PHYSICAL REVIEW D VOLUME 4, NUMBER 2 15 JULY 1971

Motion of Particles in Einstein's Relativistic Field Theory. I. Introduction and General Theory

Coates R. Johnson
Department of Physics and Astronomy, University of Georgia, Athens, Georgia 30601
(Received 22 December 1969)

If we represent particles through singularities in the field—we shall call such particles ideal particles—the field equations of Einstein's theory of the nonsymmetric field (unified field theory) can be solved step by step with respect to the powers of a parameter which measures the "strength" of the singularities associated with each particle. In paper I we develop the above-mentioned approximation procedure and show that ideal particles in an isolated region of space-time interact with each other as if each particle were acted on by a force which depends on the particle's kinematic and structural properties and on a certain external field in the neighborhood of the particle. The approximation procedure allows one to find, to any order of approximation desired, the equations of motion satisfied by each particle, the equations of change satisfied by those quantities which characterize the structure of each particle, and the field equations satisfied by the external field which acts on each particle. The equations are Lorentz-covariant in form.

I. INTRODUCTION

In the year 1915 Albert Einstein completed his relativistic theory of the gravitational field.[1] Einstein regarded the construction of this theory as a first step in an effort by him to construct a theory of the total field, a theory which would describe all of nature through one continuous field. In such a theory the particles of nature appear as limited portions of the space-time continuum in which the field strength is particularly high or the field very inhomogeneous. Such a theory has sometimes been called a unified field theory. It was not until the last years of his life that Einstein felt that he had found the natural generalization of his gravitational theory and thus a logically satisfactory theory of the total field. Einstein called this generalization of his gravitational theory the relativistic theory of the nonsymmetric field.[2] This is the theory we shall be investigating.

In the relativistic theory of the nonsymmetric field, nature is described through a second-rank tensor field which satisfies certain logically simple structural laws. These structural laws take the form of nonlinear partial differential equations. No nonsingular solutions satisfying realistic boundary conditions have ever been found to these partial differential equations, and whether such solutions exist, is not known. In this paper we shall assume that such solutions do exist, and furthermore, that they can often be approximated throughout a finite region of the continuum, except near a particle's center, by solutions in which each particle in the region is associated with a singularity in the field. By considering a particle as being in a certain four-dimensional region of the continuum we mean that we are only concerned with that segment of the particle's world line which is enclosed in that region. Particles associated with singularities in the field will be known as ideal particles, providing the particles are stable and are located in a perfectly isolated region of the continuum. The terms "stable particle," "isolated region," and "perfectly isolated region" will be defined later.

If we represent real particles in an isolated region of the continuum by ideal particles in a perfectly isolated region of the continuum, an approximation procedure can be found which allows one to solve Einstein's field equations step by step with respect to the powers of a parameter which measures the "strength" of the singularities associated with each particle. Such an approximation procedure can of course only be valid if the particles are not too near one another. In this paper we develop the above-mentioned approximation procedure[3] and show that ideal particles interact with each other as if each particle were acted on by a force which depends on the particle's kinematic and structural properties and on a certain external field in the neighborhood of the particle. The external field can be regarded as produced by the other particles in the region. The approximation procedure allows one to find, to any order of approximation desired, the equations of motion satisfied by each particle in the region, the equations of change satisfied by those quantities which characterize the structure of each particle in the region, and the field equations satisfied by the external field which acts on each particle in the region.

In paper II we apply the above-mentioned approximation procedure to the case of certain simple ideal particles located in a perfectly isolated region of the continuum and we find the equations of motion satisfied by these particles to certain low orders of approximation. We also find the equations of change satisfied by those quantities which characterize the structure of each particle in the region and the field equations satisfied by the external field which acts on each particle. We find that there are simple ideal particles which, in a finite region of space-time, interact with each other, to a good approximation, according to the laws of Maxwell-Lorentz electrodynamics. In a higher order of approximation we find an additional interaction among the particles which can be looked upon as a Lorentz-covariant generalization of the interaction given by Newton's gravitational theory.

A. General Discussion

The body of this paper begins (in Sec. II) with a description of Einstein's theory of the nonsymmetric field. In this section certain concepts and definitions needed for a general understanding of Einstein's theory are presented.

As indicated in Sec. II, the physical meaning of the various field quantities which appear in Einstein's theory has been until now vague and speculative. One reason for the writing of this paper is therefore to find physical meaning for the various field quantities which appear in Einstein's theory. In particular, one reason for the author's investigation of the interaction of particles in Einstein's theory is to find out (assuming the theory is correct) what field quantities in the theory correspond to the quantities we observe in nature as the electromagnetic field and the gravitational field. Since the electromagnetic and gravitational fields are in practice defined through the observed interaction of particles, one reasonable way of finding what field quantities in Einstein's theory correspond to these fields is to investigate the interaction of particles in the theory and to compare the results with observation.

The representation of particles through singularities in the field (which is discussed in Sec. III) has been used in the paper for three basic reasons. First, interacting particles interacting over macroscopic distances seem to be well represented by point particles (Newton's gravitational theory and classical electrodynamics can be considered as based upon the concept of the point particle). Second, the representation of particles through singularities gives satisfactory results in Einstein's theory of the pure gravitational field (neutral particles interacting over macroscopic distances). Third, there is no known way to systematically derive nonsingular solutions in Einstein's theory which might represent interacting particles. The author would like to point out that this idealization (particles represented by singularities) is used extensively in general-relativistic theories and is well known.

In order to solve the field equations of Einstein's theory over a region of the continuum containing singularities (particles), the author is forced to expand the field in powers of a parameter which measures the "strength" of the singularities in the field. (This is developed in Sec. IV.) This type of expansion is, in effect, used in all "fast-motion" approximation schemes.[4,5] The author then shows (in Sec. V) that the resulting equations of Einstein's unified field theory can be solved step by step to any order of approximation desired. The equations, when solved, lead to equations of motion and equations to be satisfied by several quantities which characterize each singularity. Other of the quantities which characterize the singularities can be chosen arbitrarily. The technique is developed for the most general kind of singularity which can represent a particle (a discussion of the meaning of this statement is given in Sec. V and in the footnotes). The technique presented in the paper for obtaining the equations of motion of interacting particles is not quasistatic and never involves integrals over surfaces. It is not the conventional Einstein-Infeld-Hoffmann (EIH) technique[6] applied to a new theory. It is a new "fast-motion" approximation technique. It is Lorentz-covariant at each step[7] and applied with complete generality (most general singularities) to a theory never before (to the author's knowledge) analyzed using a Lorentz-covariant technique.

The published Lorentz-covariant technique, used to analyze the Einstein-Maxwell equations and due to Kerr,[4] might be considered a "natural" Lorentz-covariant generalization of the EIH technique, as it uses surface integrals to obtain equations of motion. The author's technique avoids such integrals and the troubles they bring. The equations of motion obtained by Kerr in the Einstein-Maxwell theory, if they are carried out far enough, seem to involve infinite divergences which must be argued away (renormalization). This problem also occurs in the published Lorentz-covariant technique developed by Havas and Goldberg[5] for pole particles in the pure gravitational theory.[8] The author regards such troubles (infinite divergence) as due to unsatisfactory analysis. The author's technique never gives rise to such divergences. The author regards his technique as the only logically satisfactory Lorentz-covariant technique of which he is aware. It is also the only Lorentz-covariant technique of which he is aware ever developed for Einstein's unified field theory, a theory which is not identical with the Einstein-Maxwell theory or the pure gravitational theory and thus demands a separate analysis.

Also, in contrast to the other Lorentz-covariant techniques, it appears that the author's technique can actually be used to find the equations of motion of particles to any order of approximation desired. All differential equations which need to be solved when using the technique appear to be of a form for which an explicit solution can always be found. There is also, as mentioned earlier, no ambiguity resulting from infinite divergences. Only tedious labor seems to be involved in finding the equations of motion of particles to higher and higher orders of approximation. There is, of course, always a certain ambiguity in the higher orders of approximation due to the freedom one has in choosing coordinate systems.[9]

Because there seems to be some confusion with respect to this point in the literature on approximation techniques, the author would like to emphasize the fact that in his approximation technique the equations of motion of particles (and thus the motion of particles) are not changed from one order of approximation to another. The equations of motion (and thus the motion of particles) are just found, using his technique, step by step to higher orders of approximation. The above statement also applies to the equations of change satisfied by quantities which characterize the structure of particles.

In this paper, the author specifically avoids assigning any specific (exact) correlation between the fields appearing in the theory and the physical fields of nature until after he has obtained the form of the equations of motion for the most general singularity in the theory. In paper II, the author finds that singularities will interact as the particles of conventional classical electrodynamics (over certain distances and in the lowest order of approximation), and only then does the author identify certain quantities characterizing the singularities with what are known in nature as charge, mass, etc.[10] This assignment is of course based on the assumption that Einstein's theory is correct and that particles can, at least over macroscopic interaction distances, be represented by singularities in the field. The author, in this way, also finds a field which can be identified with the electromagnetic field observed in nature. The field to be identified with the gravitational field is found in an analogous way. Of course, because of the relationship of the theory of the nonsymmetric field to the theory of the gravitational field, there are no surprises here.

The author does – and this may lead to some confusion – assign names (labels) to the various quantities characterizing the most general singularities (particles) and to certain field variables appearing in the theory before their relationship to any quantity appearing in nature is shown. This freedom to assign arbitrary names is used throughout this paper. Only at the end of paper II, where the theory is compared with observation, are the concepts and quantities given names in the theory by the author seen to correspond generally to the identically named concepts and quantities in nature.

To sum up, the author feels that his papers are of interest for three basic reasons. First, the author develops a Lorentz-covariant technique for finding the equations of motion to any order of approximation desired for the most general particle (represented by a singularity) in Einstein's unified field theory. No such general technique for investigating Einstein's theory has ever been published to the author's knowledge. A general Lorentz-covariant technique was developed by Kerr for investigating the Einstein-Maxwell equations. His scheme makes use of surface integrals and leads to divergent terms in the equations of motion. Another Lorentz-covariant technique due to Havas and Goldberg is only developed for especially simple particles (pole singularities) and only in the pure gravitational theory. Their technique also leads to divergent terms in the equations of motion. One of the purposes of the author's papers is to develop a completely general technique (general singularities) for finding equations of motion which, because it makes use of only well understood mathematical procedures, can lead to no divergent terms in the equations of motion. The author has accomplished this task for Einstein's unified field theory and because it is a special case of the unified field theory, for Einstein's pure gravitational theory. The technique can easily be taken over and applied to the Einstein-Maxwell equations.

The second reason the author feels that his papers are of interest is that prominent physicists have used as an argument against the unified field theory a "proof that Einstein's so-called unified field theory leads to the wrong equations of motion for charged concentrations of mass-energy, in the sense that the object moves in external electromagnetic fields as if completely uncharged – an argument against that theory." This proof is based on Callaway's analysis[11] using the EIH approximation technique of the equations of motion of particles represented by singularities in the unified field theory. The author has shown that Callaway's analysis was not general enough and that there are singularities which not only interact through the Coulomb force[12] but interact in the lowest order of approximation over macroscopic distances through the complete laws of classical electrodynamics. The radiation reaction force even comes out correctly and there are no divergent terms in the equations of motion. The above argument used against the theory is not valid.

A third reason for an interest in the papers is that the author's technique for finding the equations of motion of particles in Einstein's theory allows one to find, in addition to the classical interactions (electromagnetic and gravitational), the corrections to these interactions which are demanded by Einstein's theory. One of these corrections may in fact be observable over astronomical distances. The possibility of observing this "astronomical interaction" is discussed in some detail in two appendixes of paper II. In addition to this interaction one also finds that there is a very complicated interaction between elementary particles which is likely to become important over distances $\lesssim 10^{-6}$ cm. An estimate of its range and strength is given in one of the appendixes of paper II.[13] The author is at present attempting to find the explicit form of this interaction. Does this interaction have any relation to quantum effects or does it indicate the breakdown of the idealization used in the paper (particles represented by singularities) or the incorrectness of the theory itself? The point of all of this is that the approximation procedure developed in this paper allows one to systematically investigate the interaction of particles in the Einsten unified field theory and to compare the results with observation. The author knows of no other analysis of the theory which provides such an opportunity.

In the final analysis the author believes that the physical meaning of the quantities appearing in Einstein's theory can only be determined through attempts to correlate the theory with observations. In order to do this in such a mathematically complex theory as Einstein's unified field theory, one has to make certain idealizations and approximations. The ones the author has made in his papers do not seem to the author to be unreasonable. In any case the author has shown that all of classical physics (electromagnetism and gravitation) can be obtained in a natural way from Einstein's unified field theory. The author has also opened the way to much further investigation of possible observable consequences of the theory.

B. Notation

We shall use the following notation in this paper. Unless otherwise stated, all lower-case Latin indices will take the values 1, 2, and 3, and will refer to space coordinates only. Lower-case Greek indices will refer to both space and time and will run through the values 1, 2, 3, and 4. Repetition of indices will imply summation. Unless otherwise stated, all raising and lowering of indices will be performed with the Minkowski metric.

The Minkowski metric $\eta_{\mu\nu}$ will be defined through the equations

$$\eta_{st} = -\delta_{st}, \quad \eta_{s4} = \eta_{4s} = 0, \quad \eta_{44} = 1, \tag{1.1}$$

where δ_{st} is the three-dimensional Kronecker δ. By definition, $\eta^{\mu\nu} = \eta_{\mu\nu}$. The four-dimensional Kronecker δ will be denoted by $\delta_\mu{}^\nu$. The Levi-Civita symbols $\epsilon_{\mu\nu\rho\sigma}$, $\epsilon^{\mu\nu\rho\sigma}$, and ϵ_{stk} will be chosen so that

$$\epsilon^{1234} = 1, \quad \epsilon_{1234} = -1, \tag{1.2}$$

$$\epsilon_{123} = 1. \tag{1.3}$$

If $A_{\mu\nu}\ldots$ is a field possessing at least two indices, we shall use the notation

$$A_{(\mu\nu)}\ldots = \tfrac{1}{2}(A_{\mu\nu}\ldots + A_{\nu\mu}\ldots),$$
$$A_{[\mu\nu]}\ldots = \tfrac{1}{2}(A_{\mu\nu}\ldots - A_{\nu\mu}\ldots). \tag{1.4}$$

Note that

$$A_{\mu\nu}\ldots = A_{(\mu\nu)}\ldots + A_{[\mu\nu]}\ldots \, . \tag{1.5}$$

If the field $A_{\mu\nu}\ldots$ is symmetric with respect to the interchange of the indices μ and ν, we shall often use the notation $A_{(\mu\nu)}\ldots$ to indicate this fact. If the field $A_{\mu\nu}\ldots$ is antisymmetric with respect to the interchange of the indices μ and ν, the notation $A_{[\mu\nu]}\ldots$ will be used. If the indices associated with any quantity are enclosed in parentheses, that quantity is understood to be symmetric with respect to the interchange of those indices.

Labels after a comma signify variables with respect to which differentiation is performed. Thus

$$\psi_{,\mu} = \frac{\partial\psi}{\partial x^\mu}\,. \tag{1.6}$$

The abbreviations

$$\Box^2\psi = \eta^{\mu\nu}\psi_{,\mu\nu}\,, \tag{1.7}$$

$$\nabla^2\psi = \delta_{st}\psi_{,st} \tag{1.8}$$

will be used in this paper. If $A_{\mu\nu}$ is a field possessing only two indices, the abbreviation

$$A_{[\mu\nu,\rho]} = A_{[\mu\nu],\rho} + A_{[\nu\rho],\mu} + A_{[\rho\mu],\nu} \tag{1.9}$$

will be used.

II. THE CONTINUUM

A. Structural Laws

In Einstein's relativistic field theory, nature is regarded as a four-dimensional space-time continuum whose structure is described through a fundamental tensor field $g_{\mu\nu}$. Simple structural laws are assumed to exist in this continuum. They can be expressed through the general covariant equation system

$$\Gamma^{\nu}_{[\mu\nu]}=0, \tag{2.1a}$$

$$R_{[\mu\nu,\rho]}=0, \tag{2.1b}$$

$$R_{(\mu\nu)}=0, \tag{2.1c}$$

where $\Gamma^{\rho}_{\mu\nu}$ and $R_{\mu\nu}$ are defined through the equations

$$g_{\mu(+)\nu(-);\rho}(=g_{\mu\nu,\rho}-g_{\sigma\nu}\Gamma^{\sigma}_{\mu\rho}-g_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu})=0, \tag{2.2}$$

$$R_{\mu\nu}=\Gamma^{\rho}_{\mu\nu,\rho}-\Gamma^{\rho}_{\mu\rho,\nu}-\Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu}+\Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma}. \tag{2.3}$$

Although the field $\Gamma^{\rho}_{\mu\nu}$ is not a tensor with respect to general (analytic) coordinate transformations, both $\Gamma^{\rho}_{[\mu\nu]}$ and $R_{\mu\nu}$ are tensors with respect to such transformations. The fields $\Gamma^{\rho}_{\mu\nu}$ and $R_{\mu\nu}$ are known, respectively, as the displacement field and the contracted curvature tensor.

B. Space and Time

The parameters x^1, x^2, and x^3 are chosen to be the space coordinates within the continuum. The parameter x^4 will be the timelike coordinate. Time t will be defined through the equation

$$x^4=ct, \tag{2.4}$$

where c is a constant having the dimensions of a velocity and a magnitude which depends on the units used to measure time. We shall later show that weak electromagnetic disturbances (light) propagate in empty space with the velocity c.

Occasionally we shall denote the timelike indices associated with a tensor by a zero. Thus

$$g_{44}=g_{00}. \tag{2.5}$$

Differentiation with respect to time will always be denoted by a zero. Thus

$$g_{\mu\nu,4}=\lambda g_{\mu\nu,0}, \tag{2.6}$$

where

$$\lambda=c^{-1}. \tag{2.7}$$

C. Particles and Physical Fields

A region of the continuum will be called flat if a coordinate system can be found so that

$$g_{\mu\nu}=\eta_{\mu\nu} \tag{2.8}$$

throughout the region. Particles are limited portions of the continuum – limited at least in the spatial directions – which have a very nonflat structure. Those portions of the continuum between the particles and possessing a nearly flat structure will be known as empty space or vacuum. The slight deviations from flatness in such portions of space-time will be taken to indicate the presence of an electromagnetic field if $g_{[\mu\nu]}\neq 0$, and of a gravitational field if $g_{(\mu\nu)}\neq\eta_{\mu\nu}$.

D. Isolated Regions

The structure of a region of the continuum can be determined through Eqs. (2.1) only if the boundary conditions over the three-dimensional surface of the region are known. In general one does not know these boundary conditions. There are certain cases, however, where the boundary conditions over a region of the continuum can be adequately specified without a detailed knowledge of the continuum external to the region. If a region is sufficiently limited in size and sufficiently distant from those particles which do not pass through it, then, to a good approximation, any deviation from flatness within the region can be considered to depend only on the structure of those particles which are within the region or which pass through it. A region of the continuum for which this approximation is adequate will be known as an isolated region. An isolated region containing no particles can be considered, to a good approximation, flat.

