)A_\rho^i(x)])\,. \tag{A12}$$

APPENDIX B: DIFFERENTIAL EQUATIONS RELATING THE FUNCTIONS M^{ij}, N^{ij}, AND R^{ij}

$$\tfrac{1}{2}(t_{kml}S^m\partial_l M^{ij}-t_{jml}S^m\partial_l M^{ik})+(M^{lk}\partial_l M^{ij}-M^{lj}\partial_l M^{ik})+(t_{kml}P^m\tilde{\partial}_l M^{ij}-t_{jml}P^m\tilde{\partial}_l M^{ik})+(N^{lk}\tilde{\partial}_l M^{ij}-N^{lj}\tilde{\partial}_l M^{ik}) =\tfrac{1}{4}t_{jkl}t_{ilm}S^m+t_{jkl}M^{il}+\tfrac{1}{2}t_{ikl}M^{jl}-\tfrac{1}{2}t_{ijl}M^{kl}\,, \tag{B1}$$

$$\tfrac{1}{2}t_{kml}S^m\tilde{\partial}_l M^{ij}+R^{lk}\partial_l M^{ij}+N^{kl}\partial_l M^{ij}-\tfrac{1}{2}t_{jml}S^m\partial_l N^{ik}-M^{lj}\partial_l N^{ik}-t_{jml}P^m\tilde{\partial}_l N^{ik}-N^{lj}\tilde{\partial}_l N^{ik}=t_{jkl}N^{li}-\tfrac{1}{2}t_{ijl}N^{kl}\,, \tag{B2}$$

$$(\tfrac{1}{2}t_{kml}S^m\partial_l N^{ij}-\tfrac{1}{2}t_{jml}S^m\partial_l N^{ik})+(M^{lk}\partial_l N^{ij}-M^{lj}\partial_l N^{ik})+(t_{kml}P^m\tilde{\partial}_l N^{ij}-t_{jml}P^m\tilde{\partial}_l N^{ik})+(N^{lk}\tilde{\partial}_l N^{ij}-N^{lj}\tilde{\partial}_l N^{ik}) =t_{jkl}N^{il}+t_{ikl}N^{lj}-t_{ijl}N^{lk}\,, \tag{B3}$$

$$\tfrac{1}{2}t_{kml}S^m\tilde{\partial}_l N^{ij}+R^{lk}\partial_l N^{ij}+N^{kl}\partial_l N^{ij}-\tfrac{1}{2}t_{jml}S^m\partial_l R^{ik}-M^{lj}\partial_l R^{ik}-t_{jml}P^m\tilde{\partial}_l R^{ik}-N^{lj}\tilde{\partial}_l R^{ik} =\tfrac{1}{4}t_{ikl}t_{jlm}S^m+t_{jkl}R^{il}+\tfrac{1}{2}t_{ikl}M^{lj}-t_{ijl}R^{lk}\,, \tag{B4}$$

$$(\tfrac{1}{2}t_{kml}S^m\tilde{\partial}_l R^{ij}-\tfrac{1}{2}t_{jml}S^m\partial_l R^{ik})+(R^{lk}\partial_l R^{ij}-R^{lj}\partial_l R^{ik})+(N^{kl}\partial_l R^{ij}-N^{jl}\partial_l R^{ik}) =t_{jkl}t_{ilm}P^m+t_{jkl}N^{il}+\tfrac{1}{2}t_{ikl}N^{jl}-\tfrac{1}{2}t_{ijl}N^{kl}\,. \tag{B5}$$

APPENDIX C: STUDY OF A CLASS OF OPERATORS CARRYING GIVEN ISOSPIN AND HYPERCHARGE QUANTUM NUMBERS

We will consider only the case of SU(3) but the same reasoning can apply to SU(2).

Let us define an infinite polynomial function of the nonzero S's by

$$T=\sum_{q,r,s,t} C_{q,r,s,t}(S_1^+)^q(S_1^-)^r(S_2^+)^s(S_2^-)^t \tag{C1}$$

with the following properties:

$$[T, V^3]=i\tfrac{1}{2}mT,\quad [T, V^8]=i\tfrac{1}{2}\sqrt{3}nT. \tag{C2}$$

Since the S's transform linearly with respect to the conserved generators one obtains

$$q-r-s+t=m,$$

$$q-r+s-t=n$$

or

$$q-r=\tfrac{1}{2}(m+n),\quad t-s=\tfrac{1}{2}(m-n). \tag{C3}$$

Let us consider the case $m+n>0$, $m-n>0$. One can write

$$T=\sum_{r,s}\tilde{C}_{r,s}(S_1^+)^{r+(m+n)/2}(S_1^-)^r(S_2^+)^s(S_2^-)^{s+(m-n)/2}, \tag{C4}$$

where $\tilde{C}_{r,s}=C_{r+(m+n)/2,r,s,s+(m-n)/2}$.

But (C4) can be written as

$$T=(S_1^+)^{(m+n)/2}(S_2^-)^{(m-n)/2}\sum_{r,s}\tilde{C}_{r,s}(S_1^+S_1^-)^r(S_2^+S_2^-)^s. \tag{C5}$$

Putting $X=S_1^+S_1^-$, $Y=S_2^+S_2^-$,

$$g(X, Y)=\sum_{r,s}\tilde{C}_{r,s}(X)^r(Y)^s, \tag{C6}$$

one has

$$T=(S_1^+)^{(m+n)/2}(S_2^-)^{(m-n)/2}g(X, Y). \tag{C7}$$

Following the same reasoning for different signs of $m+n$ and $m-n$, one obtains the desired result (3.31).

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