Harmonic coordinates. When investigating the structure of an isolated region of the continuum, we shall find it convenient to use a specific set of coordinate systems within the region. The coordinate systems we shall find convenient are called harmonic coordinate systems. A harmonic coordinate system is a coordinate system in which the coordinates x^μ satisfy the equations

$$\Box x^\mu\{=(-g)^{-1/2}[(-g)^{1/2}g^{(\rho\sigma)}x^\mu{}_{,\rho}]_{,\sigma}\}=0, \tag{2.9}$$

where $g^{\mu\nu}$ is defined through

$$g_{\mu\rho}g^{\nu\rho}=g_{\rho\mu}g^{\rho\nu}=\delta_\mu^{\ \nu}, \tag{2.10}$$

and g denotes the determinant of $g_{\mu\nu}$. We are assuming $g<0$ throughout the continuum.

Equations (2.9) are easily seen to be equivalent to

$$\mathfrak{g}^{(\mu\nu)}{}_{,\nu}=0, \tag{2.11}$$

where

$$\mathfrak{g}^{\mu\nu}=(-g)^{1/2}g^{\mu\nu}. \tag{2.12}$$

Under a transformation from one coordinate system (unprimed) to another coordinate system (primed), the field components $\mathfrak{g}^{(\mu\nu)}{}_{,\nu}$ satisfy the transformation law[14]

$$(-g')^{-1/2}\mathfrak{g}'^{(\mu\nu)}{}_{,\nu}=\frac{\partial x'^\mu}{\partial x^\rho}(-g)^{-1/2}\mathfrak{g}^{(\rho\nu)}{}_{,\nu}+\frac{\partial^2 x'^\mu}{\partial x^\rho\partial x^\sigma}(-g)^{-1/2}\mathfrak{g}^{(\rho\sigma)}. \tag{2.13}$$

Since, by definition, (2.11) is valid in every harmonic coordinate system, it follows from (2.13) that the transformation equations between two harmonic systems must satisfy the differential equations

$$\mathfrak{g}^{(\rho\sigma)}\frac{\partial^2 x'^{\mu}}{\partial x^\rho \partial x^\sigma}=0. \qquad (2.14)$$

Field equations. If we use harmonic coordinates when investigating an isolated region of the continuum, the field $g_{\mu\nu}$ will be subject to the coordinate conditions (2.11) in addition to the structural equations (2.1). This means, making use of the identity[15]

$$\mathfrak{g}^{(\mu\nu)}\Gamma^{\rho}_{[\nu\rho]}=\mathfrak{g}^{[\mu\nu]}{}_{,\nu}, \qquad (2.15)$$

that in a harmonic coordinate system the field $g_{\mu\nu}$ must satisfy the equations

$$\mathfrak{g}^{[\mu\nu]}{}_{,\nu}=0, \qquad (2.16a)$$

$$R_{[\mu\nu,\rho]}=0, \qquad (2.16b)$$

$$\mathfrak{g}^{(\mu\nu)}{}_{,\nu}=0, \qquad (2.16c)$$

$$R_{(\mu\nu)}=0. \qquad (2.16d)$$

It should be noted that the field equations (2.16) are not covariant under the group of all continuous (analytic) coordinate transformations but are covariant under all continuous (analytic) coordinate transformations which satisfy (2.14).

III. PERFECTLY ISOLATED REGIONS

A. Ideal Particles

In order to solve the field equations over an isolated region of the continuum, we shall make certain assumptions about the structure of the region with which we are concerned and about the boundary conditions over its surface. First we shall assume that the region is perfectly isolated. By this we shall mean that any deviation from flatness within the region can be considered to depend only on the structure of those particles which are within the region, or which pass through it. Secondly we shall assume that the particles within the region are stable so that each particle within the region is an individual thread of very nonflat structure running through the region. Finally we shall assume that the deviations from flatness which make up the threads of very nonflat structure running through the region become not only very large, but infinite, along a world line within the thread. These world lines are assumed never to cross. Particles which are within a perfectly isolated region of the continuum and which satisfy the abovementioned conditions will be known as ideal particles.

Both the assumption that the region is perfectly isolated and the assumption that the particles within the region are ideal must be considered as idealizations. Regions of the continuum are never perfectly isolated, and if Einstein's ideas are correct, only nonsingular solutions to the field equations are realized in nature. This means that we must assume, if our solutions are to be realistic, that there are acceptable nonsingular solutions to the field equations, describing an isolated region of the continuum containing stable particles, which can be approximated, except near the center of a particle, by singular solutions, solutions in which each particle in the region is represented by an ideal particle.

The use of ideal particles in a perfectly isolated region to represent real particles in an isolated region may eliminate the possibility of understanding quantum effects through Einstein's relativistic field theory, as the understanding of these effects may depend on an accurate knowledge of the structure of a particle near its center. In fact, if Einstein's theory could provide us with an understanding of quantum effects, it is quite possible that these effects would be found to be closely associated with the very requirement that no singularity appear in the continuum.

The use of ideal particles in a perfectly isolated region of the continuum to represent real particles in an isolated region of the continuum almost certainly eliminates the possibility of understanding the actual values of mass, charge, spin, etc., associated with the particles observed in nature.

IV. METHOD OF APPROXIMATION

A. Basic Assumptions

In our analysis we shall make the assumption that the region of the space-time continuum with which we are concerned is perfectly isolated and contains only ideal particles. We shall also assume that the field $g_{\mu\nu}$ within the region, excluding of course the world lines associated with the particles in the region, depends continuously on a positive parameter κ in such a way that as $\kappa\to 0$ the region becomes flat. Furthermore, we assume that there is a harmonic coordinate system in which the field $g_{\mu\nu}$, at points in the region sufficiently far from any world line, can be expanded in a power series in κ such that

$$g_{\mu\nu}=\sum_{k=0}^{\infty}\kappa^{k}\,{}_{(k)}g_{\mu\nu}, \qquad (4.1)$$

where

$${}_{(0)}g_{\mu\nu}=\eta_{\mu\nu}. \qquad (4.2)$$

We shall always work in such harmonic coordinate systems.

Although it will not usually be stated, whenever we use the expansion (4.1) it is to be understood that we are investigating the field $g_{\mu\nu}$ only at points which are sufficiently far from the world lines (singularities) associated with the particles in the region, so that (4.1) is valid. The order of magnitude of such "distances" will be discussed later.

Field equations. If we expand $\mathfrak{g}^{\mu\nu}$ and $R_{\mu\nu}$ in a power series in κ, we have, using (4.1) and (4.2) along with (2.2), (2.3), (2.10), and (2.12),

$$\mathfrak{g}^{\mu\nu}=\sum_{k=0}^{\infty}\kappa^{k}{}_{(k)}\mathfrak{g}^{\mu\nu}, \tag{4.3}$$

$$R_{\mu\nu}=\sum_{k=0}^{\infty}\kappa^{k}{}_{(k)}R_{\mu\nu}; \tag{4.4}$$

where

$$\begin{aligned}&{}_{(0)}\mathfrak{g}^{\mu\nu}=\eta^{\mu\nu},\\ &{}_{(k)}\mathfrak{g}^{\mu\nu}=-\eta^{\mu\rho}\eta^{\nu\sigma}{}_{(k)}g_{\sigma\rho}+\tfrac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}{}_{(k)}g_{\rho\sigma}+{}_{(k)}\mathfrak{g}^{N\mu\nu}\quad(k>0),\end{aligned} \tag{4.5}$$

$$\begin{aligned}&{}_{(0)}R_{\mu\nu}=0,\\ &{}_{(k)}R_{\mu\nu}=\tfrac{1}{2}\eta^{\rho\sigma}({}_{(k)}g_{\rho\mu,\nu\sigma}+{}_{(k)}g_{\rho\nu,\mu\sigma}\\ &\qquad -{}_{(k)}g_{\nu\mu,\rho\sigma}-{}_{(k)}g_{\rho\sigma,\mu\nu})+{}_{(k)}R^{N}_{\mu\nu}\quad(k>0).\end{aligned} \tag{4.6}$$

The fields ${}_{(k)}\mathfrak{g}^{N\mu\nu}$ and ${}_{(k)}R^{N}_{\mu\nu}$ contain only terms nonlinear in ${}_{(k')}g_{\mu\nu}$ $(0<k'<k)$. The subscripts to the left of a field indicate the order.

Using (4.3) and (4.4), we find that the field equations (2.16) take the form

$${}_{(k)}\mathfrak{g}^{[\mu\nu]}{}_{,\nu}=0, \tag{4.7a}$$

$${}_{(k)}R_{[\mu\nu,\rho]}=0, \tag{4.7b}$$

$${}_{(k)}\mathfrak{g}^{(\mu\nu)}{}_{,\nu}=0, \tag{4.7c}$$

$${}_{(k)}R_{(\mu\nu)}=0. \tag{4.7d}$$

We have succeeded in breaking the field equations up into equations for each order.

B. The Field $\gamma_{\mu\nu}$

Since the field $\mathfrak{g}^{\mu\nu}$ transforms as a contravariant tensor under Lorentz transformations, the quantities $\mathfrak{g}^{\mu\nu}-\eta^{\mu\nu}$ can be regarded as the contravariant components of a second-rank tensor field with respect to such transformations. This field will be denoted by $\gamma^{\mu\nu}$. Expanding the field $\gamma^{\mu\nu}$ in a power series in κ, we have

$$\gamma^{\mu\nu}=\sum_{k=0}^{\infty}\kappa^{k}{}_{(k)}\gamma^{\mu\nu}, \tag{4.8}$$

where

$${}_{(0)}\gamma^{\mu\nu}=0,\quad {}_{(k)}\gamma^{\mu\nu}={}_{(k)}\mathfrak{g}^{\mu\nu}\quad(k>0). \tag{4.9}$$

A field $\gamma_{\mu\nu}$ can be defined as follows:

$$\gamma_{\mu\nu}=\eta_{\mu\rho}\eta_{\nu\sigma}\gamma^{\rho\sigma}. \tag{4.10}$$

This field transforms as a covariant tensor under Lorentz transformations. From the definition of $\gamma_{\mu\nu}$ and using (4.3) and (4.5) we see that

$$\gamma_{\mu\nu}=\sum_{k=0}^{\infty}\kappa^{k}{}_{(k)}\gamma_{\mu\nu}, \tag{4.11}$$

where

$$\begin{aligned}&{}_{(0)}\gamma_{\mu\nu}=0,\\ &{}_{(k)}\gamma_{\mu\nu}=-{}_{(k)}g_{\nu\mu}+\tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}{}_{(k)}g_{\rho\sigma}+{}_{(k)}\gamma^{N}_{\mu\nu}\quad(k>0),\\ &{}_{(k)}\gamma^{N}_{\mu\nu}=\eta_{\mu\rho}\eta_{\nu\sigma}{}_{(k)}\mathfrak{g}^{N\rho\sigma}\quad(k>0).\end{aligned} \tag{4.12}$$

The fields ${}_{(k)}\gamma^{N}_{\mu\nu}$ contain only terms nonlinear in ${}_{(k')}g_{\mu\nu}$ $(0<k'<k)$. Using (4.11) and (4.12) we find that

$$g_{\mu\nu}=\sum_{k=0}^{\infty}\kappa^{k}{}_{(k)}g_{\mu\nu}, \tag{4.13}$$

where

$$\begin{aligned}&{}_{(0)}g_{\mu\nu}=\eta_{\mu\nu},\\ &{}_{(k)}g_{\mu\nu}=-{}_{(k)}\gamma_{\nu\mu}+\tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}{}_{(k)}\gamma_{\rho\sigma}+{}_{(k)}g^{N'}_{\mu\nu}\quad(k>0),\\ &{}_{(k)}g^{N'}_{\mu\nu}=\eta_{\mu\rho}\eta_{\nu\sigma}{}_{(k)}\mathfrak{g}^{N\sigma\rho}-\tfrac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}{}_{(k)}\mathfrak{g}^{N\rho\sigma}\quad(k>0).\end{aligned} \tag{4.14}$$

The fields ${}_{(k)}g^{N'}_{\mu\nu}$ contain only terms nonlinear in ${}_{(k')}\gamma_{\mu\nu}$ $(0<k'<k)$. Their explicit form can be found as needed through the use of an iteration process involving (4.14).

Field equations. The form of Eqs. (2.16) and (4.7) suggests that it might be easier to solve them in terms of the field $\gamma_{\mu\nu}$ than in terms of the field $g_{\mu\nu}$. After solving the equations for $\gamma_{\mu\nu}$, one can then use (4.13) and (4.14) to determine $g_{\mu\nu}$ from $\gamma_{\mu\nu}$.

In terms of $\gamma_{\mu\nu}$, the field equations (4.7) take the form

$${}_{(k)}\gamma_{[\mu\nu]}{}^{,\nu}=0, \tag{4.15a}$$

$${}_{(k)}R_{[\mu\nu,\rho]}=0, \tag{4.15b}$$

$${}_{(k)}\gamma_{(\mu\nu)}{}^{,\nu}=0, \tag{4.15c}$$

$${}_{(k)}R_{(\mu\nu)}=0, \tag{4.15d}$$

where

$$\begin{aligned}&{}_{(0)}R_{\mu\nu}=0,\\ &{}_{(k)}R_{\mu\nu}=\tfrac{1}{2}\Box^{2}({}_{(k)}\gamma_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}{}_{(k)}\gamma_{\rho\sigma})\\ &\qquad -\tfrac{1}{2}{}_{(k)}\gamma_{\mu\rho}{}^{,\rho}{}_{,\nu}-\tfrac{1}{2}{}_{(k)}\gamma_{\nu\rho}{}^{,\rho}{}_{,\mu}+{}_{(k)}R^{N'}_{\mu\nu}\quad(k>0).\end{aligned} \tag{4.16}$$

The fields ${}_{(k)}R^{N'}_{\mu\nu}$ contain those terms of ${}_{(k)}R_{\mu\nu}$ which are nonlinear in ${}_{(k')}\gamma_{\mu\nu}$ $(0<k'<k)$.

If we introduce ${}_{(k)}K_{\mu\nu}$, where

$${}_{(k)}K_{\mu\nu}=-2({}_{(k)}R_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}{}_{(k)}R_{\rho\sigma}), \tag{4.17}$$

we find that Eqs. (4.15) are equivalent to the set of equations

$${}_{(k)}\gamma_{[\mu\nu]}{}^{,\nu}=0, \tag{4.18a}$$

$${}_{(k)}K_{[\mu\nu,\rho]}=0, \tag{4.18b}$$

$${}_{(k)}\gamma_{(\mu\nu)}{}^{,\nu}=0, \tag{4.18c}$$

$${}_{(k)}K_{(\mu\nu)}=0, \tag{4.18d}$$

where

$$\begin{aligned}&{}_{(k)}K_{\mu\nu}=-\Box^{2}{}_{(k)}\gamma_{\mu\nu}+{}_{(k)}\gamma_{\mu\rho}{}^{,\rho}{}_{,\nu}+{}_{(k)}\gamma_{\nu\rho}{}^{,\rho}{}_{,\mu}\\ &\qquad -\eta_{\mu\nu}{}_{(k)}\gamma_{\rho\sigma}{}^{,\rho\sigma}+{}_{(k)}K^{N'}_{\mu\nu},\\ &{}_{(k)}K^{N'}_{\mu\nu}=-2({}_{(k)}R^{N'}_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}{}_{(k)}R^{N'}_{\rho\sigma}).\end{aligned} \tag{4.19}$$

If we then write out explicitly the terms which are linear in ${}_{(k)}\gamma_{\mu\nu}$, Eqs. (4.18) take the form

$$_{(k)}\gamma_{[\mu\nu]}{}^{,\nu}=0, \tag{4.20a}$$

$$\Box^2{}_{(k)}\gamma_{[\mu\nu,\rho]}={}_{(k)}K^{N'}_{[\mu\nu,\rho]}, \tag{4.20b}$$

$$_{(k)}\gamma_{(\mu\nu)}{}^{,\nu}=0, \tag{4.20c}$$

$$\Box^2{}_{(k)}\gamma_{(\mu\nu)}={}_{(k)}K^{N'}_{(\mu\nu)}. \tag{4.20d}$$

If we introduce the fields $\gamma^{*[\mu\nu]}$ and $\gamma^*_{[\mu\nu]}$, where

$$\gamma^{*[\mu\nu]}=\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\gamma_{[\rho\sigma]},\quad \gamma^*_{[\mu\nu]}=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\gamma^{[\rho\sigma]}, \tag{4.21}$$

and make use of the power-series expansions

$$\gamma^{*[\mu\nu]}=\sum_{k=0}^{\infty}\kappa^k{}_{(k)}\gamma^{*[\mu\nu]},\quad \gamma^*_{[\mu\nu]}=\sum_{k=0}^{\infty}\kappa^k{}_{(k)}\gamma^*_{[\mu\nu]}, \tag{4.22}$$

we find that the field equations for $\gamma_{\mu\nu}$ can be put in the simple form

$$\Box^2{}_{(k)}j_\mu={}_{(k)}s_\mu, \tag{4.23a}$$

$$_{(k)}j_\mu{}^{,\mu}=0, \tag{4.23b}$$

$$_{(k)}\gamma^*_{[\mu\nu]}{}^{,\nu}={}_{(k)}j_\mu, \tag{4.23c}$$

$$_{(k)}\gamma^*_{[\mu\nu,\rho]}=0, \tag{4.23d}$$

$$\Box^2{}_{(k)}\gamma_{(\mu\nu)}={}_{(k)}t_{\mu\nu}, \tag{4.23e}$$

$$_{(k)}\gamma_{(\mu\nu)}{}^{,\nu}=0, \tag{4.23f}$$

where

$$_{(k)}s_\mu=\tfrac{1}{6}\eta_{\mu\rho}\epsilon^{\rho\sigma\kappa\lambda}{}_{(k)}K^{N'}_{[\kappa\lambda,\sigma]},\quad {}_{(k)}t_{\mu\nu}={}_{(k)}K^{N'}_{(\mu\nu)}. \tag{4.24}$$

Making use of the definitions

$$s_\mu=\sum_{k=0}^{\infty}\kappa^k{}_{(k)}s_\mu,\quad j_\mu=\sum_{k=0}^{\infty}\kappa^k{}_{(k)}j_\mu,\quad t_{\mu\nu}=\sum_{k=0}^{\infty}\kappa^k{}_{(k)}t_{\mu\nu}, \tag{4.25}$$

we can write Eqs. (4.23) in the more compact form

$$\Box^2 j_\mu=s_\mu, \tag{4.26a}$$

$$j_\mu{}^{,\mu}=0, \tag{4.26b}$$

$$\gamma^*_{[\mu\nu]}{}^{,\nu}=j_\mu, \tag{4.26c}$$

$$\gamma^*_{[\mu\nu,\rho]}=0, \tag{4.26d}$$

$$\Box^2\gamma_{(\mu\nu)}=t_{\mu\nu}, \tag{4.26e}$$

$$\gamma_{(\mu\nu)}{}^{,\nu}=0. \tag{4.26f}$$

We have from Eqs. (4.21)

$$\gamma_{[\mu\nu]}=-\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\gamma^{*[\rho\sigma]},\quad \gamma^{[\mu\nu]}=-\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\gamma^*_{[\rho\sigma]}. \tag{4.27}$$

The form of Eqs. (4.26) suggests that we identify $\gamma^*_{[\mu\nu]}$,

$$\gamma^*_{[\mu\nu]}=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{G}^{[\rho\sigma]}, \tag{4.28}$$

with the electromagnetic field and $\gamma_{(\mu\nu)}$,

$$\gamma_{(\mu\nu)}=\eta_{\mu\rho}\eta_{\nu\sigma}\mathcal{G}^{(\rho\sigma)}-\eta_{\mu\nu}, \tag{4.29}$$

with the gravitational field. We shall later see that this is a reasonable identification.

If we make the above identification, the electric current density is given by j_μ. From (4.26b) we see that electric charge is conserved.

From (4.26d) we see that there can be no magnetic current density except at points where the electromagnetic field is singular. In fact, it follows rigorously from (2.16a) that there can be no nonsingular solution to Einstein's field equations in which a magnetic current exists. This means that only those ideal particles which do not give rise to a magnetic current can be considered to approximate the real particles of nature.

Integrability conditions. Equations (4.26a)–(4.26d) can be solved at each order only if the conditions

$$_{(k)}s_\mu{}^{,\mu}=0 \tag{4.30}$$

are satisfied. These conditions are always satisfied. This follows from the definition of ${}_{(k)}s_\mu$ given in (4.24).

Equations (4.26e) and (4.26f) can be solved at each order only if the conditions

$$_{(k)}t_{\mu\nu}{}^{,\nu}=0 \tag{4.31}$$

are satisfied. We shall show that these conditions are always satisfied if the field equations (4.26) have been satisfied in all lower orders.

Einstein has shown that if the fields $\Gamma^\rho_{\mu\nu}$ and $R_{\mu\nu}$, defined through (2.2) and (2.3), are subject to (2.1a) and (2.1b), then one has the identity[16]

$$g^{\rho\sigma}(R_{(\rho(+)\sigma(-));\mu}-R_{(\rho(+)\mu(-));\sigma}-R_{(\mu(-)\sigma(-));\rho})=0, \tag{4.32}$$

where

$$R_{(\mu(+)\nu(-));\rho}=R_{(\mu\nu),\rho}-R_{(\sigma\nu)}\Gamma^\sigma_{\mu\rho}-R_{(\mu\sigma)}\Gamma^\sigma_{\rho\nu}, \tag{4.33}$$

$$R_{(\mu(+)\nu(+));\rho}=R_{(\mu\nu),\rho}-R_{(\sigma\nu)}\Gamma^\sigma_{\mu\rho}-R_{(\mu\sigma)}\Gamma^\sigma_{\nu\rho}, \tag{4.34}$$

$$R_{(\mu(-)\nu(-));\rho}=R_{(\mu\nu),\rho}-R_{(\sigma\nu)}\Gamma^\sigma_{\rho\mu}-R_{(\mu\sigma)}\Gamma^\sigma_{\rho\nu}. \tag{4.35}$$

From this identity it follows that if we have solved Eqs. (4.26a)–(4.26d) up to the kth order, and Eqs. (4.26e) and (4.26f) up to the $(k-1)$th order, then

$$\eta^{\rho\sigma}({}_{(k)}R_{(\rho\sigma),\mu}-2\,{}_{(k)}R_{(\mu\rho),\sigma})=0, \tag{4.36}$$

and from the definition of ${}_{(k)}K_{\mu\nu}$ in (4.17),

$$_{(k)}K_{(\mu\nu)}{}^{,\nu}=0. \tag{4.37}$$

From (4.19) we see that

$$_{(k)}K_{(\mu\nu)}{}^{,\nu}={}_{(k)}K^{N'}_{(\mu\nu)}{}^{,\nu}, \tag{4.38}$$

so that (4.37) implies

$$_{(k)}K^{N'}_{(\mu\nu)}{}^{,\nu}=0. \tag{4.39}$$

Since ${}_{(k)}K^{N'}_{(\mu\nu)}$ contains only terms nonlinear in ${}_{(k')}\gamma_{\mu\nu}$

$(k'<k)$, and thus does not depend on ${}_{(k)}\gamma_{\mu\nu}$, Eqs. (4.39) will be satisfied as long as the field equations (4.26) have been satisfied up to the $(k-1)$th order. From (4.24) and (4.39) it then follows that

$$ {}_{(k)}t_{\mu\nu}{}^{,\nu}=0, \tag{4.40}$$

if the field equations (4.26) have been satisfied in all lower orders.

A procedure for finding $\gamma_{\mu\nu}$. We shall now show how Eqs. (4.26) can be used to find the field $\gamma_{\mu\nu}$ to any order of approximation desired. Let us assume that we have found the field $\gamma_{\mu\nu}$ to the $(k-1)$th order of approximation so that the fields ${}_{(k')}\gamma_{\mu\nu}$ $(k'<k)$ are all known. From Eqs. (4.26a) and (4.26b) we can find ${}_{(k)}j_\mu$. This is possible because ${}_{(k)}s_\mu$ is nonlinear in ${}_{(k')}\gamma_{\mu\nu}$ and thus depends only on the known ${}_{(k')}\gamma_{\mu\nu}$. Having found ${}_{(k)}j_\mu$ we can use (4.26c) and (4.26d) to find ${}_{(k)}\gamma^*_{[\mu\nu]}$, and the use of (4.27) will give ${}_{(k)}\gamma_{[\mu\nu]}$. The field ${}_{(k)}\gamma_{(\mu\nu)}$ can be found from Eqs. (4.26e) and (4.26f). This is possible because ${}_{(k)}t_{\mu\nu}$ is nonlinear in ${}_{(k')}\gamma_{\mu\nu}$ and therefore depends only on the known ${}_{(k')}\gamma_{\mu\nu}$. Thus we have ${}_{(k)}\gamma_{\mu\nu}$. By proceeding step by step, assuming such a procedure converges, we can determine the field $\gamma_{\mu\nu}$ to any order of approximation we desire.

V. METHOD OF SOLUTION

A. Electric and Magnetic Potentials

We shall split the field $\gamma^*_{[\mu\nu]}$ into what we shall call an electric part $\gamma^{*E}_{[\mu\nu]}$ and a magnetic part $\gamma^{*M}_{[\mu\nu]}$. This will be done in such a way that both the electric part and the magnetic part can be derived from a potential. We have

$$\gamma^*_{[\mu\nu]}=\gamma^{*E}_{[\mu\nu]}+\gamma^{*M}_{[\mu\nu]}, \tag{5.1}$$

where

$$\gamma^{*E}_{[\mu\nu]}=\gamma^E_{\mu,\nu}-\gamma^E_{\nu,\mu},\quad \gamma^{*M}_{[\mu\nu]}=\epsilon_{\mu\nu\rho\sigma}\gamma^{M\sigma,\rho}. \tag{5.2}$$

The field γ^E_μ will be called the electric potential and the field γ^M_μ will be called the magnetic potential. In terms of these potentials the field equations (4.26) take the form

$$\Box^2 j_\mu=s_\mu, \tag{5.3a}$$

$$j_\mu{}^{,\mu}=0, \tag{5.3b}$$

$$\Box^2\gamma^E_\mu=j_\mu, \tag{5.3c}$$

$$\gamma^{E,\mu\nu}_\mu=0, \tag{5.3d}$$

$$\Box^2\gamma^M_\mu=0, \tag{5.3e}$$

$$\gamma^{M,\mu\nu}_\mu=0, \tag{5.3f}$$

$$\Box^2\gamma_{(\mu\nu)}=t_{\mu\nu}, \tag{5.3g}$$

$$\gamma_{(\mu\nu)},^{\nu}=0. \tag{5.3h}$$

The field $\gamma_{[\mu\nu]}$ can also be regarded as consisting of an electric and magnetic part. From (4.27), (5.1), and (5.2) we have

$$\gamma_{[\mu\nu]}=\gamma^E_{[\mu\nu]}+\gamma^M_{[\mu\nu]}, \tag{5.4}$$

where

$$\gamma^E_{[\mu\nu]}=\epsilon_{\mu\nu\rho\sigma}\gamma^{E\sigma,\rho},\quad \gamma^M_{[\mu\nu]}=\gamma^M_{\nu,\mu}-\gamma^M_{\mu,\nu}. \tag{5.5}$$

B. Homogeneous and Inhomogeneous Equations

We shall find it convenient to divide the field equations (5.3) into two sets of equations, a set of homogeneous equations

$$\Box^2 j^H_\mu=0, \tag{5.6a}$$

$$\Box^2\gamma^{EH}_\mu=0, \tag{5.6b}$$

$$\Box^2\gamma^{MH}_\mu=0, \tag{5.6c}$$

$$\Box^2\gamma^H_{(\mu\nu)}=0, \tag{5.6d}$$

and a set of inhomogeneous equations

$$\Box^2 j^I_\mu=s_\mu, \tag{5.7a}$$

$$j^{I,\mu}_\mu=-j^{H,\mu}_\mu, \tag{5.7b}$$

$$\Box^2\gamma^{EI}_\mu=j_\mu, \tag{5.7c}$$

$$\gamma^{EI,\mu\nu}_\mu=-\gamma^{EH,\mu\nu}_\mu, \tag{5.7d}$$

$$\gamma^{MI}_\mu=0, \tag{5.7e}$$

$$\gamma^{MI,\mu\nu}_\mu=-\gamma^{MH,\mu\nu}_\mu, \tag{5.7f}$$

$$\Box^2\gamma^I_{(\mu\nu)}=t_{\mu\nu}, \tag{5.7g}$$

$$\gamma^I_{(\mu\nu)}{}^{,\nu}=-\gamma^H_{(\mu\nu)}{}^{,\nu}, \tag{5.7h}$$

where

$$j_\mu=j^H_\mu+j^I_\mu,$$
$$\gamma^E_\mu=\gamma^{EH}_\mu+\gamma^{EI}_\mu,\quad \gamma^M_\mu=\gamma^{MH}_\mu+\gamma^{MI}_\mu, \tag{5.8}$$
$$\gamma_{(\mu\nu)}=\gamma^H_{(\mu\nu)}+\gamma^I_{(\mu\nu)}.$$

Taken together, these two sets of equations are entirely equivalent to (5.3).

The character of the entire solution to the field equations (5.6)–(5.8) can be considered to depend on the choice of functions we take as solutions to the set of homogeneous equations. We shall therefore regard the solutions to the set of homogeneous equations as determining the basic structure of the particles within the region under consideration, while the solutions to the set of inhomogeneous equations give corrections to this basic structure. It is to be understood that no new quantities characterizing the structure of a particle will be introduced when solving the inhomogeneous equations.

The field $\gamma_{\mu\nu}$ will be regarded as consisting of two parts, a field $\gamma^H_{\mu\nu}$ and a field $\gamma^I_{\mu\nu}$. We have

$$\gamma_{\mu\nu}=\gamma^H_{\mu\nu}+\gamma^I_{\mu\nu}, \tag{5.9}$$

where

$$\gamma^H_{\mu\nu} = \gamma^H_{(\mu\nu)} + \gamma^H_{[\mu\nu]}\ ,\quad \gamma^I_{\mu\nu} = \gamma^I_{(\mu\nu)} + \gamma^I_{[\mu\nu]} \tag{5.10}$$

and

$$\gamma^H_{[\mu\nu]} = \gamma^{EH}_{[\mu\nu]} + \gamma^{MH}_{[\mu\nu]}\ ,\quad \gamma^I_{[\mu\nu]} = \gamma^{EI}_{[\mu\nu]}\ , \tag{5.11}$$

$$\gamma^{EH}_{[\mu\nu]} = \epsilon_{\mu\nu\rho\sigma}\gamma^{EH\sigma,\rho}\ ,\qquad \gamma^{MH}_{[\mu\nu]} = \gamma^{MH}_{\nu,\mu} - \gamma^{MH}_{\mu,\nu}\ ,\quad \gamma^{EI}_{[\mu\nu]} = \epsilon_{\mu\nu\rho\sigma}\gamma^{EI\sigma,\rho}\ . \tag{5.12}$$

Solutions to the homogeneous equations. Before we discuss any solutions to the homogeneous equations we shall introduce some notation. The number of particles within a region of the continuum with which we are concerned will be denoted by N. The coordinates of the singularity associated with the pth particle in the region will be denoted by ${}^{(p)}\xi^\mu$. The points ${}^{(p)}\xi^\mu$ will be considered as forming the world line of the pth particle and will be parametrized by a quantity ${}^{(p)}\tau$ defined through the equation

$$d\,{}^{(p)}\tau^2 = \eta_{\mu\nu}\, d\,{}^{(p)}\xi^\mu d\,{}^{(p)}\xi^\nu\ . \tag{5.13}$$

Finally, we introduce the notation

$${}^{(p)}r^\mu = x^\mu - {}^{(p)}\xi^\mu\ ,\quad {}^{(p)}u^\mu = {}^{(p)}\dot{\xi}^\mu = d\,{}^{(p)}\xi^\mu / d\,{}^{(p)}\tau\ , \tag{5.14}$$

$${}^{(p)}(r^2) = {}^{(p)}r_\rho\, {}^{(p)}r^\rho\ ,\quad {}^{(p)}(ru) = {}^{(p)}r_\rho\, {}^{(p)}u^\rho\ . \tag{5.15}$$

A superscript (p) to the left of an expression means that those quantities in the expression which are associated with a particle are to be associated with the pth particle. A dot over a quantity associated with the pth particle means the derivative of that quantity with respect to ${}^{(p)}\tau$.

As a solution to Eqs. (5.6) we can choose

$$j^H_\mu = \sum_{p=1}^{N} {}^{(p)}j^H_\mu\ , \tag{5.16}$$

$$\gamma^{EH}_\mu = \sum_{p=1}^{N} {}^{(p)}\gamma^{EH}_\mu\ ,\quad \gamma^{MH}_\mu = \sum_{p=1}^{N} {}^{(p)}\gamma^{MH}_\mu\ , \tag{5.17}$$

$$\gamma^H_{(\mu\nu)} = \sum_{p=1}^{N} {}^{(p)}\gamma^H_{(\mu\nu)}\ , \tag{5.18}$$

where

$${}^{(p)}j^H_\mu = {}^{(p)}j^H_{\mu\ \mathrm{ret}} + {}^{(p)}j^H_{\mu\ \mathrm{adv}}, \tag{5.19}$$

$${}^{(p)}\gamma^{EH}_\mu = {}^{(p)}\gamma^{EH}_{\mu\ \mathrm{ret}} + {}^{(p)}\gamma^{EH}_{\mu\ \mathrm{adv}},\qquad {}^{(p)}\gamma^{MH}_\mu = {}^{(p)}\gamma^{MH}_{\mu\ \mathrm{ret}} + {}^{(p)}\gamma^{MH}_{\mu\ \mathrm{adv}}, \tag{5.20}$$

$${}^{(p)}\gamma^H_{(\mu\nu)} = {}^{(p)}\gamma^H_{(\mu\nu)\,\mathrm{ret}} + {}^{(p)}\gamma^H_{(\mu\nu)\,\mathrm{adv}}, \tag{5.21}$$

and

$${}^{(p)}j^H_{\mu\ \mathrm{ret(adv)}} = \frac{1}{c^2}\,{}^{(p)}a^D_{\mathrm{ret(adv)}}\,{}^{(p)}[(e^D/l^2)u_\mu(ru)^{-1}]_{\mathrm{ret(adv)}} + \frac{1}{c^2}\sum_{i=1}^{\infty}{}^{(p)}a^{D_i}_{\mathrm{ret(adv)}}\,{}^{(p)}[(e^D_{[\mu\sigma_1]\cdots\sigma_i}/l^2)(ru)^{-1}]_{\mathrm{ret(adv)}}{}^{,\sigma_1\cdots\sigma_i}, \tag{5.22}$$

$${}^{(p)}\gamma^{EH}_{\mu\ \mathrm{ret(adv)}} = \frac{1}{c^2}\,{}^{(p)}a^E_{\mathrm{ret(adv)}}\,{}^{(p)}[e^E u_\mu(ru)^{-1}]_{\mathrm{ret(adv)}} + \frac{1}{c^2}\sum_{i=1}^{\infty}{}^{(p)}a^{E_i}_{\mathrm{ret(adv)}}\,{}^{(p)}[e^E_{[\mu\sigma_1]\cdots\sigma_i}(ru)^{-1}]_{\mathrm{ret(adv)}}{}^{,\sigma_1\cdots\sigma_i}, \tag{5.23}$$

$${}^{(p)}\gamma^{MH}_{\mu\ \mathrm{ret(adv)}} = \frac{1}{c^3}\,{}^{(p)}a^M_{\mathrm{ret(adv)}}\,{}^{(p)}[e^M u_\mu(ru)^{-1}]_{\mathrm{ret(adv)}} + \frac{1}{c^3}\sum_{i=1}^{\infty}{}^{(p)}a^{M_i}_{\mathrm{ret(adv)}}\,{}^{(p)}[e^M_{[\mu\sigma_1]\cdots\sigma_i}(ru)^{-1}]_{\mathrm{ret(adv)}}{}^{,\sigma_1\cdots\sigma_i}, \tag{5.24}$$

$$\begin{aligned}{}^{(p)}\gamma^H_{(\mu\nu)\,\mathrm{ret(adv)}} = {} & \frac{1}{c^2}\,{}^{(p)}a^G_{\mathrm{ret(adv)}}\,{}^{(p)}[(m^G u_\mu u_\nu + \tfrac{1}{2}\dot{S}_{\mu\rho}u^\rho u_\nu + \tfrac{1}{2}\dot{S}_{\nu\rho}u^\rho u_\mu)(ru)^{-1}]_{\mathrm{ret(adv)}} \\ & + \frac{1}{c^2}\,{}^{(p)}a^G_{\mathrm{ret(adv)}}\,{}^{(p)}[(\tfrac{1}{2}S_{\mu\rho}u_\nu + \tfrac{1}{2}S_{\nu\rho}u_\mu)(ru)^{-1}]_{\mathrm{ret(adv)}}{}^{,\rho} \\ & + \frac{1}{c^2}\sum_{i=2}^{\infty}{}^{(p)}a^{G_i}_{\mathrm{ret(adv)}}\,{}^{(p)}[m^G_{[\mu\sigma_1][\nu\sigma_2]\cdots\sigma_i}(ru)^{-1}]_{\mathrm{ret(adv)}}{}^{,\sigma_1\sigma_2\cdots\sigma_i}\ ,\end{aligned} \tag{5.25}$$

with

$${}^{(p)}S_{\mu\nu} = -{}^{(p)}S_{\nu\mu}\ ,\qquad {}^{(p)}m^G_{[\mu\sigma_1][\nu\sigma_2]\cdots\sigma_i} = {}^{(p)}m^G_{[\nu\sigma_2][\mu\sigma_1]\cdots\sigma_i}\ . \tag{5.26}$$

The quantities

$${}^{(p)}e^D,\ {}^{(p)}e^D_{[\mu\sigma_1]\cdots\sigma_i},\ {}^{(p)}e^E,\ {}^{(p)}e^E_{[\mu\sigma_1]\cdots\sigma_i},\ {}^{(p)}e^M,\ {}^{(p)}e^M_{[\mu\sigma_1]\cdots\sigma_i},\ {}^{(p)}m^G,\ {}^{(p)}S_{\mu\nu},\ {}^{(p)}m^G_{[\mu\sigma_1][\nu\sigma_2]\cdots\sigma_i} \tag{5.27}$$

in (5.22)–(5.26) describe the structure of the pth particle and may be functions of ${}^{(p)}\tau$. The quantities ${}^{(p)}a_{\mathrm{ret}}$ and ${}^{(p)}a_{\mathrm{adv}}$, which further describe this structure, are constants. They satisfy the relations

$${}^{(p)}a^D_{\mathrm{ret}} + {}^{(p)}a^D_{\mathrm{adv}} = 1,\quad {}^{(p)}a^E_{\mathrm{ret}} + {}^{(p)}a^E_{\mathrm{adv}} = 1,\quad {}^{(p)}a^M_{\mathrm{ret}} + {}^{(p)}a^M_{\mathrm{adv}} = 1,\quad {}^{(p)}a^G_{\mathrm{ret}} + {}^{(p)}a^G_{\mathrm{adv}} = 1, \tag{5.28}$$

$${}^{(p)}a^{D_i}_{\mathrm{ret}} + {}^{(p)}a^{D_i}_{\mathrm{adv}} = 1,\quad {}^{(p)}a^{E_i}_{\mathrm{ret}} + {}^{(p)}a^{E_i}_{\mathrm{adv}} = 1,\quad {}^{(p)}a^{M_i}_{\mathrm{ret}} + {}^{(p)}a^{M_i}_{\mathrm{adv}} = 1,\quad {}^{(p)}a^{G_i}_{\mathrm{ret}} + {}^{(p)}a^{G_i}_{\mathrm{adv}} = 1. \tag{5.29}$$

The quantity l has been introduced into (5.22) for dimensional reasons and will be treated as a universal constant. The subscript ret indicates that in the expression in brackets, quantities associated with the pth particle are to be evaluated at the "retarded point"

$$^{(p)}(r^2)=0,\quad {}^{(p)}r^4>0, \tag{5.30}$$

while the subscript adv indicates that the expression in brackets is to be multiplied by −1 and then those quantities associated with the pth particle are to be evaluated at the "advanced point"

$$^{(p)}(r^2)=0,\quad {}^{(p)}r^4<0. \tag{5.31}$$

The quantities ${}^{(p)}e^D$, ${}^{(p)}e^E$, ${}^{(p)}e^M$, ${}^{(p)}m^G$, and ${}^{(p)}S_{\mu\nu}$ will be known, respectively, as the coefficient of diffuse electric charge, the localized electric charge, the localized magnetic charge, the mass, and the spin of the pth particle. A particle will be considered neutral if ${}^{(p)}e^D={}^{(p)}e^E={}^{(p)}e^M=0$. The quantity l will be known as the electromagnetic dispersion length.

If we wish a solution of the field equations (4.26) to approximate a nonsingular solution to the field equations (2.16), we must choose

$$^{(p)}e^M=0,\quad {}^{(p)}e^M_{[\mu\sigma_1]\cdots\sigma_i}=0. \tag{5.32}$$

The reason for this has been discussed in Sec. IV.

From (5.16)–(5.26) we find that

$$j_\mu^{H,\mu}=\sum_{p=1}^N {}^{(p)}a^D_{\rm ret}\,{}^{(p)}[C^{DH}(ru)^{-1}]_{\rm ret}+\sum_{p=1}^N {}^{(p)}a^D_{\rm adv}\,{}^{(p)}[C^{DH}(ru)^{-1}]_{\rm adv}, \tag{5.33}$$

$$\gamma_\mu^{EH,\mu\nu}=\sum_{p=1}^N {}^{(p)}a^E_{\rm ret}\,{}^{(p)}[C^{EH}(ru)^{-1}]_{\rm ret}{}^{,\nu}+\sum_{p=1}^N {}^{(p)}a^E_{\rm adv}\,{}^{(p)}[C^{EH}(ru)^{-1}]_{\rm adv}{}^{,\nu}, \tag{5.34}$$

$$\gamma_\mu^{MH,\mu\nu}=\sum_{p=1}^N {}^{(p)}a^M_{\rm ret}\,{}^{(p)}[C^{MH}(ru)^{-1}]_{\rm ret}{}^{,\nu}+\sum_{p=1}^N {}^{(p)}a^M_{\rm adv}\,{}^{(p)}[C^{MH}(ru)^{-1}]_{\rm adv}{}^{,\nu}, \tag{5.35}$$

$$\gamma^H_{(\mu\nu)}{}^{,\nu}=\sum_{p=1}^N {}^{(p)}a^G_{\rm ret}\,{}^{(p)}[C^H_\mu(ru)^{-1}]_{\rm ret}+\sum_{p=1}^N {}^{(p)}a^G_{\rm adv}\,{}^{(p)}[C^H_\mu(ru)^{-1}]_{\rm adv} +\sum_{p=1}^N {}^{(p)}a^G_{\rm ret}\,{}^{(p)}[C^H_{[\mu\nu]}(ru)^{-1}]_{\rm ret}{}^{,\nu}+\sum_{p=1}^N {}^{(p)}a^G_{\rm adv}\,{}^{(p)}[C^H_{[\mu\nu]}(ru)^{-1}]_{\rm adv}{}^{,\nu}, \tag{5.36}$$

where

$$^{(p)}C^{DH}={}^{(p)}\dot e^D/c^2l^2, \tag{5.37}$$

$$^{(p)}C^{EH}={}^{(p)}\dot e^E/c^2,\quad {}^{(p)}C^{MH}={}^{(p)}\dot e^M/c^3, \tag{5.38}$$

$$^{(p)}C^H_\mu={}^{(p)}\dot P_\mu/c^2,\qquad {}^{(p)}C^H_{[\mu\nu]}={}^{(p)}(\dot S_{\mu\nu}-\dot S_{\mu\rho}u^\rho u_\nu+\dot S_{\nu\rho}u^\rho u_\mu)/2c^2, \tag{5.39}$$

and

$$^{(p)}P_\mu={}^{(p)}(m^G u_\mu+\dot S_{\mu\rho}u^\rho). \tag{5.40}$$

Note that

$$^{(p)}C^H_{[\mu\nu]}{}^{(p)}u^\nu=0. \tag{5.41}$$

The quantity ${}^{(p)}P_\mu$ will be known as the momentum of the pth particle.

Fields j^H_μ, γ^{EH}_μ, γ^{MH}_μ, and $\gamma^H_{(\mu\nu)}$, of the forms given in (5.16)–(5.26), are not the most general solutions to the homogeneous equations (5.6). We shall assume, however, that in any region of the continuum which we investigate a harmonic coordinate system can be introduced so that j^H_μ, γ^{EH}_μ, γ^{MH}_μ, and $\gamma^H_{(\mu\nu)}$ do take these forms.[17] Such a region will be considered to contain only standard ideal particles if the structure of each particle in the region can be described through a finite number of the quantities (5.27). The possibility that a region of the continuum may contain a more general kind of ideal particle will not be considered here. Our investigation will be restricted to finite regions of the continuum containing only standard ideal particles, and we shall always work in a harmonic coordinate system such that (5.16)–(5.26) are valid.[18]

Solutions to the inhomogeneous equations. We shall introduce some further notation. The subscript $\{k\}$ to the left of a field will mean that the field contains only those terms which are of order k or lower in the power-series expansion in κ of the field without the subscript. For example,

$$_{\{k\}}\gamma_{\mu\nu}=\sum_{k'=0}^k \kappa^{k'}{}_{(k')}\gamma_{\mu\nu}\,. \tag{5.42}$$

The world line of a particle and the quantities (5.27) describing a particle's structure will depend on κ, and we are assuming that they can be expanded in a power series in κ. The subscript $[k]$ to the left of a field will mean that the field is identical to the field with the subscript $\{k\}$ to the left of it, except that the world lines of the particles in the region and the quantities (5.27) associated with the particles' structure will be treated as exact, and not expanded in a power series in κ. In terms of these fields the field equations (5.7) take the form

$$\Box^2\,{}_{[k]}j^I_\mu={}_{[k]}s_\mu+O(\kappa^{k+1}), \tag{5.43a}$$

$$_{[k]}j^{I}_\mu{}^{,\mu}=-\,{}_{[k]}j^{H}_\mu{}^{,\mu}+O(\kappa^{k+1}), \tag{5.43b}$$

$$\Box^2{}_{[k]}\gamma^{EI}_{\mu} = {}_{[k]}j_{\mu} + O(\kappa^{k+1}), \tag{5.43c}$$

$$_{[k]}\gamma^{EI,\mu\nu}_{\mu} = -{}_{[k]}\gamma^{EH,\mu\nu}_{\mu} + O(\kappa^{k+1}), \tag{5.43d}$$

$$_{[k]}\gamma^{MI}_{\mu} = 0, \tag{5.43e}$$

$$_{[k]}\gamma^{MI,\mu\nu}_{\mu} = -{}_{[k]}\gamma^{MH,\mu\nu}_{\mu} + O(\kappa^{k+1}), \tag{5.43f}$$

$$\Box^2{}_{[k]}\gamma^{I}_{(\mu\nu)} = {}_{[k]}t_{\mu\nu} + O(\kappa^{k+1}), \tag{5.43g}$$

$$_{[k]}\gamma^{I}_{(\mu\nu)}{}^{,\nu} = -{}_{[k]}\gamma^{H}_{(\mu\nu)}{}^{,\nu} + O(\kappa^{k+1}), \tag{5.43h}$$

where the expression $O(\kappa^{k+1})$ means a term whose power-series expansion in κ begins with κ^{k+1}. The conditions

$$_{[k]}s_{\mu}{}^{,\mu} = O(\kappa^{k+1}), \tag{5.44}$$

$$_{[k]}j_{\mu}{}^{,\mu} = O(\kappa^{k+1}), \tag{5.45}$$

hold true by definition, and the condition

$$_{[k]}t_{\mu\nu}{}^{,\nu} = O(\kappa^{k+1}) \tag{5.46}$$

holds true if the field equations (5.43) have been satisfied in all lower orders.

We shall now show how the set of equations (5.43) can be solved step by step with respect to k to any order of approximation desired. In order to do this we shall assume that at each step with respect to k Eqs. (5.43a), (5.43c), and (5.43g), equations of the form of d'Alembert's equation, have solutions which when expressed in terms of x^s and t can be expanded in a power series in c^{-1}. This seems to be a reasonable assumption as we can always, at least formally, construct such solutions step by step with respect to powers of c^{-1}.

The expression "kth order of approximation" will mean from now on an approximate solution which differs from the exact solution by terms of order $k+1$ or higher. The zeroth-order solutions to (5.43) are

$$_{[0]}j^{I}_{\mu} = 0, \tag{5.47}$$

$$_{[0]}\gamma^{EI}_{\mu} = 0, \quad {}_{[0]}\gamma^{MI}_{\mu} = 0, \tag{5.48}$$

$$_{[0]}\gamma^{I}_{(\mu\nu)} = 0. \tag{5.49}$$

From the work of Appendix A, an appendix in which we assume that at each order Eqs. (5.43a), (5.43c), and (5.43g) have solutions which can be expanded in a power series in c^{-1}, it follows that if all lower-order field equations in the set (5.43) have been solved, then the kth-order equations of (5.43a) have a solution over the region we are investigating such that

$$_{[k]}j^{I}_{\mu}{}^{,\mu} = \sum_{p=1}^{N} {}^{(p)}a^{D}_{\text{ret}}\,{}^{(p)}[{}_{[k]}C^{DI}(ru)^{-1}]_{\text{ret}} + \sum_{p=1}^{N} {}^{(p)}a^{D}_{\text{adv}}\,{}^{(p)}[{}_{[k]}C^{DI}(ru)^{-1}]_{\text{adv}} + O(\kappa^{k+1}). \tag{5.50}$$

The ${}^{(p)}_{[k]}C^{DI}$ in (5.50) are functions only of ${}^{(p)}\tau$. If (5.43b) is to be satisfied, we must choose

$${}^{(p)}_{[k]}C^{D} = O(\kappa^{k+1}), \tag{5.51}$$

where

$${}^{(p)}_{[k]}C^{D} = {}^{(p)}_{[k]}C^{DH} + {}^{(p)}_{[k]}C^{DI}. \tag{5.52}$$

The equation

$${}^{(p)}_{[k]}C^{D} = 0 \tag{5.53}$$

will be known as the kth-order equation of diffuse electric charge associated with the pth particle.

From the work of Appendix A we find that if Eqs. (5.43) have been solved to the $(k-1)$th order, and Eqs. (5.43a) and (5.43b) to the kth order, then Eqs. (5.43c) have a kth-order solution over the region we are investigating such that

$$_{[k]}\gamma^{EI,\mu\nu}_{\mu} = \sum_{p=1}^{N} {}^{(p)}a^{E}_{\text{ret}}\,{}^{(p)}[{}_{[k]}C^{EI}(ru)^{-1}]_{\text{ret}}{}^{,\nu} + \sum_{p=1}^{N} {}^{(p)}a^{E}_{\text{adv}}\,{}^{(p)}[{}_{[k]}C^{EI}(ru)^{-1}]_{\text{adv}}{}^{,\nu} + O(\kappa^{k+1}). \tag{5.54}$$

The ${}^{(p)}_{[k]}C^{EI}$ in (5.54) are functions only of ${}^{(p)}\tau$. We know that Eqs. (5.43e) have the solution ${}_{[k]}\gamma^{MI}_{\mu} = 0$, for which

$$_{[k]}\gamma^{MI,\mu\nu}_{\mu} = 0. \tag{5.55}$$

If (5.43d) and (5.43f) are to be satisfied, we must have

$${}^{(p)}_{[k]}C^{E} = O(\kappa^{k+1}), \quad {}^{(p)}_{[k]}C^{M} = O(\kappa^{k+1}), \tag{5.56}$$

where

$${}^{(p)}_{[k]}C^{E} = {}^{(p)}_{[k]}C^{EH} + {}^{(p)}_{[k]}C^{EI}, \quad {}^{(p)}_{[k]}C^{M} = {}^{(p)}_{[k]}C^{MH}. \tag{5.57}$$

The equations

$${}^{(p)}_{[k]}C^{E} = 0, \quad {}^{(p)}_{[k]}C^{M} = 0, \tag{5.58}$$

will be known, respectively, as the kth-order equation of localized electric charge and the kth-order equation of localized magnetic charge associated with the pth particle.

It is shown in Appendix A that if Eqs. (5.43) have been solved to the $(k-1)$th order, then Eqs. (5.43g) have a kth-order solution over the region we are investigating such that

$$_{[k]}\gamma^{I}_{(\mu\nu)}{}^{,\nu} = \sum_{p=1}^{N} {}^{(p)}a^{G}_{\text{ret}}\,{}^{(p)}[{}_{[k]}C^{I}_{\mu}(ru)^{-1}]_{\text{ret}} + \sum_{p=1}^{N} {}^{(p)}a^{G}_{\text{ret}}\,{}^{(p)}[{}_{[k]}C^{I}_{\mu}(ru)^{-1}]_{\text{adv}} + \sum_{p=1}^{N} {}^{(p)}a^{G}_{\text{ret}}\,{}^{(p)}[{}_{[k]}C^{I}_{[\mu\nu]}(ru)^{-1}]_{\text{ret}}{}^{,\nu} + \sum_{p=1}^{N} {}^{(p)}a^{G}_{\text{ret}}\,{}^{(p)}[{}_{[k]}C^{I}_{[\mu\nu]}(ru)^{-1}]_{\text{adv}}{}^{,\nu} + O(\kappa^{k+1}), \tag{5.59}$$

where ${}^{(p)}_{[k]}C^I_\mu$ and ${}^{(p)}_{[k]}C^I_{[\mu\nu]}$ are functions only of ${}^{(p)}\tau$ and

$$ {}^{(p)}_{[k]}C^I_{[\mu\nu]}\,{}^{(p)}u^\nu = 0. \tag{5.60}$$

If (5.43h) is to be satisfied, we must choose

$$ {}^{(p)}_{[k]}C_\mu = O(\kappa^{k+1}), \quad {}^{(p)}_{[k]}C_{[\mu\nu]} = O(\kappa^{k+1}), \tag{5.61}$$

where

$$ {}^{(p)}_{[k]}C_\mu = {}^{(p)}_{[k]}C^H_\mu + {}^{(p)}_{[k]}C^I_\mu, \qquad {}^{(p)}_{[k]}C_{[\mu\nu]} = {}^{(p)}_{[k]}C^H_{[\mu\nu]} + {}^{(p)}_{[k]}C^I_{[\mu\nu]}. \tag{5.62}$$

The equations

$$ {}^{(p)}_{[k]}C_\mu = 0, \quad {}^{(p)}_{[k]}C_{[\mu\nu]} = 0, \tag{5.63}$$

will be known respectively as the kth-order equations of mass and motion and the kth-order equations of spin associated with the pth particle.

It is clear that the inhomogeneous equations, and thus the field equations (5.3), can be solved step by step to any order of approximation desired if, and only if, the standard ideal particles within the region with which we are concerned satisfy the equations of diffuse electric charge

$$ {}^{(p)}C^D = 0, \tag{5.64}$$

the equations of localized electric and magnetic charge

$$ {}^{(p)}C^E = 0, \quad {}^{(p)}C^M = 0, \tag{5.65}$$

and the equations of mass and motion and spin

$$ {}^{(p)}C_\mu = 0, \quad {}^{(p)}C_{[\mu\nu]} = 0. \tag{5.66}$$

We are using the notation

$$ {}^{(p)}C^D = {}^{(p)}_{[\infty]}C^D, \tag{5.67}$$

$$ {}^{(p)}C^E = {}^{(p)}_{[\infty]}C^E, \quad {}^{(p)}C^M = {}^{(p)}_{[\infty]}C^M, \tag{5.68}$$

$$ {}^{(p)}C_\mu = {}^{(p)}_{[\infty]}C_\mu, \quad {}^{(p)}C_{[\mu\nu]} = {}^{(p)}_{[\infty]}C_{[\mu\nu]}, \tag{5.69}$$

in (5.64)–(5.66).[19]

C. Inertial Systems

Within a perfectly isolated region of the continuum which contains only standard ideal particles, we shall always use the especially convenient harmonic coordinate system discussed earlier in this paper in which j^H_μ, γ^{EH}_μ, γ^{MH}_μ, and $\gamma^H_{(\mu\nu)}$ take the forms given in (5.16)–(5.26). We shall call such a harmonic coordinate system a Cartesian inertial system or, more simply, an inertial system.

From this definition of an inertial system, and using (2.14), we see that coordinate transformations which lead from one inertial system to another have the form

$$ x'^\mu = \alpha^\mu + \alpha^\mu{}_\nu x^\nu + \epsilon^\mu, \tag{5.70}$$

where α^μ and $\alpha^\mu{}_\nu$ are constants satisfying the equations

$$ \alpha^\mu{}_\rho \alpha^\nu{}_\sigma \eta^{\rho\sigma} = \eta^{\mu\nu}, \tag{5.71}$$

and ϵ^μ is a quantity nonlinear in the particle parameters (5.27) and satisfying the equations

$$ \mathfrak{g}^{(\rho\sigma)}\epsilon^\mu{}_{,\rho\sigma} = 0. \tag{5.72}$$

Coordinate transformations of the form

$$ x'^\mu = \alpha^\mu + \alpha^\mu{}_\nu x^\nu, \tag{5.73}$$

where α^μ and $\alpha^\mu{}_\nu$ are constants and satisfy (5.71) are known as Lorentz transformations. Coordinate transformations of the form

$$ x'^\mu = x^\mu + \epsilon^\mu, \tag{5.74}$$

where ϵ^μ has a power-series expansion in κ such that $\epsilon^\mu \to 0$ as $\kappa \to 0$, are known as gauge transformations. The condition (5.72) along with the fact that ϵ^μ in (5.70) is nonlinear in the particle parameters (5.27) restricts the gauge transformations with which we shall be concerned to gauge transformations between inertial systems.

VI. EQUATIONS OF CHARGE, MASS, MOTION, AND SPIN

A. Chargers, Force, and Torque

If we introduce the notation

$$ {}^{(p)}s^D = -c^2 l^2\,{}^{(p)}C^{DI}, \tag{6.1}$$

$$ {}^{(p)}s^E = -c^2\,{}^{(p)}C^{EI}, \tag{6.2}$$

$$ {}^{(p)}f_\mu = -c^2\,{}^{(p)}C^I_\mu, \quad {}^{(p)}n^S_{\mu\nu} = -2c^2\,{}^{(p)}C^I_{[\mu\nu]}, \tag{6.3}$$

we find in an inertial system in a region of the continuum containing only standard ideal particles that the equation of diffuse electric charge associated with the pth particle takes the form

$$ {}^{(p)}\dot{e}^D = {}^{(p)}s^D, \tag{6.4}$$

the equations of localized electric and magnetic charge take the forms

$$ {}^{(p)}\dot{e}^E = {}^{(p)}s^E, \quad {}^{(p)}\dot{e}^M = 0, \tag{6.5}$$

and the equations of mass and motion and spin take the forms

$$ {}^{(p)}\dot{P}_\mu = {}^{(p)}f_\mu, \tag{6.6}$$

$$ {}^{(p)}(\dot{S}_{\mu\nu} - \dot{S}_{\mu\rho}u^\rho u_\nu + \dot{S}_{\nu\rho}u^\rho u_\mu) = {}^{(p)}n^S_{\mu\nu}, \tag{6.7}$$

where

$$ {}^{(p)}P_\mu = {}^{(p)}(m^G u_\mu + \dot{S}_{\mu\rho}u^\rho) \tag{6.8}$$

and

$$ {}^{(p)}n^S_{\mu\nu}\,{}^{(p)}u^\nu = 0. \tag{6.9}$$

The quantity ${}^{(p)}s^D$ will be known as the diffuse electric charger acting on the pth particle, ${}^{(p)}s^E$ will be known as the localized electric charger acting on the pth particle, ${}^{(p)}f_\mu$ will be known as the force acting on the pth particle, and ${}^{(p)}n^S_{\mu\nu}$ will be known as the torque acting on the pth particle about the particle's center ${}^{(p)}\xi^\mu$. The quantity ${}^{(p)}n^S_{\mu\nu}$ will often be called the spin torque acting on the pth particle.

The equations of spin can be put in a more familiar form. Introducing ${}^{(p)}L_{\mu\nu}$, the orbital angular momentum of the pth particle with respect to the point x^μ, through the equations

$${}^{(p)}L_{\mu\nu} = {}^{(p)}(r_\nu P_\mu - r_\mu P_\nu), \tag{6.10}$$

and introducing ${}^{(p)}n^L_{\mu\nu}$, the orbital torque acting on this particle with respect to the point x^μ, by

$${}^{(p)}n^L_{\mu\nu} = {}^{(p)}(r_\nu f_\mu - r_\mu f_\nu), \tag{6.11}$$

we find, through the use of (6.6), that Eqs. (6.7) take the form

$${}^{(p)}\dot{J}_{\mu\nu} = {}^{(p)}n_{\mu\nu}, \tag{6.12}$$

where

$${}^{(p)}J_{\mu\nu} = {}^{(p)}L_{\mu\nu} + {}^{(p)}S_{\mu\nu}, \tag{6.13}$$

$${}^{(p)}n_{\mu\nu} = {}^{(p)}n^L_{\mu\nu} + {}^{(p)}n^S_{\mu\nu}. \tag{6.14}$$

The quantity ${}^{(p)}J_{\mu\nu}$ will be known as the total angular momentum of the pth particle with respect to the point x^μ. The quantity ${}^{(p)}n_{\mu\nu}$ will be known as the total torque acting on the pth particle with respect to the point x^μ.

We see from (6.4) and (6.5) that the electric charge, both diffuse and localized, which is associated with a particle does not in general remain constant but changes with time. Of course total electric charge is conserved. This follows from (4.26b).

We shall now investigate the functional dependence of ${}^{(p)}s^D$, ${}^{(p)}s^E$, ${}^{(p)}f_\mu$, and ${}^{(p)}n^S_{\mu\nu}$. A study of the field equations for $\gamma_{\mu\nu}$ ($\gamma_{\mu\nu}$ is here understood as computed using the approximation method of this paper) shows that the field $\gamma_{\mu\nu}$ over an isolated region of the continuum containing N standard ideal particles can be split into two parts, a part ${}^{(p)}\gamma^{\text{self}}_{\mu\nu}$, which is singular along the world line of the pth particle, and a part ${}^{(p)}\gamma^{\text{ext}}_{\mu\nu}$, which is a solution to the field equations where one ignores the existence of the pth particle except as it affects the coefficients of diffuse electric charge, the localized electric charges, the masses, the spins, and the motion of the other $N-1$ particles in the region.[20] The field ${}^{(p)}\gamma^{\text{ext}}_{\mu\nu}$ is nonsingular along the world line of the pth particle but is singular along the world lines of the other $N-1$ particles. We are assuming the world lines of two different particles never cross.

In the neighborhood of the world line of the pth particle we thus have

$$\gamma_{\mu\nu} = {}^{(p)}\gamma^{\text{self}}_{\mu\nu} + {}^{(p)}\gamma^{\text{ext}}_{\mu\nu}, \tag{6.15}$$

where we shall call ${}^{(p)}\gamma^{\text{self}}_{\mu\nu}$ the self field and ${}^{(p)}\gamma^{\text{ext}}_{\mu\nu}$ the external field in the neighborhood of the pth particle. A study of the field equations for j_μ, γ^E_μ, γ^M_μ, and $\gamma_{(\mu\nu)}$ shows that if these fields are evaluated at a point in the neighborhood of ${}^{(p)}\xi^\mu$ (where ${}^{(p)}\xi^\mu$ is a point on the world line of the pth particle), then these fields, in addition to depending on ${}^{(p)}r^\mu$, will depend on three kinds of quantities. First, they will depend on those quantities which describe the basic structure of the pth particle in the neighborhood of ${}^{(p)}\xi^\mu$, that is, on the quantities (5.27) and their derivatives with respect to ${}^{(p)}\tau$ all evaluated at ${}^{(p)}\xi^\mu$ and on the constants ${}^{(p)}a_{\text{ret}}$ and ${}^{(p)}a_{\text{adv}}$. Second, they will depend on those quantities which describe the world line of the pth particle in the neighborhood of ${}^{(p)}\xi^\mu$, that is, on ${}^{(p)}u^\mu$ and its derivatives with respect to ${}^{(p)}\tau$ all evaluated at ${}^{(p)}\xi^\mu$. Third, they will depend on those quantities which describe the external field in the neighborhood of the pth particle, that is, on the fields ${}^{(p)}\gamma^{\text{ext}}_{\mu\nu}$, ${}^{(p)}\gamma^{\text{ext}}_{\mu\nu,\rho}$, etc. all evaluated at ${}^{(p)}\xi^\mu$.

The chargers ${}^{(p)}s^D$ and ${}^{(p)}s^E$, the force ${}^{(p)}f_\mu$, and the spin torque ${}^{(p)}n^S_{\mu\nu}$ can be regarded as depending exclusively on the three kinds of quantities described above. This follows from their definitions in (6.1)–(6.3), and from (5.50), (5.54), and (5.59).

Two procedures for finding the chargers, force, and spin torque. We have seen that the chargers, force, and spin torque acting on a standard ideal particle will depend on the quantities which characterize the particle's structure and motion and on the external field in the neighborhood of the particle. In this section we shall describe two practical procedures for finding this functional dependence to any order of approximation desired.

A straightforward way to obtain this dependence is to solve Eqs. (5.43) step by step to the order of approximation desired. Such a procedure, however, appears to be impractical. A practical procedure for obtaining this functional dependence is as follows.

Expand the solutions to the homogeneous equations (5.16)–(5.26) in a power series around the position of the pth particle at time t, that is, in a power series in ${}^{(p)}|\vec{r}|$, where

$${}^{(p)}|\vec{r}| = {}^{(p)}(r^s r^s)^{1/2}. \tag{6.16}$$

Solve Eqs. (5.43) in the neighborhood of the pth particle step by step to the order of approximation desired, which we shall denote by k, keeping only those terms in ${}_{[k]}j^I_\mu$, ${}_{[k]}\gamma^{EI}_\mu$, and ${}_{[k]}\gamma^I_{(\mu\nu)}$ which, when one forms ${}_{[k]}j^{I}_{\mu}{}^{,\mu}$, ${}_{[k]}\gamma^{EI}_{\mu}{}^{,\mu}$, and ${}_{[k]}\gamma^I_{(\mu\nu)}{}^{,\nu}$ in an inertial system in which the pth particle is at rest, give rise to terms which become infinite as ${}^{(p)}|\vec{r}|\to 0$. This procedure for solving Eqs. (5.43) in the neighborhood of the pth particle can be carried out with a finite amount of labor as one need keep only a finite number of terms in the power-series expansion in ${}^{(p)}|\vec{r}|$. Next, form ${}_{[k]}j^{I}_{\mu}{}^{,\mu}$, ${}_{[k]}\gamma^{EI,\mu}_{\mu}$, and ${}_{[k]}\gamma^{I}_{(\mu\nu)}{}^{,\nu}$ in an inertial system in which the pth particle is at rest, keeping only those terms which become infinite as ${}^{(p)}|\vec{r}|\to 0$. We see from (5.50), (5.54), and (5.59) that we will find, if Eqs. (5.43) have been solved properly,

$$ {}_{[k]}j^{I}_{\mu}{}^{,\mu} = {}^{(p)}_{[k]}C^{DI(p)}|\vec{r}|^{-1} + O(\kappa^{k+1}), \tag{6.17} $$

$$ {}_{[k]}\gamma^{I}_{\mu}{}^{,\mu} = {}^{(p)}_{[k]}C^{EI(p)}|\vec{r}|^{-1} + O(\kappa^{k+1}) + \text{const}, $$

$$ {}_{[k]}\gamma^{I}_{(\mu\nu)}{}^{,\nu} = {}^{(p)}[{}_{[k]}C^{I}_{\mu} - \tfrac{1}{2}{}_{[k]}C^{I}_{[\mu s]}\xi^{t}{}_{,44}(\delta_{st} + r^{s}r^{t}/|\vec{r}|^{2})] \times {}^{(p)}|\vec{r}|^{-1} - {}^{(p)}_{[k]}C^{I}_{[\mu s]}\,{}^{(p)}|\vec{r}|^{-1}{}_{,s} + O(\kappa^{k+1}). \tag{6.18} $$

In (6.18) we have made use of the fact that in an inertial system in which the pth particle is at rest,

$$ {}^{(p)}_{[k]}C_{[\mu 0]} = 0, \quad {}^{(p)}_{[k]}C_{[\mu 4],4} = -{}^{(p)}_{[k]}C_{[\mu s]}\,{}^{(p)}\xi^{s}{}_{,44}. \tag{6.19} $$

These equations follow from (5.60).

We have described a procedure for finding to the kth order ${}^{(p)}_{[k]}C^{DI}$, ${}^{(p)}_{[k]}C^{EI}$, ${}^{(p)}_{[k]}C^{I}_{\mu}$, and ${}^{(p)}_{[k]}C^{I}_{[\mu\nu]}$ in an inertial system in which the pth particle is at rest at time t. The quantities ${}^{(p)}_{[k]}C^{DI}$, ${}^{(p)}_{[k]}C^{EI}$, ${}^{(p)}_{[k]}C^{I}_{\mu}$, and ${}^{(p)}_{[k]}C^{I}_{[\mu\nu]}$ in (6.17) and (6.18) are found as functions of the quantities which describe the structure and motion of the pth particle and of the external field in the neighborhood of the pth particle. Since the transformation properties with respect to Lorentz transformations of ${}^{(p)}_{[k]}C^{DI}$, ${}^{(p)}_{[k]}C^{EI}$, ${}^{(p)}_{[k]}C^{I}_{\mu}$, and ${}^{(p)}_{[k]}C^{I}_{[\mu\nu]}$ and of all the terms appearing in them are known, it is possible, through the use of a Lorentz transformation, to find ${}^{(p)}_{[k]}C^{DI}$, ${}^{(p)}_{[k]}C^{EI}$, ${}^{(p)}_{[k]}C^{I}_{\mu}$, and ${}^{(p)}_{[k]}C^{I}_{[\mu\nu]}$ in any inertial system having the same gauge as the original rest system. The use of (6.1)–(6.3) will then give to the kth order ${}^{(p)}_{[k]}s^{D}$, ${}^{(p)}_{[k]}s^{E}$, ${}^{(p)}_{[k]}f_{\mu}$, and ${}^{(p)}_{[k]}n^{S}_{\mu\nu}$ in such a coordinate system.

After having completed most of the work presented in this paper the author came to realize that the procedure described above for finding ${}^{(p)}_{[k]}s^{D}$, ${}^{(p)}_{[k]}s^{E}$, ${}^{(p)}_{[k]}f_{\mu}$, and ${}^{(p)}_{[k]}n^{S}_{\mu\nu}$ in an arbitrary inertial system is not necessarily the most practical procedure to use when seeking these quantities. It certainly is not the most elegant. A more elegant procedure and a procedure which makes use of Lorentz-covariant notation at each step of the analysis is as follows.

Expand the solutions to the homogeneous equations (5.16)–(5.26) in a power series around the world line of the pth particle, that is in a power series in ${}^{(p)}\epsilon$, where

$$ {}^{(p)}\epsilon = {}^{(p)}(-r_{\rho}r^{\rho})^{1/2}. \tag{6.20} $$

The vector ${}^{(p)}r^{\mu}$ which appears in (6.20) should always be chosen to be a spacelike vector. Solve Eqs. (5.43) in the neighborhood of the world line of the pth particle step by step to the order of approximation desired, which we shall denote by k, keeping only those terms in ${}_{[k]}j^{I}_{\mu}$, ${}_{[k]}\gamma^{EI}_{\mu}$, and ${}_{[k]}\gamma^{I}_{(\mu\nu)}$ which, when one forms ${}_{[k]}j^{I}_{\mu}{}^{,\mu}$, ${}_{[k]}\gamma^{EI,\mu}_{\mu}$, and ${}_{[k]}\gamma^{I}_{(\mu\nu)}{}^{,\nu}$, give rise to terms which become infinite as ${}^{(p)}\epsilon\to 0$. This procedure for solving Eqs. (5.43) in the neighborhood of the world line of the pth particle can be carried out with a finite amount of labor, as one need keep only a finite number of terms in the power-series expansion in ${}^{(p)}\epsilon$. Next, form ${}_{[k]}j^{I}_{\mu}{}^{,\mu}$, ${}_{[k]}\gamma^{EI,\mu}_{\mu}$, and ${}_{[k]}\gamma^{I}_{(\mu\nu)}{}^{,\nu}$, keeping only those terms which become infinite as ${}^{(p)}\epsilon\to 0$. From (5.50), (5.54), and (5.59), we know that we will find, if Eqs. (5.43) have been solved properly,

$$ {}_{[k]}j^{I}_{\mu}{}^{,\mu} = {}^{(p)}\{{}_{[k]}C^{DI}[1+(\alpha u)^{2}]^{-1/2}\}\,{}^{(p)}\epsilon^{-1} + O(\kappa^{k+1}), \tag{6.21} $$

$$ {}_{[k]}\gamma^{EI,\mu}_{\mu} = {}^{(p)}\{{}_{[k]}C^{EI}[1+(\alpha u)^{2}]^{-1/2}\}\,{}^{(p)}\epsilon^{-1} + O(\kappa^{k+1}) + \text{const}, \tag{6.22} $$

$$ {}_{[k]}\gamma^{I}_{(\mu\nu)}{}^{,\nu} = {}^{(p)}\{[{}_{[k]}C^{I}_{\mu} + \tfrac{1}{2}{}_{[k]}C^{I}_{[\mu\nu]}\alpha^{\nu}(\alpha\dot{u}) + \tfrac{1}{2}{}_{[k]}C^{I}_{[\mu\nu]}\dot{u}^{\nu} + {}_{[k]}\dot{C}^{I}_{[\mu\nu]}u^{\nu} + O((\alpha u))][1+(\alpha u)^{2}]^{-5/2}\}\,{}^{(p)}\epsilon^{-1} + {}^{(p)}\{[{}_{[k]}C^{I}_{[\mu\nu]}\alpha^{\nu} + O((\alpha u))][1+(\alpha u)^{2}]^{-3/2}\}\,{}^{(p)}\epsilon^{-2} + O(\kappa^{k+1}), \tag{6.23} $$

where

$$ {}^{(p)}\alpha^{\mu} = {}^{(p)}r^{\mu}\,{}^{(p)}\epsilon^{-1}, \tag{6.24} $$

$$ {}^{(p)}(\alpha u) = {}^{(p)}\alpha_{\rho}\,{}^{(p)}u^{\rho}. \tag{6.25} $$

In arriving at (6.23) from (5.59) we have not made use of the fact that ${}^{(p)}_{[k]}C^{I}_{[\mu\nu]}$ is antisymmetric in μ and ν and that (5.60) is valid.[21]

We have just described a second procedure for finding to the kth order ${}^{(p)}_{[k]}C^{DI}$, ${}^{(p)}_{[k]}C^{EI}$, ${}^{(p)}_{[k]}C^{I}_{\mu}$, and ${}^{(p)}_{[k]}C^{I}_{[\mu\nu]}$ in an arbitrary inertial system. Note that this procedure uses Lorentz-covariant notation at each step of the analysis. The use of (6.1)–(6.3) will give ${}^{(p)}_{[k]}s^{D}$, ${}^{(p)}_{[k]}s^{E}$, ${}^{(p)}_{[k]}f_{\mu}$, and ${}^{(p)}_{[k]}n^{S}_{\mu\nu}$.

B. Simple Ideal Particles

The equations of charge satisfied by a standard ideal particle form a system of three independent equations for the three independent quantities which describe the charge associated with the particle. These three independent quantities are the coefficient of diffuse electric charge, the localized electric charge, and the localized magnetic charge.

The equations of mass, motion, and spin satisfied by a standard ideal particle form a system of seven independent equations for the ten independent quantities which describe the mass, motion, and spin associated with the particle. The motion of a particle is understood as being described by the particle's velocity. If no additional conditions are placed on the structure of a particle, then its world line will not necessarily be restricted by these seven independent equations. If, however, the particle's structure is not regarded as arbitrary, but at all times the structure changes in such a way that the particle has no mass dipole moment, that is, if

$$ {}^{(p)}S_{\mu\nu}\,{}^{(p)}u^{\nu}=0, \tag{6.26}$$

then the equations of mass, motion, and spin do restrict the motion of the particle. This is so because in this case the equations of mass, motion and spin, along with (6.26), form a system of ten independent equations for the ten independent quantities which describe the mass, motion, and spin of the particle.[22]

We note that of the quantities given in (5.27) characterizing the structure of a standard ideal particle, only ${}^{(p)}e^{D}$, ${}^{(p)}e^{E}$, ${}^{(p)}e^{M}$, ${}^{(p)}m^{G}$, and ${}^{(p)}S_{\mu\nu}$ must satisfy restricting equations. The mass dipole moment ${}^{(p)}S_{\mu\nu}\,{}^{(p)}u^{\nu}$ along with the quantities ${}^{(p)}e^{D}_{[\mu\sigma_1]\cdots\sigma_i}$, ${}^{(p)}e^{E}_{[\mu\sigma_1]\cdots\sigma_i}$, ${}^{(p)}e^{M}_{[\mu\sigma_1]\cdots\sigma_i}$, and ${}^{(p)}m^{G}_{[\mu\sigma_1][\nu\sigma_2]\cdots\sigma_i}$ may be chosen arbitrarily. A standard ideal particle for which these quantities, which can be chosen arbitrarily, are zero and for which

$$\begin{aligned} {}^{(p)}a^{D}_{\text{ret}}&={}^{(p)}a^{E}_{\text{ret}}={}^{(p)}a^{M}_{\text{ret}}={}^{(p)}a^{G}_{\text{ret}},\\ {}^{(p)}a^{D}_{\text{adv}}&={}^{(p)}a^{E}_{\text{adv}}={}^{(p)}a^{M}_{\text{adv}}={}^{(p)}a^{G}_{\text{adv}}, \end{aligned}\tag{6.27}$$

will be known as a simple ideal particle. We shall study simple ideal particles in more detail in paper II.

ACKNOWLEDGMENT

The author would like to thank Dr. Jon Nance for his interest in the problem, for his reading of most of the manuscript, and for much stimulating discussion.

APPENDIX A: ON SOLUTIONS TO THE FIELD EQUATIONS

The inhomogeneous field equations (5.43) are very complicated and must be solved step by step with respect to k. In this appendix we shall show that in a perfectly isolated region of the continuum containing only standard ideal particles, the inhomogeneous field equations can be solved step by step to any order of approximation desired. In order to do this we shall assume that at each order, over a perfectly isolated region containing only standard ideal particles, Eqs. (5.43a), (5.43c), and (5.43g), equations of the form of d'Alembert's equation, always have solutions which, when expressed in terms of x^s and t, can be expanded in a power series in c^{-1}. We shall restrict our investigation to such solutions.

We regard the inhomogeneous field equations as having the zeroth-order solutions (5.47)–(5.49). This means that if we wish to show that the inhomogeneous field equations can be solved to any order of approximation desired, we need only show that one can always solve the kth-order field equations ($k>0$) of (5.43) if one has already solved all lesser-order field equations. This we shall do.

Let us first look at Eqs. (5.43a) and (5.43b). The assumption that the field equations have been solved to the $(k-1)$th order means that ${}_{[k]}s_{\mu}$ is known and satisfies the kth-order equation

$$ {}_{[k]}s_{\mu}{}^{,\mu}=O(\kappa^{k+1}). \tag{A1}$$

If we expand the fields ${}_{[k]}s_{\mu}$ and ${}_{[k]}j_{\mu}$ in a power series in λ, where $\lambda=c^{-1}$,

$$ {}_{[k]}s_0=\sum_{l=0}^{\infty}\lambda^{l+2}{}_{(l+2)[k]}s_0,\quad {}_{[k]}s_m=\sum_{l=0}^{\infty}\lambda^{l+3}{}_{(l+3)[k]}s_m, \tag{A2}$$

$$ {}_{[k]}j_0=\sum_{l=0}^{\infty}\lambda^{l+2}{}_{(l+2)[k]}j_0,\quad {}_{[k]}j_m=\sum_{l=0}^{\infty}\lambda^{l+3}{}_{(l+3)[k]}j_m, \tag{A3}$$

we find that the kth-order equations of (5.43a) and (5.43b) take the form

$$\nabla^2{}_{(l+2)[k]}j^{I}_{0}={}_{(l)[k]}j^{I}_{0,00}-{}_{(l+2)[k]}s_0+O(\kappa^{k+1}), \tag{A4}$$

$$\nabla^2{}_{(l+3)[k]}j^{I}_{m}={}_{(l+1)[k]}j^{I}_{m,00}-{}_{(l+3)[k]}s_m+O(\kappa^{k+1}), \tag{A5a}$$

$${}_{(l+3)[k]}j^{I}_{m,m}={}_{(l+2)[k]}j^{I}_{0,0}-\left({}_{(l+3)[k]}j^{H}_{m,m}-{}_{(l+2)[k]}j^{H}_{0,0}\right)+O(\kappa^{k+1}), \tag{A5b}$$

where

$${}_{(l+3)[k]}s_{m,m}={}_{(l+2)[k]}s_{0,0}+O(\kappa^{k+1}). \tag{A6}$$

Using (A6) we see that any set of solutions to (A4) and (A5a) will satisfy the equations

$$\nabla^2\left({}_{(l+3)[k]}j^{I}_{m,m}-{}_{(l+2)[k]}j^{I}_{0,0}\right)=\left({}_{(l+1)[k]}j^{I}_{m,m}-{}_{(l)[k]}j^{I}_{0,0}\right)_{,00}+O(\kappa^{k+1}), \tag{A7}$$

or, what is the same thing,

$$\Box^2\,{}_{[k]}j^{\prime I,\mu}_{\mu}=O(\kappa^{k+1}). \tag{A8}$$

Equations (A4) can always be solved step by step with respect to l. This can be seen from the fact that at each step the right-hand side of the equation is known so that the equation to be solved takes the form of Poisson's equation, an equation which, over a finite region of the continuum, always possesses a solution. Let us denote by ${}_{[k]}j^{\prime I}_{0}$ a particular solution to (A4) which we obtain in this way. Equations (A5a) can also be solved step by step with respect to l. The question we must answer is whether Eqs. (A5a) can be solved step by step with respect to l so that (A5b) is satisfied. In the next few paragraphs we shall show that they can.

Let us start with the case where $l=0$. We can always construct a solution to (A5a). Denoting this solution by ${}_{(3)[k]}j^{\prime I}_{m}$ we see from (A7) that it will satisfy the equation

$$\nabla^2({}_{(3)[k]}j^{\prime I}_{m,m}-{}_{(2)[k]}j^{\prime I}_{0,0})=O(\kappa^{k+1}), \tag{A9}$$

and thus we can write

$${}_{(2)[k]}j^{\prime I}_{0,0}-{}_{(3)[k]}j^{\prime I}_{m,m}=\sum_{p=1}^{N}{}^{(p)}_{(3)[k]}C^{DI}\,{}^{(p)}|\vec{r}|^{-1}+\sum_{p=1}^{N}\sum_{i=1}^{\infty}{}^{(p)}({}_{(3)[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}|\vec{r}|^{-1})_{,\sigma_1\cdots\sigma_i}+{}_{(3)[k]}\chi^{\prime D}+O(\kappa^{k+1}), \tag{A10}$$

where the coefficients ${}^{(p)}_{(3)[k]}C^{DI}$ and ${}^{(p)}_{(3)[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}$ may be functions of t and ${}_{(3)[k]}\chi^{\prime D}$ is a nonsingular function of x^{μ} over the region we are investigating. We are using the notation

$${}^{(p)}|\vec{r}|={}^{(p)}(r_s r_s)^{1/2}. \tag{A11}$$

Next let us look at the cases where $l=1$ and $l=2$. We can always construct solutions ${}_{(4)[k]}j^{\prime I}_{m}$ and ${}_{(5)[k]}j^{\prime I}_{m}$ to (A5a). Through the application of (A7) we see that ${}_{(3)[k]}j^{\prime I}_{0,0}-{}_{(4)[k]}j^{\prime I}_{m,m}$ and ${}_{(4)[k]}j^{\prime I}_{0,0}-{}_{(5)[k]}j^{\prime I}_{m,m}$ can always be written in the forms

$${}_{(3)[k]}j^{\prime I}_{0,0}-{}_{(4)[k]}j^{\prime I}_{m,m}=\sum_{p=1}^{N}{}^{(p)}_{(4)[k]}C^{DI}\,{}^{(p)}|\vec{r}|^{-1}+\sum_{p=1}^{N}\sum_{i=1}^{\infty}{}^{(p)}({}_{(4)[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}|\vec{r}|^{-1})_{,\sigma_1\cdots\sigma_i}+{}_{(4)[k]}\chi^{\prime D}+O(\kappa^{k+1}), \tag{A12}$$

$$\begin{aligned}{}_{(4)[k]}j^{\prime I}_{0,0}-{}_{(5)[k]}j^{\prime I}_{m,m}=&\sum_{p=1}^{N}{}^{(p)}_{(5)[k]}C^{DI}\,{}^{(p)}|\vec{r}|^{-1}+\sum_{p=1}^{N}\frac{1}{2}\frac{d^2}{dt^2}{}^{(p)}({}_{(3)[k]}C^{DI}|\vec{r}|)+\sum_{p=1}^{N}\sum_{i=1}^{\infty}{}^{(p)}({}_{(5)[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}|\vec{r}|^{-1})_{,\sigma_1\cdots\sigma_i}\\&+\sum_{p=1}^{N}\sum_{i=1}^{\infty}\frac{1}{2}\frac{d^2}{dt^2}{}^{(p)}({}_{(3)[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}|\vec{r}|)_{,\sigma_1\cdots\sigma_i}+{}_{(5)[k]}\chi^{\prime D}+O(\kappa^{k+1}),\end{aligned} \tag{A13}$$

where ${}^{(p)}_{(4)[k]}C^{DI}$, ${}^{(p)}_{(4)[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}$, ${}^{(p)}_{(5)[k]}C^{DI}$, and ${}^{(p)}_{(5)[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}$ are functions of t only, and ${}_{(4)[k]}\chi^{\prime D}$ and ${}_{(5)[k]}\chi^{\prime D}$ are nonsingular functions of x^{μ} over the region we are investigating.

If we continue step by step to higher order in λ and apply (A7) at each step, we find that ${}_{[k]}j^{\prime I,\mu}_{\mu}$ itself can always be written in the form

$$\begin{aligned}{}_{[k]}j^{\prime I,\mu}_{\mu}=&\sum_{p=1}^{N}\sum_{l=0}^{\infty}\frac{\lambda^{2l}}{(2l)!}\frac{d^{2l}}{dt^{2l}}{}^{(p)}({}_{[k]}C^{DI}|\vec{r}|^{2l-1})+\sum_{p=1}^{N}\sum_{l=0}^{\infty}\sum_{i=1}^{\infty}\frac{\lambda^{2l}}{(2l)!}\frac{d^{2l}}{dt^{2l}}{}^{(p)}({}_{[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}|\vec{r}|^{2l-1})_{,\sigma_1\cdots\sigma_i}\\&+{}_{[k]}\chi^{\prime D}+O(\kappa^{k+1}),\end{aligned} \tag{A14}$$

where the quantities ${}^{(p)}_{[k]}C^{DI}$ and ${}^{(p)}_{[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}$,

$$\begin{aligned}{}^{(p)}_{[k]}C^{DI}&=\sum_{l=0}^{\infty}\lambda^{l+3}\,{}^{(p)}_{(l+3)[k]}C^{DI},\\ {}^{(p)}_{[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}&=\sum_{l=0}^{\infty}\lambda^{l+3}\,{}^{(p)}_{(l+3)[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)},\end{aligned} \tag{A15}$$

are functions of t only, and ${}_{[k]}\chi^{\prime D}$ is a nonsingular function of x^{μ} over the region we are investigating. If we make use of the power-series expansions

$${}^{(p)}[{}_{[k]}C^{DI}(ru)^{-1}]_{\text{adv}}+{}^{(p)}[{}_{[k]}C^{DI}(ru)^{-1}]_{\text{ret}}=2\sum_{l=0}^{\infty}\frac{\lambda^{2l}}{(2l)!}\frac{d^{2l}}{dt^{2l}}{}^{(p)}({}_{[k]}C^{DI}|\vec{r}|^{2l-1}), \tag{A16}$$

$${}^{(p)}[{}_{[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}(ru)^{-1}]_{\text{adv}}+{}^{(p)}[{}_{[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}(ru)^{-1}]_{\text{ret}}=2\sum_{l=0}^{\infty}\frac{\lambda^{2l}}{(2l)!}\frac{d^{2l}}{dt^{2l}}{}^{(p)}({}_{[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}|\vec{r}|^{2l-1}), \tag{A17}$$

${}_{[k]}j^{\prime I,\mu}_{\mu}$ can be written in the form

$$j'^{I,\mu}_{\mu} = \tfrac{1}{2}\sum_{p=1}^{N} {}^{(p)}[{}_{[k]}C^{DI}(ru)^{-1}]_{\text{ret}} + \tfrac{1}{2}\sum_{p=1}^{N} {}^{(p)}[{}_{[k]}C^{DI}(ru)^{-1}]_{\text{adv}} + \tfrac{1}{2}\sum_{p=1}^{N}\sum_{i=1}^{\infty} {}^{(p)}[{}_{[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}(ru)^{-1}]_{\text{ret}},{}^{\sigma_1\cdots\sigma_i}$$
$$+ \tfrac{1}{2}\sum_{p=1}^{N}\sum_{i=1}^{\infty} {}^{(p)}[{}_{[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}(ru)^{-1}]_{\text{adv}},{}^{\sigma_1\cdots\sigma_i} + {}_{[k]}\chi'^{D} + O(\kappa^{k+1}), \tag{A18}$$

where ${}^{(p)}_{[k]}C^{DI}$ and ${}^{(p)}_{[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}$ can be considered functions of ${}^{(p)}\tau$ and the field ${}_{[k]}\chi'^{D}$ satisfies the equation

$$\Box^2\, {}_{[k]}\chi'^{D} = O(\kappa^{k+1}). \tag{A19}$$

Equation (A19) follows from (A8) and the form of (A18).

Let us next introduce the field ${}_{[k]}\chi^{D}$,

$${}_{[k]}\chi^{D} = {}_{[k]}\chi'^{D} + \tfrac{1}{2}\sum_{p=1}^{N} {}^{(p)}a^{D\,(p)}\{[{}_{[k]}C^{DI}(ru)^{-1}]_{\text{ret}} - [{}_{[k]}C^{DI}(ru)^{-1}]_{\text{adv}}\}$$
$$+ \tfrac{1}{2}\sum_{p=1}^{N}\sum_{i=1}^{\infty} {}^{(p)}a^{D_i\,(p)}\{[{}_{[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}(ru)^{-1}]_{\text{ret}},{}^{\sigma_1\cdots\sigma_i} - [{}_{[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}(ru)^{-1}]_{\text{adv}},{}^{\sigma_1\cdots\sigma_i}\}, \tag{A20}$$

where the quantities ${}^{(p)}a^{D}$ and ${}^{(p)}a^{D_i}$ in (A20) are arbitrary constants. Note that ${}_{[k]}\chi^{D}$ will be nonsingular over the region we are investigating and will satisfy the equation

$$\Box^2\, {}_{[k]}\chi^{D} = O(\kappa^{k+1}). \tag{A21}$$

From (A18) and (A20) it follows that ${}_{[k]}j'^{I,\mu}_{\mu}$ can be written in the form

$${}_{[k]}j'^{I,\mu}_{\mu} = \sum_{p=1}^{N} {}^{(p)}a^{D}_{\text{ret}}\,{}^{(p)}[{}_{[k]}C^{DI}(ru)^{-1}]_{\text{ret}} + \sum_{p=1}^{N} {}^{(p)}a^{D}_{\text{adv}}\,{}^{(p)}[{}_{[k]}C^{DI}(ru)^{-1}]_{\text{adv}}$$
$$+ \sum_{p=1}^{N}\sum_{i=1}^{\infty} {}^{(p)}a^{D_i}_{\text{ret}}\,{}^{(p)}[{}_{[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}(ru)^{-1}]_{\text{ret}},{}^{\sigma_1\cdots\sigma_i}$$
$$+ \sum_{p=1}^{N}\sum_{i=1}^{\infty} {}^{(p)}a^{D_i}_{\text{adv}}\,{}^{(p)}[{}_{[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}(ru)^{-1}]_{\text{adv}},{}^{\sigma_1\cdots\sigma_i} + {}_{[k]}\chi^{D} + O(\kappa^{k+1}), \tag{A22}$$

where we are using the notation

$${}^{(p)}a^{D}_{\text{ret}} = \tfrac{1}{2}(1 - {}^{(p)}a^{D}), \quad {}^{(p)}a^{D}_{\text{adv}} = \tfrac{1}{2}(1 + {}^{(p)}a^{D}), \tag{A23}$$

$${}^{(p)}a^{D_i}_{\text{ret}} = \tfrac{1}{2}(1 - {}^{(p)}a^{D_i}), \quad {}^{(p)}a^{D_i}_{\text{adv}} = \tfrac{1}{2}(1 + {}^{(p)}a^{D_i}). \tag{A24}$$

The constants ${}^{(p)}a^{D}_{\text{ret}}$, ${}^{(p)}a^{D}_{\text{adv}}$, ${}^{(p)}a^{D_i}_{\text{ret}}$, and ${}^{(p)}a^{D_i}_{\text{adv}}$ in (A22) can be chosen arbitrarily except that they must satisfy the equations

$${}^{(p)}a^{D}_{\text{ret}} + {}^{(p)}a^{D}_{\text{adv}} = 1, \quad {}^{(p)}a^{D_i}_{\text{ret}} + {}^{(p)}a^{D_i}_{\text{adv}} = 1. \tag{A25}$$

We have just shown that the field ${}_{[k]}j'^{I,\mu}_{\mu}$ associated with any kth-order solution ${}_{[k]}j'^{I}_{\mu}$ of (5.43a) (assuming that ${}_{[k]}j'^{I}_{\mu}$ can be expanded in a power series in c^{-1}) can be always written in the form (A22) where (A21) and (A25) are satisfied. Except for the fact that they must satisfy (A25), the constants ${}^{(p)}a^{D}_{\text{ret}}$, ${}^{(p)}a^{D}_{\text{adv}}$, ${}^{(p)}a^{D_i}_{\text{ret}}$, and ${}^{(p)}a^{D_i}_{\text{adv}}$ in (A22) can be chosen arbitrarily.

Next, note that the equation

$$\nabla^2\, {}_{[k]}\varphi^{D}(x^1, x^2, x^3) = -\,{}_{[k]}\chi^{D},{}_4(x^1, x^2, x^3, 0), \tag{A26}$$

being Poisson's equation, can always be solved over a finite region of the space-time continuum for a nonsingular function ${}_{[k]}\varphi^{D}$ which can be used to define a nonsingular field ${}_{[k]}j'^{H}_{\mu}$,

$${}_{[k]}j'^{H}_{s} = 0, \quad {}_{[k]}j'^{H}_{4} = \int_0^{x^4} {}_{[k]}\chi^{D}(x^1, x^2, x^3, \alpha)\,d\alpha + {}_{[k]}\varphi^{D}(x^1, x^2, x^3), \tag{A27}$$

which satisfies the equations

$$\Box^2\, {}_{[k]}j'^{H}_{\mu} = O(\kappa^{k+1}), \quad {}_{[k]}j'^{H,\mu}_{\mu} = -\,{}_{[k]}\chi^{D}. \tag{A28}$$

This means that if a kth-order solution to Eqs. (5.43a) exists which can be expanded in a power series in c^{-1}, then we can always find another kth-order solution ${}_{[k]}j^{I}_{\mu}$ to the equations,

$$ {}_{[k]}j^{I}_{\mu} = {}_{[k]}j'^{I}_{\mu} + {}_{[k]}j'^{H}_{\mu} - \sum_{p=1}^{N}\sum_{i=1}^{\infty} {}^{(p)}a^{D_i}_{\mathrm{ret}}\, {}^{(p)}[{}_{[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}(ru)^{-1}]_{\mathrm{ret}},{}^{\sigma_1\cdots\sigma_i} - \sum_{p=1}^{N}\sum_{i=1}^{\infty} {}^{(p)}a^{D_i}_{\mathrm{adv}}\, {}^{(p)}[{}_{[k]}C^{DI}_{(\sigma_1\cdots\sigma_i)}(ru)^{-1}]_{\mathrm{adv}},{}^{\sigma_1\cdots\sigma_i} \tag{A29} $$

for which

$$ {}_{[k]}j^{I,\mu}_{\mu} = \sum_{p=1}^{N} {}^{(p)}a^{D}_{\mathrm{ret}}\, {}^{(p)}[{}_{[k]}C^{DI}(ru)^{-1}]_{\mathrm{ret}} + \sum_{p=1}^{N} {}^{(p)}a^{D}_{\mathrm{ret}}\, {}^{(p)}[{}_{[k]}C^{DI}(ru)^{-1}]_{\mathrm{adv}} + O(\kappa^{k+1}). \tag{A30} $$

The field ${}_{[k]}j^{I}_{\mu}$, which is by construction a solution to the kth-order equations of (5.43a), will also be a solution to the kth-order equations of (5.43b) if we choose the quantities ${}^{(p)}a^{D}_{\mathrm{ret}}$ and ${}^{(p)}a^{D}_{\mathrm{adv}}$ in (A30) to be those appearing in (5.22), and if the coefficients of diffuse electric charge ${}^{(p)}e^{D}$ associated with the standard ideal particles in the region we are investigating satisfy the equations of charge

$$ {}^{(p)}_{[k]}\dot{e}^{D} = -c^2 l^2\, {}^{(p)}_{[k]}C^{I} + O(\kappa^{k+1}). \tag{A31} $$

Since Eqs. (A31) can be always satisfied step by step with respect to k, we see that a solution to the kth-order equations of (5.43a) and (5.43b) can always be constructed if one has already solved all lesser-order field equations.

Next we must look at Eqs. (5.43c) and (5.43d). We are assuming that the field equations (5.43) have been solved to the $(k-1)$th order, and because of the work above, we can also assume that the field ${}_{[k]}j^{I}_{\mu}$ is known and satisfies Eq. (5.45). If we apply the same sort of arguments which were used in analyzing (5.43a) to the kth-order equations of (5.43c), it is clear that if a solution to these equations exists which can be expanded in a power series in c^{-1}, then we can always construct a kth-order solution ${}_{[k]}\gamma^{EI}_{\mu}$ to Eqs. (5.43c) so that ${}_{[k]}\gamma^{EI,\mu\nu}_{\mu}$ will take the form

$$ {}_{[k]}\gamma^{EI,\mu\nu}_{\mu} = \sum_{p=1}^{N} {}^{(p)}a^{E}_{\mathrm{ret}}\, {}^{(p)}[{}_{[k]}C^{EI}(ru)^{-1}]_{\mathrm{ret}},{}^{\nu} + \sum_{p=1}^{N} {}^{(p)}a^{E}_{\mathrm{adv}}\, {}^{(p)}[{}_{[k]}C^{EI}(ru)^{-1}]_{\mathrm{adv}},{}^{\nu} + O(\kappa^{k+1}), \tag{A32} $$

where ${}^{(p)}_{[k]}C^{EI}$ is a function of ${}^{(p)}\tau$ only, and the constants ${}^{(p)}a^{E}_{\mathrm{ret}}$ and ${}^{(p)}a^{E}_{\mathrm{adv}}$ are arbitrary except that they must satisfy the equations

$$ {}^{(p)}a^{E}_{\mathrm{ret}} + {}^{(p)}a^{E}_{\mathrm{adv}} = 1. \tag{A33} $$

The field ${}_{[k]}\gamma^{EI}_{\mu}$ will be a solution to the kth-order equations of (5.43d) if we choose the quantities ${}^{(p)}a^{E}_{\mathrm{ret}}$ and ${}^{(p)}a^{E}_{\mathrm{adv}}$ in (A32) to be those appearing in (5.23) and if the localized electric charges associated with the standard ideal particles in the region we are investigating satisfy the equations of charge

$$ {}^{(p)}_{[k]}\dot{e}^{E} = -c^{2}\, {}^{(p)}_{[k]}C^{EI} + O(\kappa^{k+1}). \tag{A34} $$

These equations can always be satisfied.

Next we look at Eqs. (5.43e)–(5.43f). Equations (5.43e)–(5.43f) will be satisfied to all orders of approximation if the localized magnetic charges associated with the standard ideal particles in the region we are investigating satisfy the equations of charge

$$ {}^{(p)}\dot{e}^{M} = 0. \tag{A35} $$

These equations can always be satisfied.

Finally we must look at Eqs. (5.43g)–(5.43h). The assumption that the field equations have been solved to the $(k-1)$th order means that ${}_{[k]}t_{\mu\nu}$ is known and satisfied Eqs. (5.46). The assumption that Eqs. (5.43g) have a solution that can be expanded in a power series in c^{-1} means, applying the same sort of analysis to these equations as we applied to Eqs. (5.43a), that the field ${}_{[k]}\gamma'^{I}_{(\mu\nu)},{}^{\nu}$ associated with such a solution ${}_{[k]}\gamma'^{I}_{(\mu\nu)}$ can always be written in the form

$$ \begin{aligned} {}_{[k]}\gamma'^{I}_{(\mu\nu)},{}^{\nu} = {} & \sum_{p=1}^{N} {}^{(p)}a^{G}_{\mathrm{ret}}\, {}^{(p)}[{}_{[k]}C'^{I}_{\mu}(ru)^{-1}]_{\mathrm{ret}} + \sum_{p=1}^{N} {}^{(p)}a^{G}_{\mathrm{adv}}\, {}^{(p)}[{}_{[k]}C'^{I}_{\mu}(ru)^{-1}]_{\mathrm{adv}} \\ & + \sum_{p=1}^{N}\sum_{i=1}^{\infty} {}^{(p)}a^{G_i}_{\mathrm{ret}}\, {}^{(p)}[{}_{[k]}C^{I}_{\mu(\sigma_1\cdots\sigma_i)}(ru)^{-1}]_{\mathrm{ret}},{}^{\sigma_1\cdots\sigma_i} \\ & + \sum_{p=1}^{N}\sum_{i=1}^{\infty} {}^{(p)}a^{G_i}_{\mathrm{adv}}\, {}^{(p)}[{}_{[k]}C^{I}_{\mu(\sigma_1\cdots\sigma_i)}(ru)^{-1}]_{\mathrm{adv}},{}^{\sigma_1\cdots\sigma_i} + {}_{[k]}\chi_{\mu} + O(\kappa^{k+1}), \end{aligned} \tag{A36} $$

where ${}^{(p)}_{[k]}C'^{I}_{\mu}$ and ${}^{(p)}_{[k]}C^{I}_{\mu(\sigma_1\cdots\sigma_i)}$ are functions of ${}^{(p)}\tau$ only, and the constants ${}^{(p)}a^{G}_{\mathrm{ret}}$, ${}^{(p)}a^{G}_{\mathrm{adv}}$, ${}^{(p)}a^{G_i}_{\mathrm{ret}}$, and ${}^{(p)}a^{G_i}_{\mathrm{adv}}$ are arbitrary except that they must satisfy the equations

$$ {}^{(p)}a^{G}_{\text{ret}} + {}^{(p)}a^{G}_{\text{adv}} = 1, \quad {}^{(p)}a^{G_i}_{\text{ret}} + {}^{(p)}a^{G_i}_{\text{adv}} = 1. \tag{A37} $$

For $i > 2$, the coefficients ${}^{(p)}_{[k]}C^{I}_{\mu(\sigma_1 \cdots \sigma_i)}$ in (A36) may be replaced by the coefficients ${}^{(p)}_{[k]}C^{I}_{(\mu\sigma_1)(\sigma_2 \cdots \sigma_i)}$,

$$ {}^{(p)}_{[k]}C^{I}_{(\mu\sigma_1)(\sigma_2\cdots\sigma_i)} = {}^{(p)}_{[k]}C^{I}_{\mu(\sigma_1\sigma_2\cdots\sigma_i)} + {}^{(p)}_{[k]}C^{I}_{\sigma_1(\mu\sigma_2\sigma_3\cdots\sigma_i)} - {}^{(p)}_{[k]}C^{I}_{\sigma_2(\mu\sigma_1\sigma_3\cdots\sigma_i)}, \tag{A38} $$

which are symmetric in μ and σ_1. This means that we can write

$$ \begin{aligned} \gamma'^{I}_{(\mu\nu)}{}^{,\nu} = {} & \sum_{p=1}^{N} {}^{(p)}a^{G}_{\text{ret}}\, {}^{(p)}[{}_{[k]}C'^{I}_{\mu}(ru)^{-1}]_{\text{ret}} + \sum_{p=1}^{N} {}^{(p)}a^{G}_{\text{adv}}\, {}^{(p)}[{}_{[k]}C'^{I}_{\mu}(ru)^{-1}]_{\text{adv}} \\ & + \sum_{p=1}^{N} {}^{(p)}a^{G}_{\text{ret}}\, {}^{(p)}[{}_{[k]}C'^{I}_{\mu\nu}(ru)^{-1}]_{\text{ret}}{}^{,\nu} + \sum_{p=1}^{N} {}^{(p)}a^{G}_{\text{adv}}\, {}^{(p)}[{}_{[k]}C'^{I}_{\mu\nu}(ru)^{-1}]_{\text{adv}}{}^{,\nu} \\ & + \sum_{p=1}^{N} \sum_{i=2}^{\infty} {}^{(p)}a^{G_i}_{\text{ret}}\, {}^{(p)}[{}_{[k]}C^{I}_{(\mu\sigma_1)(\sigma_2\cdots\sigma_i)}(ru)^{-1}]_{\text{ret}}{}^{,\sigma_1\cdots\sigma_i} \\ & + \sum_{p=1}^{N} \sum_{i=2}^{\infty} {}^{(p)}a^{G_i}_{\text{adv}}\, {}^{(p)}[{}_{[k]}C^{I}_{(\mu\sigma_1)(\sigma_2\cdots\sigma_i)}(ru)^{-1}]_{\text{adv}}{}^{,\sigma_1\cdots\sigma_i} + {}_{[k]}\chi_\mu + O(\kappa^{k+1}), \end{aligned} \tag{A39} $$

where we have chosen ${}^{(p)}a^{G_1}_{\text{ret}} = {}^{(p)}a^{G}_{\text{ret}}$, ${}^{(p)}a^{G_1}_{\text{adv}} = {}^{(p)}a^{G}_{\text{adv}}$ and are using the notation

$$ {}^{(p)}_{[k]}C'^{I}_{\mu\nu} = {}^{(p)}_{[k]}C^{I}_{\mu(\nu)}. \tag{A40} $$

Next note that the equations

$$ \begin{aligned} & \Box^2\, {}_{[k]}\varphi_1(x^2, x^3, x^4) = {}_{[k]}\chi_{1,1}(0, x^2, x^3, x^4), \\ & \Box^2\, {}_{[k]}\varphi_2(x^1, x^3, x^4) = {}_{[k]}\chi_{2,2}(x^1, 0, x^3, x^4), \\ & \Box^2\, {}_{[k]}\varphi_3(x^1, x^2, x^4) = {}_{[k]}\chi_{3,3}(x^1, x^2, 0, x^4), \\ & \nabla^2\, {}_{[k]}\varphi_4(x^1, x^2, x^3) = -{}_{[k]}\chi_{4,4}(x^1, x^2, x^3, 0), \end{aligned} \tag{A41} $$

can, over a finite region of space-time, always be solved for a set of nonsingular functions ${}_{[k]}\varphi_\mu$ and that these functions can be used to define a field ${}_{[k]}\gamma'^{H}_{(\mu\nu)}$,

$$ \begin{aligned} & {}_{[k]}\gamma'^{H}_{(11)} = {}_{[k]}\varphi_1 - \int_0^{x^1} {}_{[k]}\chi_1(\alpha, x^2, x^3, x^4)\, d\alpha, \\ & {}_{[k]}\gamma'^{H}_{(22)} = {}_{[k]}\varphi_2 - \int_0^{x^2} {}_{[k]}\chi_2(x^1, \alpha, x^3, x^4)\, d\alpha, \\ & {}_{[k]}\gamma'^{H}_{(33)} = {}_{[k]}\varphi_3 - \int_0^{x^3} {}_{[k]}\chi_3(x^1, x^2, \alpha, x^4)\, d\alpha, \\ & {}_{[k]}\gamma'^{H}_{(44)} = {}_{[k]}\varphi_4 + \int_0^{x^4} {}_{[k]}\chi_4(x^1, x^2, x^3, \alpha)\, d\alpha, \\ & {}_{[k]}\gamma'^{H}_{(4s)} = {}_{[k]}\gamma'^{H}_{(s4)} = 0, \end{aligned} \tag{A42} $$

which satisfies the equations

$$ \Box^2{}_{[k]}\gamma'^{H}_{(\mu\nu)} = O(\kappa^{k+1}), \quad {}_{[k]}\gamma'^{H}_{(\mu\nu)}{}^{,\nu} = -{}_{[k]}\chi_\mu. \tag{A43} $$

This means that the field ${}_{[k]}\gamma^{I}_{(\mu\nu)}$,

$$ \begin{aligned} {}_{[k]}\gamma^{I}_{(\mu\nu)} = {} & {}_{[k]}\gamma'^{I}_{(\mu\nu)} + {}_{[k]}\gamma'^{H}_{(\mu\nu)} - \sum_{p=1}^{N} {}^{(p)}a^{G}_{\text{ret}}\, {}^{(p)}[({}_{[k]}C'^{I}_{(\mu\nu)} - {}_{[k]}C'^{I}_{[\mu\rho]}u^\rho u_\nu - {}_{[k]}C'^{I}_{[\nu\rho]}u^\rho u_\mu)(ru)^{-1}]_{\text{ret}} \\ & - \sum_{p=1}^{N} {}^{(p)}a^{G}_{\text{adv}}\, {}^{(p)}[({}_{[k]}C'^{I}_{(\mu\nu)} - {}_{[k]}C'^{I}_{[\mu\rho]}u^\rho u_\nu - {}_{[k]}C'^{I}_{[\nu\rho]}u^\rho u_\mu)(ru)^{-1}]_{\text{adv}} \\ & - \sum_{p=1}^{N} \sum_{i=2}^{\infty} {}^{(p)}a^{G_i}_{\text{ret}}\, {}^{(p)}[{}_{[k]}C^{I}_{(\mu\nu)(\sigma_2\cdots\sigma_i)}(ru)^{-1}]_{\text{ret}}{}^{,\sigma_2\cdots\sigma_i} \\ & - \sum_{p=1}^{N} \sum_{i=2}^{\infty} {}^{(p)}a^{G_i}_{\text{adv}}\, {}^{(p)}[{}_{[k]}C^{I}_{(\mu\nu)(\sigma_2\cdots\sigma_i)}(ru)^{-1}]_{\text{adv}}{}^{,\sigma_2\cdots\sigma_i}, \end{aligned} \tag{A44} $$

is a solution to Eqs. (5.43g) for which

$$ {}_{[k]}\gamma^{I}_{(\mu\nu)}{}^{,\nu}=\sum_{p=1}^{N}{}^{(p)}a^{G}_{\text{ret}}\,{}^{(p)}[{}_{[k]}C^{I}_{\mu}(ru)^{-1}]_{\text{ret}}+\sum_{p=1}^{N}{}^{(p)}a^{G}_{\text{adv}}\,{}^{(p)}[{}_{[k]}C^{I}_{\mu}(ru)^{-1}]_{\text{adv}} $$
$$ +\sum_{p=1}^{N}{}^{(p)}a^{G}_{\text{ret}}\,{}^{(p)}[{}_{[k]}C^{I}_{[\mu\nu]}(ru)^{-1}]_{\text{ret}}{}^{,\nu}+\sum_{p=1}^{N}{}^{(p)}a^{G}_{\text{adv}}\,{}^{(p)}[{}_{[k]}C^{I}_{[\mu\nu]}(ru)^{-1}]_{\text{adv}}{}^{,\nu}+O(\kappa^{k+1}), \tag{A45} $$

where

$$ {}^{(p)}_{[k]}C^{I}_{\mu}={}^{(p)}({}_{[k]}C'^{I}_{\mu}+2{}_{[k]}\dot{C}'^{I}_{[\mu\rho]}u^{\rho}+2{}_{[k]}C'^{I}_{[\mu\rho]}\dot{u}^{\rho}), \tag{A46} $$

$$ {}^{(p)}_{[k]}C^{I}_{[\mu\nu]}={}^{(p)}({}_{[k]}C'^{I}_{[\mu\nu]}-{}_{[k]}C'^{I}_{[\mu\rho]}u^{\rho}u_{\nu}+{}_{[k]}C'^{I}_{[\nu\rho]}u^{\rho}u_{\mu}). \tag{A47} $$

Note that

$$ {}^{(p)}_{[k]}C^{I}_{[\mu\nu]}\,{}^{(p)}u^{\nu}=0. \tag{A48} $$

The field ${}_{[k]}\gamma^{I}_{(\mu\nu)}$ above, which is by construction a solution to the kth-order equations of (5.43g), will also be a solution to the kth-order equations of (5.43h) if we choose the constants ${}^{(p)}a^{G}_{\text{ret}}$ and ${}^{(p)}a^{G}_{\text{adv}}$ to be those appearing in (5.25) and if the mass, motion, and spin associated with each standard ideal particle in the region we are investigating satisfy the equations of mass and motion

$$ {}^{(p)}_{[k]}\dot{P}_{\mu}=-c^{2}\,{}^{(p)}_{[k]}C^{I}_{\mu}+O(\kappa^{k+1}), \tag{A49} $$

and the equations of spin

$$ {}^{(p)}({}_{[k]}\dot{S}_{\mu\nu}-{}_{[k]}\dot{S}_{\mu\rho}u^{\rho}u_{\nu}+{}_{[k]}\dot{S}_{\nu\rho}u^{\rho}u_{\mu})=-2c^{2}\,{}^{(p)}_{[k]}C^{I}_{[\mu\nu]}+O(\kappa^{k+1}). \tag{A50} $$

Since these equations can always be satisfied at each order of approximation we have shown that the kth-order equations of (5.42g) and (5.42h) can always be solved if one has already solved all lesser-order field equations.

This completes our demonstration that one can always solve the kth-order field equations of (5.43) if one has already solved all lesser-order field equations. Note that we have had to assume that at each order Eqs. (5.43a), (5.43c), and (5.43g), equations of the form of d'Alembert's equation, always have solutions which can be expanded in a power series in c^{-1}. This assumption appears to be a reasonable one as at each order Eqs. (5.43a), (5.43c), and (5.43g) can always be solved, at least formally, step by step with respect to powers of c^{-1}.

APPENDIX B: ON THE FIELDS ${}^{(p)}\gamma^{\text{self}}_{\mu\nu}$ AND ${}^{(p)}\gamma^{\text{ext}}_{\mu\nu}$

We shall illustrate some of the arguments which can be used to show that the field $\gamma_{\mu\nu}$ can be split into the two fields ${}^{(p)}\gamma^{\text{self}}_{\mu\nu}$ and ${}^{(p)}\gamma^{\text{ext}}_{\mu\nu}$.

First we look at ${}_{[1]}\gamma_{\mu\nu}$. This field can clearly be split into the two parts ${}^{(p)}_{[1]}\gamma^{\text{self}}_{\mu\nu}$ and ${}^{(p)}_{[1]}\gamma^{\text{ext}}_{\mu\nu}$ since the field ${}_{[1]}\gamma_{\mu\nu}$ can always be written

$$ {}_{[1]}\gamma_{\mu\nu}={}^{(p)}_{[1]}\gamma^{\text{self}}_{\mu\nu}+{}^{(p)}_{[1]}\gamma^{\text{ext}}_{\mu\nu}, \tag{B1} $$

where

$$ {}^{(p)}_{[1]}\gamma^{\text{self}}_{\mu\nu}={}^{(p)}_{[1]}\gamma_{\mu\nu}, \qquad {}^{(p)}_{[1]}\gamma^{\text{ext}}_{\mu\nu}=\sum_{p'\neq p}{}^{(p')}_{[1]}\gamma_{\mu\nu}. \tag{B2} $$

The field ${}^{(p)}_{[1]}\gamma^{\text{self}}_{\mu\nu}$ in (B2) is obviously singular along the world line of the pth particle while the field ${}^{(p)}_{[1]}\gamma^{\text{ext}}_{\mu\nu}$ is not singular along this world line. Note that the field ${}^{(p)}_{[1]}\gamma^{\text{ext}}_{\mu\nu}$ is a solution to the first-order field equations for $\gamma_{\mu\nu}$, where one ignores the existence of the pth particle.

We next investigate the field ${}_{[2]}\gamma_{(\mu\nu)}$. This field satisfies the differential equations

$$ \Box^{2}\,{}_{[2]}\gamma_{(\mu\nu)}={}_{[2]}t_{\mu\nu}+O(\kappa^{3}), \tag{B3} $$

where ${}_{[2]}t_{\mu\nu}$ is bilinear in ${}_{[1]}\gamma_{\mu\nu}$. The quantity ${}_{[2]}t_{\mu\nu}$ can be split into the two parts ${}^{(p)}_{[2]}t^{\text{self}}_{\mu\nu}$ and ${}^{(p)}_{[2]}t^{\text{ext}}_{\mu\nu}$,

$$ {}_{[2]}t_{\mu\nu}={}^{(p)}_{[2]}t^{\text{self}}_{\mu\nu}+{}^{(p)}_{[2]}t^{\text{ext}}_{\mu\nu}, \tag{B4} $$

where ${}^{(p)}_{[2]}t^{\text{ext}}_{\mu\nu}$ contains those terms in ${}_{[2]}t_{\mu\nu}$ which are bilinear in ${}^{(p)}_{[1]}\gamma^{\text{ext}}_{\mu\nu}$, and ${}^{(p)}_{[2]}t^{\text{self}}_{\mu\nu}$ contains all other terms. The quantity ${}^{(p)}_{[2]}t^{\text{self}}_{\mu\nu}$ is singular along the world line of the pth particle while ${}^{(p)}_{[2]}t^{\text{ext}}_{\mu\nu}$ is not singular along this world line.

From the above we see that ${}_{[2]}\gamma_{(\mu\nu)}$ can be split into the two fields ${}^{(p)}_{[2]}\gamma^{\text{self}}_{(\mu\nu)}$ and ${}^{(p)}_{[2]}\gamma^{\text{ext}}_{(\mu\nu)}$,

$$ {}_{[2]}\gamma_{(\mu\nu)}={}^{(p)}_{[2]}\gamma^{\text{self}}_{(\mu\nu)}+{}^{(p)}_{[2]}\gamma^{\text{ext}}_{(\mu\nu)}, \tag{B5} $$

where

$$ \Box^{2}\,{}^{(p)}_{[2]}\gamma^{\text{self}}_{(\mu\nu)}={}^{(p)}_{[2]}t^{\text{self}}_{\mu\nu}+O(\kappa^{3}), \qquad \Box^{2}\,{}^{(p)}_{[2]}\gamma^{\text{ext}}_{(\mu\nu)}={}^{(p)}_{[2]}t^{\text{ext}}_{\mu\nu}+O(\kappa^{3}). \tag{B6} $$

The field ${}^{(p)}_{[2]}\gamma^{\text{self}}_{(\mu\nu)}$ in (B6) is singular along the world line of the pth particle while the field ${}^{(p)}_{[2]}\gamma^{\text{ext}}_{(\mu\nu)}$ can be chosen so as not to be singular along this world line. We are assuming that the equations of charge, mass, motion, and spin given in (5.64)–(5.66) are satisfied. We note that ${}^{(p)}_{[2]}\gamma^{\text{ext}}_{(\mu\nu)}$ is a solution to the second-order field equations where one ignores the existence of the pth particle except as it affects the coefficients of diffuse electric charge, the localized electric and magnetic charges, the masses, the spins, and the motion of the other $N-1$ particles in the region under investigation.

Applying the same sort of analysis as above to the equations satisfied by ${}_{[2]}\gamma_{[\mu\nu]}$, one easily sees that ${}_{[2]}\gamma_{[\mu\nu]}$ can also be split into the two parts ${}^{(p)}_{[2]}\gamma^{\text{self}}_{[\mu\nu]}$ and ${}^{(p)}_{[2]}\gamma^{\text{ext}}_{[\mu\nu]}$.

If we continue to apply the same sort of analysis as above to the field equations at each order of approximation, it is clear that we shall find, assuming our approximation procedure converges, that $\gamma_{\mu\nu}$ itself can be split into the two fields ${}^{(p)}\gamma^{\text{self}}_{\mu\nu}$ and ${}^{(p)}\gamma^{\text{ext}}_{\mu\nu}$.

---

[1] A. Einstein, Sitzber. Preuss. Akad. Wiss. 778 (1915); 779 (1915); 831 (1915); 844 (1915); Ann. Physik 49, 769 (1916).

[2] The works in which Einstein develops this theory are (a) A. Einstein, Sitzber. Preuss. Akad. Wiss. 414 (1925); (b) Ann. Math. 46, 578 (1945); (c) A. Einstein and E. G. Straus, Ann. Math. 47, 731 (1946); (d) A. Einstein, Rev. Modern Phys. 20, 35 (1948); (e) Can. J. Math. 2, 120 (1950); (f) *The Meaning of Relativity* (Princeton Univ. Press, Princeton, 1950), 3rd ed., Appendix II, pp. 133–162; (g) A. Einstein and B. Kaufman, Contribution to the volume *Louis de Broglie, physicien et penseur* (Albin Michel, Paris, 1953), pp. 321–342; (h) A. Einstein, *The Meaning of Relativity* (Princeton Univ. Press, Princeton, 1953), 4th ed., Appendix II, pp. 133–165; (i) A. Einstein and B. Kaufman, Ann. Math. 59, 230 (1954); (j) Ann. Math. 62, 128 (1955); (k) A. Einstein, *The Meaning of Relativity* (Princeton Univ. Press, Princeton, 1955), 5th ed., Appendix II, pp. 133–166.

[3] The author makes use of the previous work of many other authors. Some papers which proved especially helpful are (a) J. Lubański, Acta Phys. Polon. 6, 163 (1937); (b) L. Infeld and P. R. Wallace, Phys. Rev. 57, 797 (1940); (c) A. Einstein and L. Infeld, Can. J. Math. 1, 209 (1949); (d) V. V. Narlikar and B. R. Rao, Proc. Natl. Inst. Sci. India A21, 409 (1955); (e) H. Treder, Ann. Physik 19, 369 (1957); (f) E. Clauser, Nuovo Cimento 7, 764 (1958); (g) R. Sachs and P. G. Bergmann, Phys. Rev. 112, 674 (1958); (h) R. P. Kerr, Nuovo Cimento 13, 469 (1959); 13, 492 (1959); 13, 673 (1959); (i) R. P. Kerr, *ibid.* 16, 26 (1960); (j) P. Havas and J. N. Goldberg, Phys. Rev. 128, 398 (1962). References to other works which proved helpful can be found in the above papers.

[4] See Ref. 3(h).

[5] See Ref. 3(j).

[6] See Ref. 3(c).

[7] In discussing two explicit procedures for obtaining at each order certain tensors (under Lorentz transformations) entering into the equations of motion, the author discusses a procedure in his paper which is not Lorentz-covariant at each stage of the analysis. One also has the freedom, as discussed in the paper, to use a completely Lorentz-covariant procedure. Such a procedure is also described in the paper. The final result will be identical whichever procedure is used and Lorentz-covariant at each order. In Appendixes A and B of paper II the first procedure is used to find the needed tensors. In a work which he hopes will soon be published, the author has used the second procedure to find the same tensors sought in Appendix B. The result is of course the same. The advantage of the second procedure over the first is, however, obvious. The reason the first procedure is used in Appendixes A and B of paper II is that at the time the author did that part of the work he had not yet discovered the second (totally Lorentz-covariant) procedure.

[8] Havas and Goldberg mention in their paper, but do not use, a method for avoiding infinite divergences. As solutions to the field equations at each order they suggest choosing "Riesz solutions" [M. Riesz, Acta Math. 81, 1 (1949)] which are finite at the positions of the singularities. The author regards the choice of such solutions as artificial, arbitrary, and just another way of renormalization. If the singularities in the fields are treated properly, as in the author's work, such renormalization (the choice of artificially constructed finite solutions) is not needed.

[9] In this paper the author works exclusively in a specific set of coordinate systems, the harmonic coordinate systems. This is not necessary. In another paper [to be published] the author has shown that with slight modification the procedure developed in the present paper for finding equations of motion can be used without imposing any conditions on the coordinates except that the field be Minkowskian in the absence of particles.

[10] Although harmonic coordinates are used in this paper, the explicit equations of motion found in paper II can be shown to hold (to the same order) in any coordinate system satisfying the condition that the field be Minkowskian in the absence of particles. See Ref. 9.

[11] J. Callaway, Phys. Rev. 92, 1567 (1953).

[12] Since Callaway's work, several authors have shown that a Coulomb force can exist between charged particles in Einstein's theory. See Refs. 3(d)–3(f).

[13] In that appendix it is mentioned that there may be another interaction between elementary particles important over distances $\lesssim 10^{-2}$ cm. Subsequent investigation has shown that this interaction does not exist.

[14] This transformation law follows from the transformation law of $g_{\mu\nu}$.

[15] See Ref. 2(e).

[16] See Ref. 2(e).

[17] If we are only interested in solutions to the homogeneous equations (5.6) which can be expanded in a power series in c^{-1} when expressed in terms of x^s and t, then it can be shown that over a perfectly isolated region of the continuum containing only ideal particles the fields (5.16)–(5.26) are the most general solutions to the homogeneous equations for which $j^{H,\mu}_{\mu}$, $\gamma^{EH,\mu\nu}_{\mu}$, $\gamma^{MH,\mu\nu}_{\mu}$, and

$\gamma^{H}_{(\mu\nu)},{}^{\nu}$ take the forms given in (5.33)–(5.36) if ${}^{(p)}C^{DH}$, ${}^{(p)}C^{EH}$, ${}^{(p)}C^{MH}$, ${}^{(p)}C^{H}_{\mu}$, and ${}^{(p)}C^{H}_{[\mu\nu]}$ are considered arbitary except that ${}^{(p)}C^{H}_{[\mu\nu]}\,{}^{(p)}u^{\nu}=0$. By the term most general we mean that any other set of solutions to the homogeneous equations will differ from those given in (5.16)–(5.26) by only terms in j^{H}_{μ}, γ^{EH}_{μ}, and γ^{MH}_{μ} which do not contribute to $\gamma^{H}_{[\mu\nu]}$ or by terms in $\gamma^{H}_{(\mu\nu)}$ which reflect the use of a different harmonic coordinate system than that used in (5.16)–(5.26). Of course one can always add to the fields (5.16)–(5.26) arbitrary nonsingular solutions to the homogeneous equations. Such additions to the fields are to be considered as representing external influences acting within the region under investigation and will be taken to be zero when dealing with a perfectly isolated region. In this footnote we have considered only fields which can be expanded in a power series in c^{-1}. The reason for this restriction is that the author has not proved that the statements made in this footnote apply to more general fields. The author has no reason, however, for believing that the statements made in this footnote do not hold for more general fields.

[18]As the regions we shall be investigating are assumed to be perfectly isolated, no external electromagnetic or gravitational field will exist in them. If we wish to investigate a region where such fields do exist we must choose a solution to the homogeneous equations (5.6) which indicates the presence of such fields. We shall not investigate such regions in this paper.

[19]If we are only interested in solutions which can be expanded in a power series in c^{-1} when expressed in terms of x^{s} and t, then the solutions to the inhomogeneous equations over a perfectly isolated region of the continuum containing only ideal particles are uniquely determined by the choice we make of solutions to the homogeneous equations. This is so because at each step two different solutions to the inhomogeneous equations will differ at most by a solution to the homogeneous equations satisfying (5.50), (5.54), (5.55), and (5.59) where (5.60) is valid. In a previous footnote we have discussed the fact that such solutions to the homogeneous equations can always be considered to take the general form (5.16)–(5.26). This means that at each order, different solutions to the inhomogeneous equations give rise to solutions to the full field equations which differ, excepting terms which depend on the harmonic coordinate system used, by at most a redefinition of those quantities which describe the structure of the ideal particles in the perfectly isolated region we are investigating.

[20]The splitting of the field $\gamma_{\mu\nu}$ into the two parts ${}^{(p)}\gamma^{\text{self}}_{\mu\nu}$ and ${}^{(p)}\gamma^{\text{ext}}_{\mu\nu}$ is discussed in more detail in Appendix B.

[21]If one has found a kth-order solution ${}_{[k]}\gamma'^{I}_{(\mu\nu)}$ to the inhomogeneous equations (5.43g) so that

$$\begin{aligned}{}_{[k]}\gamma'^{I}_{(\mu\nu)},{}^{\nu} &= \textstyle\sum_{p}{}^{(p)}a^{G}_{\text{ret}}\,{}^{(p)}[{}_{[k]}C'^{I}_{\mu}(ru)^{-1}]_{\text{ret}} \\ &+ \textstyle\sum_{p}{}^{(p)}a^{G}_{\text{adv}}\,{}^{(p)}[{}_{[k]}C'^{I}_{\mu}(ru)^{-1}]_{\text{adv}} \\ &+ \textstyle\sum_{p}{}^{(p)}a^{G}_{\text{ret}}\,{}^{(p)}[{}_{[k]}C'^{I}_{\mu\nu}(ru)^{-1}]_{\text{ret}},{}^{\nu} \\ &+ \textstyle\sum_{p}{}^{(p)}a^{G}_{\text{adv}}\,{}^{(p)}[{}_{[k]}C'^{I}_{\mu\nu}(ru)^{-1}]_{\text{adv}},{}^{\nu} + O(\kappa^{k+1}),\end{aligned}$$

it is then a simple matter to find a kth-order solution ${}_{[k]}\gamma^{I}_{(\mu\nu)}$ to (5.43g) so that (5.59) and (5.60) are valid. If we choose

$$\begin{aligned}{}_{[k]}\gamma^{I}_{(\mu\nu)} &= {}_{[k]}\gamma'^{I}_{(\mu\nu)} - \textstyle\sum_{p}{}^{(p)}a^{G}_{\text{ret}}\,{}^{(p)}[({}_{[k]}C'^{I}_{(\mu\nu)} \\ &\quad - {}_{[k]}C'^{I}_{[\mu\rho]}u^{\rho}u_{\nu} - {}_{[k]}C'^{I}_{[\nu\rho]}u^{\rho}u_{\mu})(ru)^{-1}]_{\text{ret}} \\ &\quad - \textstyle\sum_{p}{}^{(p)}a^{G}_{\text{adv}}\,{}^{(p)}[({}_{[k]}C'^{I}_{(\mu\nu)} - {}_{[k]}C'^{I}_{[\mu\rho]}u^{\rho}u_{\nu} \\ &\quad - {}_{[k]}C'^{I}_{[\nu\rho]}u^{\rho}u_{\mu})(ru)^{-1}]_{\text{adv}} + O(\kappa^{k+1}),\end{aligned}$$

we have such a solution as

$$\Box^{2}\,{}_{[k]}\gamma^{I}_{(\mu\nu)} = \Box^{2}\,{}_{[k]}\gamma'^{I}_{(\mu\nu)} + O(\kappa^{k+1}),$$

and

$$\begin{aligned}{}_{[k]}\gamma^{I}_{(\mu\nu)},{}^{\nu} &= \textstyle\sum_{p}{}^{(p)}a^{G}_{\text{ret}}\,{}^{(p)}[{}_{[k]}C^{I}_{\mu}(ru)^{-1}]_{\text{ret}} \\ &+ \textstyle\sum_{p}{}^{(p)}a^{G}_{\text{adv}}\,{}^{(p)}[{}_{[k]}C^{I}_{\mu}(ru)^{-1}]_{\text{adv}} \\ &+ \textstyle\sum_{p}{}^{(p)}a^{G}_{\text{ret}}\,{}^{(p)}[{}_{[k]}C^{I}_{[\mu\nu]}(ru)^{-1}]_{\text{ret}},{}^{\nu} \\ &+ \textstyle\sum_{p}{}^{(p)}a^{G}_{\text{adv}}\,{}^{(p)}[{}_{[k]}C^{I}_{[\mu\nu]}(ru)^{-1}]_{\text{adv}},{}^{\nu} + O(\kappa^{k+1}),\end{aligned}$$

where

$${}^{(p)}_{[k]}C^{I}_{\mu} = {}^{(p)}({}_{[k]}C'^{I}_{\mu} + 2\,{}_{[k]}\dot{C}'^{I}_{[\mu\rho]}u^{\rho} + 2\,{}_{[k]}C'^{I}_{[\mu\rho]}\dot{u}^{\rho}),$$

$${}^{(p)}_{[k]}C^{I}_{[\mu\nu]} = {}^{(p)}({}_{[k]}C'^{I}_{[\mu\nu]} - {}_{[k]}C'^{I}_{[\mu\rho]}u^{\rho}u_{\nu} + {}_{[k]}C'^{I}_{[\nu\rho]}u^{\rho}u_{\mu}).$$

Note that

$${}^{(p)}_{[k]}C^{I}_{[\mu\nu]}\,{}^{(p)}u^{\nu} = 0.$$

[22]The spin ${}^{(p)}S_{\mu\nu}$ associated with a particle can always be a considered as made up of an intrinsic spin ${}^{(p)}S^{\text{int}}_{\mu\nu}$ and a dipole spin ${}^{(p)}S^{\text{dip}}_{\mu\nu}$. It is easy to show that

$${}^{(p)}S_{\mu\nu} = {}^{(p)}S^{\text{int}}_{\mu\nu} + {}^{(p)}S^{\text{dip}}_{\mu\nu},$$

where

$${}^{(p)}S^{\text{int}}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\,{}^{(p)}S^{\sigma}\,{}^{(p)}u^{\rho},$$

$${}^{(p)}S^{\text{dip}}_{\mu\nu} = {}^{(p)}D_{\mu}\,{}^{(p)}u_{\nu} - {}^{(p)}D_{\nu}\,{}^{(p)}u_{\mu},$$

with

$${}^{(p)}S_{\mu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\,{}^{(p)}S^{\rho\sigma}\,{}^{(p)}u^{\nu}, \quad {}^{(p)}D_{\mu} = {}^{(p)}S_{\mu\nu}\,{}^{(p)}u^{\nu}.$$

Note that

$${}^{(p)}S_{\mu}\,{}^{(p)}u^{\mu} = 0, \quad {}^{(p)}D_{\mu}\,{}^{(p)}u^{\mu} = 0,$$

$${}^{(p)}S^{\text{int}}_{\mu\nu}\,{}^{(p)}u^{\nu} = 0.$$

The equations of mass, motion, and spin will restrict the motion of particles for which either ${}^{(p)}S^{\text{int}}_{\mu\nu}=0$ or ${}^{(p)}S^{\text{dip}}_{\mu\nu}=0$. The condition ${}^{(p)}S^{\text{dip}}_{\mu\nu}=0$, a condition which can be considered as equivalent to the demand that the pth particle possess no mass dipole moment ${}^{(p)}D_{\mu}$, seems to be a natural condition to place on the structure of an elementary particle.